DEGENERATE FLAT BANDS IN TWISTED BILAYER GRAPHENE

SIMON BECKER, TRISTAN HUMBERT, AND MACIEJ ZWORSKI

Abstract. We prove that in the chiral limit of the Bistritzer–MacDonald Hamiltonian, there exist magic angles at which the Hamiltonian exhibits flat bands of multiplicity four instead of two. We analyse the structure of Bloch functions associated with the bands of arbitrary multiplicity, compute the corresponding Chern number to be -1, and show that there exist infinitely many degenerate magic angles for a generic choice of tunnelling potential, including the Bistritzer–MacDonald potential. Moreover, we demonstrate for generic tunnelling potentials flat bands have only twofold or fourfold multiplicities.

1. Introduction and statement of results

Twisted bilayer graphene is a material consisting of two stacked graphene layers which are twisted with respect to each other by an angle θ. It has been predicted theoretically [BiMa11] that at a certain angle, the bands at zero energy become flat and strongly correlated electron effects dominate. This has then been experimentally confirmed that at this *magic angle*, the material exhibits phenomena such as superconductivity and a quantum Hall effect without external magnetic fields [Cao18, Ser19, Yan18]. Theoretically [TKV19] more magic angles have been expected. Contrary to common beliefs, we demonstrate that flat bands of higher multiplicity are common in this model of bilayer graphene. Higher multiplicity bands have recently also been theoretically observed in models of twisted trilayer graphene [PT23, De23]. We verify numerically that the presence of higher degenerate (almost flat) bands close to zero energy is also valid for the full (not just chiral) model, see Figure 6.

The model we consider is based on the Bistritzer-MacDonald Hamiltonian [BiMa11, CGG22, Wa*22] and its chiral limit of Tarnopolsky–Kruchkov–Vishwanath [TKV19]:

$$H(\alpha) = \begin{pmatrix} 0 & D(\alpha)^* \\ D(\alpha) & 0 \end{pmatrix} \text{ with } D(\alpha) = \begin{pmatrix} 2D_{\bar z} & \alpha U_+(z) \\ \alpha U_-(z) & 2D_{\bar z} \end{pmatrix} \tag{1.1}$$

where we use complex coordinates $z \in \mathbb{C}$ and the parameter α is proportional to the inverse relative twisting angle. Here, we write $z = x+iy$ for real x, y then $\partial_z := 1/2(\partial_x - i\partial_y)$, where ∂_x denotes differentiation with respect to x, and $D_z = -i\partial_z$. Similarly, $\partial_{\bar z} := \frac{1}{2}(\partial_x + i\partial_y)$ and $D_{\bar z} = -i\partial_{\bar z}$. Clearly, $H(\alpha) : H^1(\mathbb{C};\mathbb{C}^4) \subset L^2(\mathbb{C};\mathbb{C}^4) \to L^2(\mathbb{C};\mathbb{C}^4)$ is self-adjoint and $D(\alpha) : H^1(\mathbb{C};\mathbb{C}^2) \subset L^2(\mathbb{C};\mathbb{C}^2) \to L^2(\mathbb{C};\mathbb{C}^2)$. With $\omega = e^{2\pi i/3}$ and

$\mathbf{a} = 4\pi i(a_1\omega + a_2\bar{\omega})$, $a_j \in \mathbb{Z}$, we assume that

$$U_\pm(z+\mathbf{a}) = \omega^{\mp(a_1+a_2)}U(z), \quad U_\pm(\omega z) = \omega U_\pm(z). \tag{1.2}$$

In the physics literature, the following choice is made [BiMa11, TKV19]

$$U_+(z) = U(z), \quad U_-(z) = U(-z), \quad \overline{U(\bar{z})} = U(z), \tag{1.3}$$

see (1.12) for concrete examples.

Floquet theory for the Hamiltonian (1.1) is based on moiré translations:

$$\mathscr{L}_\mathbf{a} u(z) := \begin{pmatrix} \omega^{-(a_1+a_2)} & 0 \\ 0 & \omega^{a_1+a_2} \end{pmatrix} u(z+\mathbf{a}), \quad \mathbf{a} = 4\pi i(a_1\omega + a_2\bar{\omega}). \tag{1.4}$$

The action is extended diagonally to $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$-valued functions and we $\mathscr{L}_\mathbf{a} H(\alpha) = H(\alpha)\mathscr{L}_\mathbf{a}$.

The Floquet spectrum is given by

$$\begin{gathered} H(\alpha)u = Eu, \quad u \in H^1_k \quad H^s_k := L^2_k \cap H^s_{\mathrm{loc}}, \\ L^2_k := \{u = L^2_{\mathrm{loc}}(\mathbb{C};\mathbb{C}^4) : \mathscr{L}_\mathbf{a} u = e^{i\langle k,\mathbf{a}\rangle}u\}, \quad \langle z,w\rangle := \mathrm{Re}(z\bar{w}), \text{ and } k \in \mathbb{C}. \end{gathered} \tag{1.5}$$

The spectrum is discrete and symmetric with respect to the origin and we index it as follows (with $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$)

$$\begin{gathered} \{E_j(\alpha,k)\}_{j\in\mathbb{Z}^*}, \quad E_j(\alpha,k) = -E_{-j}(\alpha,k), \\ 0 \le E_1(\alpha,k) \le E_2(\alpha,k) \le \cdots, \quad E_1(\alpha,K) = E_1(\alpha,-K) = 0, \end{gathered} \tag{1.6}$$

see [BHZ24, §2] for more details. The points $K, -K$, $K = i$, are called the *Dirac points* and are typically denoted by K and K' in the physics literature.

Using the conjugation,

$$H_k(\alpha) := e^{i\langle z,k\rangle}H(\alpha)e^{-i\langle z,k\rangle},$$

we can equivalently study operators $H_k(\alpha)$ on L^2_0.

Definition (Magic angles and their multiplicities). *A value of α in (1.1) is called magical if $H(\alpha)$ has a flat band at zero*

$$E_1(\alpha,k) \equiv 0, \quad k \in \mathbb{C}.$$

The set of magic α's is denoted by $\mathcal{A}$ or $\mathcal{A}(U)$ if we specify the dependence on the potential. The multiplicity of a magic α is defined as

$$m(\alpha) = m_U(\alpha) = \min\{j > 0 : \max_k E_{j+1}(\alpha,k) > 0\}. \tag{1.7}$$

Magic angles are (up to physical constants) reciprocals of $\alpha \in \mathcal{A}$.

To formulate our first result on the multiplicity of flat bands, we need the following definition of multiplicity of zeros for $\mathbf{u} \in \ker(D(\alpha)+K)$:

$$M_\mathbf{u}(z_0) := \max\{m : [\partial_z^{m-1}\mathbf{u}](z_0) = 0\}, \tag{1.8}$$

with the convention that $M_{\mathbf{u}}(z_0) = 0$ if $\mathbf{u}(z_0) \neq 0$. As in [BHZ24, Lemma 3.2] we easily see that this is equivalent to $\mathbf{u}(z) = (z - z_0)^m \mathbf{u}_0(z)$, where $\mathbf{u}_0$ is smooth near z_0.

Theorem 1 (Zeros and multiplicities)**.** *For the Hamiltonian* (1.1) *with potentials satisfying* (1.2) *and* (1.3)*, let* $\mathbf{u}(\alpha) := \mathbf{u}_K(\alpha) \in \ker_{H^1_0}(D(\alpha) + K)$ *be a family of protected states (see* [Zw24, Theorem 1] *and references given there). Then for* $\alpha \in \mathbb{C}$

$$\sum_{z \in \mathbb{C}/\Lambda} M_{\mathbf{u}(\alpha)}(z) = m(\alpha). \tag{1.9}$$

For $\alpha \in \mathcal{A}$ *and* $k \in \mathbb{C}/\Lambda^*$,

$$\dim \ker_{H^1_0} H_k(\alpha) = 2 \dim \ker_{H^1_0}(D(\alpha) + k) = 2m(\alpha).$$

In particular, the zero-energy flat bands of the Hamiltonian are always spectrally gapped from the rest of the spectrum. Moreover, (1.9) *holds for any* $\mathbf{u}(\alpha) \in \ker_{H^1_0}(D(\alpha) + k) \setminus \{0\}$*,* $k \in \mathbb{C}/\Lambda^*$*.*

To formulate the next result we define

$$L^2_{0,p} := \{u \in L^2_0 : u(\omega z) = \bar{\omega}^p u(z)\}, \quad p \in \mathbb{Z}_3.$$

Using these spaces we have the following rigidity result for simple and double-degenerate magic angles:

Theorem 2 (Rigidity)**.** *Under the assumptions of Theorem 1 and with the definition of multiplicity* (1.7)*,*

$$\begin{aligned} m(\alpha) = 1 &\implies \dim \ker_{L^2_{0,2}} D(\alpha) = 1, \\ m(\alpha) = 2 &\implies \dim \ker_{L^2_{0,0}} D(\alpha) = \dim \ker_{L^2_{0,1}} D(\alpha) = 1. \end{aligned} \tag{1.10}$$

Moreover, for all $\alpha \in \mathcal{A}$*,*

$$m(\alpha) \not\equiv 0 \mod 3. \tag{1.11}$$

The first implication in (1.10) is included in [BHZ24, Theorem 2]. The multiplicity statement (1.11) which is a consequence of the proof of (1.10) was added because of a recent paper of Iugov–Nekrasov [IN25] where it was obtained using different methods.

To prove the existence of magic α's of higher multiplicities, we perform trace computations first used to show that $\mathcal{A}$ is non-empty [Be*22] and then that $|\mathcal{A}| = \infty$ [BHZ23]. The traces here refer to $\operatorname{tr} T_k^{2p}$ where T_k is a Birman–Schwinger operator with spectrum given by $\{1/\alpha : \alpha \in \mathcal{A}\}$ - see §2, [Be*22, Theorem 3], [BHZ23, Theorem 1].

Theorem 2 shows that to show the existence of degenerate α's we need to show that $\operatorname{tr}((T_0|_{L^2_{0,j}})^{2p}) \neq 0$, $j = 0, 1$ (as explained in §4 we are allowed to take $k = 0$).

Because of the symmetry of the spectrum (1.6) simple α's correspond flat bands of multiplicity 2 and double α's, to flat bands of multiplicity 4. In Figures 2 and 3, we see that the band structure for complex and real double magic angles behaves similarly

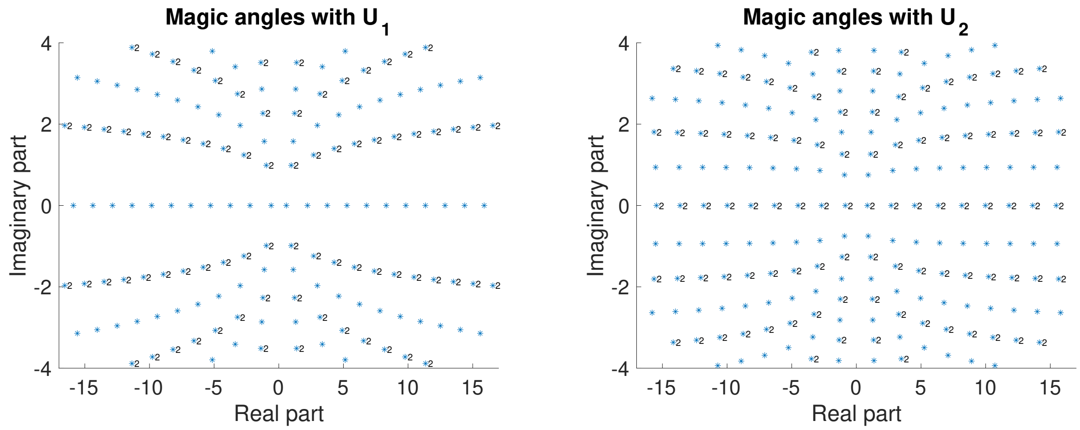

FIGURE 1. Magic angles α derived from potentials $U_\pm = U_1(\pm\bullet)$ (left) and $U_\pm = U_2(\pm\bullet)$ (right) in (1.1). The multiplicity of the flat bands u of $(D(\alpha)+k)u_k = 0$ is illustrated by the numbers (no number $\to$ simple magic angle, 2 $\to$ two-fold degenerate magic angle) in the figure. The movie https://math.berkeley.edu/~zworski/Interpolation.mp4 shows the magic angles for interpolation between these potentials: $U(z) = (\cos\theta - \sin\theta)U_1(z) + \sin\theta U_2(z)$; multiplicity one magic angles are coded by $*$ and multiplicity two by $*$.

close to the magic angle. The two bands are closest at the Γ point and are stacked on top of each other.

Examples of U's satisfying (1.2) and (1.3) are given by

$$U_1(z) = \sum_{\ell=0}^{2} \omega^\ell e^{\frac{1}{2}(z\bar\omega^\ell - \bar z\omega^\ell)} \quad \text{and} \quad U_2(z) = \tfrac{1}{\sqrt{2}}\Big(U_1(z) - \sum_{\ell=0}^{2} \omega^\ell e^{-(z\bar\omega^\ell - \bar z \omega^\ell)}\Big). \tag{1.12}$$

Numerical experiments suggest that these two potentials exhibit flat bands of different multiplicities:

$$m_{U_j}(\alpha) = j, \quad \alpha \in \mathcal{A}_{U_j} \cap \mathbb{R}, \quad j = 1, 2, \tag{1.13}$$

see Figure 1. We show (see Theorem 3 below) that the potential U_1 (the Bistritzer–MacDonald potential) has infinitely many (complex) degenerate magic α's. While in the case of U_1 all magic angles on the real axis appear to be simple, the two-fold degenerate magic angles, with non-zero imaginary parts, become real when a suitable magnetic field is added [Le22].

Theorem 3 (Degenerate magic angles)**.** *For the Bistritzer–MacDonald potential, $U_+ = U_1$ and $U_- = U_1(-\bullet)$, defined in (1.12), there exist infinitely many $\alpha \in \mathcal{A}$ which are not simple.*

k	α_k	$\alpha_k - \alpha_{k-1}$
1	0.585663	
2	2.221182	1.6355
3	3.751406	1.5302
4	5.276498	1.5251
5	6.794785	1.5183
6	8.312999	1.5182
7	9.829067	1.5161
8	11.345340	1.5163
9	12.860608	1.5153
10	14.376072	1.5155
11	15.890964	1.5149

k	α_k	$\alpha_k - \alpha_{k-1}$
1	0.853799	
2	2.691433	1.8376
3	4.507960	1.8165
4	6.332311	1.8244
5	8.157130	1.8248
6	9.983510	1.8264
7	11.809376	1.8259
8	13.635446	1.8261
9	15.460894	1.8255
10	17.286231	1.8253
11	19.111041	1.8248

TABLE 1. First 11 real magic angles, rounded to 6 digits, for $U_\pm = U_1(\pm\bullet)$ (left) and $U_\pm = U_2(\pm\bullet)$ (right). The α's for U_1 are simple and the ones on the right are double.

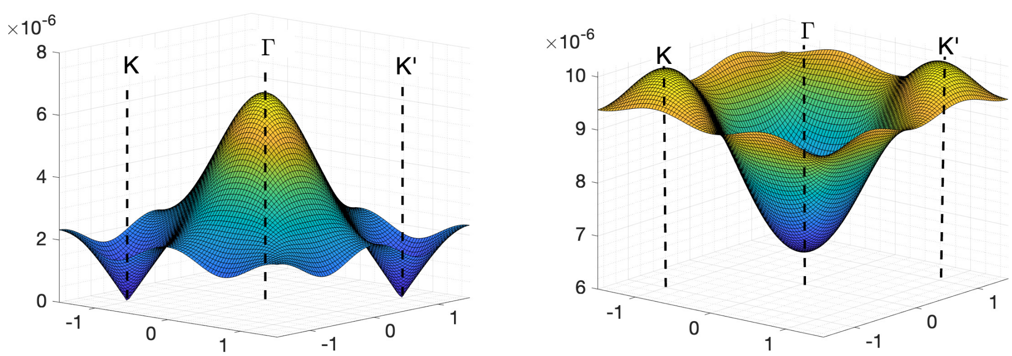

FIGURE 2. Let $\alpha \approx 0.853799$ as in Table 1, lowest two Bloch band with positive energy close to the first magic angle with $U_\pm = U_2(\pm\bullet)$. We plot $E_1(k)$ (left) and $E_2(k)$ (right).

Theorem 7 in §5 states this for a larger class of potentials satisfying the assumptions of [BHZ23, Theorem 5] with an additional non-degeneracy condition, see Theorem 6.

It is natural to ask if multiplicities always occur and if multiplicities of higher degree are also ubiquitous. If we do not demand that (1.3) holds, then, generically in the sense of Baire, magic angles are either simple or two-fold degenerate:

Theorem 4 (Generic simplicity)**.** *For Hamiltonians* (1.1) *satisfying* (1.2)*, there exists a generic subset (an intersection of open dense sets),* $\mathscr{V}_0 \subset \mathscr{V}$*, where the space of*

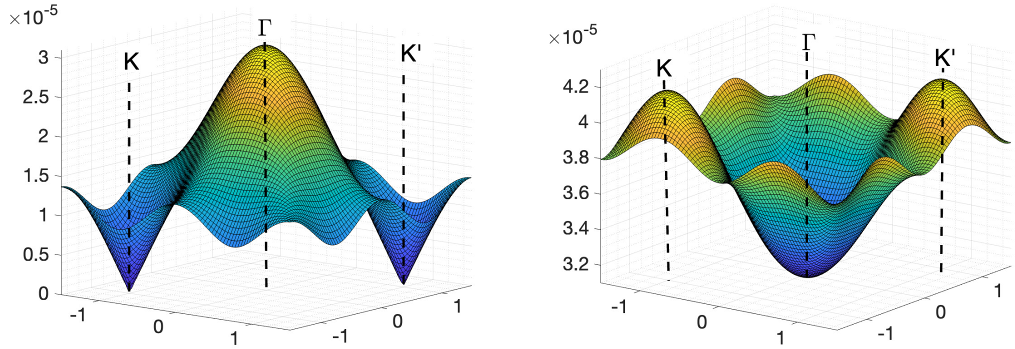

FIGURE 3. Let $\alpha \approx 0.9628 + 0.9873i$ the first complex magic angle for $U_\pm = U_1(\pm \bullet)$, lowest two Bloch band with positive energy close to the first degenerate magic angle. We plot $E_1(k)$ (left) and $E_2(k)$ (right).

matrix-valued potentials, $\mathscr{V}$, is defined in (7.3), such that if $V \in \mathscr{V}_0$ then (see (1.7))

$$m(\alpha) \leq 2.$$

More precisely, when α is simple then

$$\dim \ker_{L^2_{0,2}}(D(\alpha)) = 1 \text{ and } \dim \ker_{L^2_{0,0}}(D(\alpha)) = \dim \ker_{L^2_{0,1}}(D(\alpha)) = 0 \tag{1.14}$$

and when it is double,

$$\dim \ker_{L^2_{0,2}}(D(\alpha)) = 0 \text{ and } \dim \ker_{L^2_{0,0}}(D(\alpha)) = \dim \ker_{L^2_{0,1}}(D(\alpha)) = 1. \tag{1.15}$$

Remark. It may seem at first the conclusions (1.14),(1.15) follow from Theorem 2. However, in that theorem, we assumed also (1.3), which does not necessarily hold for potentials in $\mathscr{V}_0$.

The Chern number and Berry curvature associated to the degenerate flat band have similar properties to the case of simple flat bands. In particular, we have the following result proved in 9. More specifically, we prove that the Bloch vector bundle decomposes into a trivial bundle of rank $m(\alpha) - 1$ and a line bundle isomorphic to that of a simple band.

Theorem 5 (Flat band topology)**.** *For $\alpha \in \mathcal{A}$, the Chern number of the rank $m(\alpha)$ vector bundle E associated to $\ker_{L^2_0}(D(\alpha) + k)$ (see (9.19)) is*

$$c_1(E) = -1. \tag{1.16}$$

In addition, the trace of the curvature, H, is non-negative and satisfies $H(k) = H(\omega k)$, $H(k) = H(-k)$.

In Section 10, we collect numerical observations on the possibility of having eigenvalues of T_k of algebraic multiplicity 2 but geometric multiplicity 1 and thus corresponding to simple magic angles. We also discuss features of the Berry curvature for two-fold degenerate magic angles. That motivates the presentation of the properties of the curvature in Theorem 5.

Acknowledgements. We would like to thank Mengxuan Yang for helpful discussions.Special thanks go also to Zhongkai Tao for his help with simplifying Section 9. TH and MZ were partially supported by the National Science Foundation under the grant DMS-1901462 and by the Simons Foundation under a "Moiré Materials Magic" grant.

Data availability statement. The datasets generated during and/or analysed during the current study are available from the corresponding author on request.

Conflict of interest statement. The authors have no conflict of interest to declare.

2. Properties of the Hamiltonian

In Section 1 the moiré lattice is given by $4\pi i(\mathbb{Z}\omega \oplus \mathbb{Z}\bar{\omega})$ which is consistent with the notation in the physics literature [BiMa11, TKV19]. We followed it to make the article accessible to a broader audience. From Section 2 onwards, we introduce a simple change of variables $z_{\rm new} = \frac{4}{3}\pi i z_{\rm old}$ so that the lattice becomes $\Lambda := \mathbb{Z} \oplus \omega\mathbb{Z}$ with dual lattice $\Lambda^* := \frac{4\pi i}{\sqrt{3}}\Lambda$. In doing so, we simplify mathematical expressions involving, for instance, Jacobi theta functions. This new coordinate system has been introduced in [BZ23], see also [BHZ24, Appendix A].

Thus we work now with (1.1) but now we assume

$$U_\pm(z+\gamma) = e^{\pm i\langle \gamma, K\rangle} U_\pm(z), \quad \gamma \in \Lambda, \quad U_\pm(\omega z) = \omega U_\pm(z). \tag{2.1}$$

Here and elsewhere, $\langle z, w\rangle := \mathrm{Re}(z\bar{w})$, $\pm K$ are the nonzero points of high symmetry, $\omega K \equiv K \mod \Lambda^*$, $K = \frac{4}{3}\pi$.

The analogue of (1.3) is given by

$$U_+(z) = U(z), \quad U_-(z) = U(-z), \qquad \overline{U(\bar{z})} = -U(-z), \tag{2.2}$$

and the Bistritzer–MacDonald potential is now $U(z) = -\frac{4}{3}\pi i U_1(\frac{4}{3}\pi i z)$, where U_1 is given in (1.12).

The Hamiltonian is still of the form

$$H(\alpha) := \begin{pmatrix} 0 & D(\alpha)^* \\ D(\alpha) & 0 \end{pmatrix} \text{ with } D(\alpha) := \begin{pmatrix} 2D_{\bar{z}} & \alpha U_+(z) \\ \alpha U_-(z) & 2D_{\bar{z}} \end{pmatrix}. \tag{2.3}$$

We then define

$$\rho(z) := \mathrm{diag}(\chi_{k_j}(z)), \quad k_2 = -k_1 = K, \ \in \mathbb{C}/\Lambda^*, \quad \chi_k(z) := e^{i\langle z, k\rangle},$$

so that for $\gamma \in \Lambda$

$$V(z+\gamma) = \rho(\gamma)^{-1} V(z) \rho(\gamma), \quad V(z) := \begin{pmatrix} 0 & U_+(z) \\ U_-(z) & 0 \end{pmatrix}.$$

The modified potential, $V_\rho(z) := \rho(z) V(z) \rho(z)^{-1}$, is Λ-periodic and thus

$$\rho(z) D(\alpha) \rho(z)^{-1} = D_\rho(\alpha), \quad D_\rho(\alpha) := \operatorname{diag}((2D_{\bar z} - k_j)_{j=1}^2) + V_\rho(z).$$

Using the rotation operator $\Omega u(z) = u(\omega z)$, rotating by $2\pi/3$, satisfying $\Omega D(\alpha) = \omega D(\alpha) \Omega$, we can define $\mathscr{C} = \operatorname{diag}(1, \bar\omega)\Omega$ such that $\mathscr{C} H = H \mathscr{C}$ and translation operator $\mathscr{L}_\gamma u(z) := \rho(\gamma) u(z+\gamma)$. By using the translation $\mathscr{L}_\gamma$, we can define, for $k \in \mathbb{C}$, the spaces

$$H_k^s := H_k^s(\mathbb{C}/\Lambda, \mathbb{C}^n) := \{u \in H_{\mathrm{loc}}^s(\mathbb{C}; \mathbb{C}^n) : \mathscr{L}_\gamma u = e^{i\langle k, \gamma\rangle} u, \gamma \in \Lambda\}, \text{ with } L_k^2 := H_k^0,$$

where $n = 1$ corresponds to the first, $n = 2$ to the upper two, and $n = 4$ to all components of $\mathscr{L}_\gamma$.

When $k \in \mathcal{K} := \{K, -K, 0\} + \Lambda^*$ we also define

$$L_{k,p}^2 = L_{k,p}^2(\mathbb{C}/\Gamma; \mathbb{C}^n) := \{u \in L_k^2 : u(\omega z) = \bar\omega^p u(z)\}, \quad L_k^2 = \bigoplus_{p \in \mathbb{Z}_3} L_{k,p}^2.$$

We can then define a generalized Bloch transform

$$\mathcal{B}u(z,k) := \sum_{\gamma \in \Lambda} e^{i\langle z+\gamma, k\rangle} \mathscr{L}_\gamma u(z), \quad \mathcal{B}u(z, k+p) = e^{i\langle z, p\rangle} \mathcal{B}u(z,k), \quad p \in \Lambda^*, \quad u \in \mathscr{S}(\mathbb{C}),$$

$$\mathscr{L}_\alpha \mathcal{B}u(\bullet, k) = \sum_\gamma e^{i\langle z+\alpha+\gamma, k\rangle} \mathscr{L}_{\alpha+\gamma} u(z) = \mathcal{B}u(\bullet, k), \quad \alpha \in \Lambda$$

such that

$$\begin{gathered} \mathcal{B} D(\alpha) = (D(\alpha) - k)\mathcal{B}, \quad D(\alpha) - k = e^{i\langle z,k\rangle} D(\alpha) e^{-i\langle z,k\rangle}, \\ \mathcal{B} H(\alpha) = H_k(\alpha) \mathcal{B}, \quad H_k(\alpha) := e^{i\langle z,k\rangle} H(\alpha) e^{-i\langle z,k\rangle} = \begin{pmatrix} 0 & D(\alpha)^* - \bar k \\ D(\alpha) - k & 0 \end{pmatrix}. \end{gathered} \tag{2.4}$$

In particular, we say $H(\alpha)$ exhibits a flat band at energy zero if and only if $0 \in \bigcap_{k \in \mathbb{C}} \operatorname{Spec}(H_k(\alpha))$. To study the set of α at which $H(\alpha)$ exhibits a flat band at zero, we define the set of Dirac points $\mathcal{K}_0 := \{K, -K\} + \Lambda^*$ such that for $k \notin \mathcal{K}_0$ we can define the compact *Birman-Schwinger* operator

$$T_k = R(k) V(z) : L_0^2 \to L_0^2, \quad R(k) = (2D_{\bar z} - k)^{-1}. \tag{2.5}$$

This operator then characterizes the set of magic angles in the sense stated in the next Proposition.

Proposition 2.1 ([Be*22, Theorem 2],[BHZ24, Proposition 2.2])**.** *There exists a discrete set $\mathcal{A}$ such that*

$$\operatorname{Spec}_{L^2_0} D(\alpha) = \begin{cases} \mathcal{K}_0 & \alpha \notin \mathcal{A}, \\ \mathbb{C} & \alpha \in \mathcal{A}. \end{cases} \tag{2.6}$$

Moreover,

$$\alpha \in \mathcal{A} \iff \exists\, k \notin \mathcal{K}_0,\ \alpha^{-1} \in \operatorname{Spec}_{L^2_0} T_k \iff \forall\, k \in \mathcal{K}_0, \alpha^{-1} \in \operatorname{Spec}_{L^2_0} T_k, \tag{2.7}$$

where T_k is a compact operator given by

$$T_k := R(k)V(z) : L^2_0 \to L^2_0, \quad R(k) := (2D_{\bar{z}} - k)^{-1} \tag{2.8}$$

In particular, the spectrum of T_{k_0} is independent of $k_0 \notin \mathcal{K}_0$ and characterizes parameters $\alpha \in \mathbb{C}$ at which the Hamiltonian exhibits a flat band at zero energy. Since the parameter α is inherently connected with the twisting angle, we shall refer to α's at which (2.7) occurs as *magic* and denote their set by $\mathcal{A} \subset \mathbb{C}$. We then square the operator $T_{k_0}^2 = \operatorname{diag}(A_{k_0}, B_{k_0})$ where $A_{k_0} = R(k_0)U(z)R(k_0)U(-z)$. Setting $k_0 = 0$, we notice that T_0 leaves the subspaces $L^2_{0,j}$ invariant. By projecting the spaces $L^2_{0,j}$ onto the first component, we can define A_0 on spaces $L^2_{0,j}$.

Remark. If $\alpha \in \mathcal{A}$ is simple, then $1/\alpha$ is an eigenvalue of T_0 with eigenvalue of geometric multiplicity 1 and the Hamiltonian exhibits a *two-fold degenerate flat band* at energy zero. If $\alpha \in \mathcal{A}$ is two-fold degenerate, then $1/\alpha$ is an eigenvalue of T_0 with eigenvalue of geometric multiplicity 2 and the Hamiltonian exhibits a *four-fold degenerate flat band* at energy zero. It follows from Theorem 1 that we can drop the minima in the above definition.

Suppose that the potential $U(z)$ satisfies the symmetries given in (2.1), namely

$$U(z+\gamma) = e^{i\langle \gamma, K\rangle} U(z), \quad U(\omega z) = \omega U(z).$$

Since U is then periodic with respect to 3Λ ($3K \equiv 0 \bmod \Lambda^*$), expanding in Fourier series gives

$$U = \sum_{p \in \Lambda^*/3} a_p e^{i\langle z, p\rangle}$$

. The translational symmetry now writes:

$$\forall p \in \Lambda^*/3, \quad \forall \gamma \in \Lambda, \quad a_p e^{i\langle \gamma, p\rangle} = a_p e^{i\langle \gamma, K\rangle}.$$

Identifying the Fourier coefficients now gives that for all $p \in \Lambda^*/3$,

$$a_p \neq 0 \implies \forall\, \gamma \in \Lambda, \quad \langle \gamma, p\rangle = \langle \gamma, K\rangle \implies p \equiv K \mod \Lambda^*.$$

In other words, we see that (changing notation)

$$U(z) = \sum_{p \in \Lambda^*} a_p e^{i\langle p+K, z\rangle}. \tag{2.9}$$

We now investigate the rotational symmetry: it is equivalent to

$$\sum_{p\in\Lambda^*} a_p e^{i\langle\bar\omega p+\bar\omega K,z\rangle} = f(\omega z) = \omega f(z) = \sum_{p\in\Lambda^*} \omega a_p e^{i\langle p+K,z\rangle}.$$

Now, $\bar\omega p + \bar\omega K = \bar\omega p - r^{-1}(\omega) + K$, where we defined the rescaling map

$$z : \Lambda^* \to \Lambda, \quad z(k) := \sqrt{3}k/4\pi i. \tag{2.10}$$

(Although z is a complex variable, the notation is justified as it is a map from k-space to z-space.) Hence, the right hand side of the previous equality rewrites

$$f(\omega z) = \sum_{p\in\Lambda^*} a_{\omega p + r^{-1}(\bar\omega)} e^{i\langle p+K,z\rangle},$$

that is $a_p = \omega a_{\omega p+r^{-1}(\bar\omega)}$. The previous discussion justified the following characterization of potentials $U(z)$ satisfying the symmetries given in (2.1)

$$U(z) \text{ satisfies (2.1)} \iff U(z) = \sum_{p\in\Lambda^*} a_p e^{i\langle p+K,z\rangle} \text{ and } \forall p\in\Lambda^*, a_p = \omega a_{\omega p+r^{-1}(\bar\omega)}. \tag{2.11}$$

In other words, the values of a_p are determined on the orbits of

$$\kappa : \Lambda^* \ni p \mapsto \omega p + r^{-1}(\bar\omega), \quad \mathrm{Orb}(p) = \{p, \omega p + r^{-1}(\bar\omega), \bar\omega p - r^{-1}(\omega)\}, \quad a_{\kappa(p)} = \bar\omega a_p.$$

So, for instance, the BM potential, up to a factor, comes from the orbit of $p = 0$.

In addition, there exist a number of further anti-linear symmetries of the chiral Hamiltonian

$$Qv(z) = \overline{v(-z)}, \quad \mathscr{Q}u(z) = \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix} u(z),$$

satisfying $QD(\alpha)Q = D(\alpha)^*$ with $Q : L^2_{k,p}(\mathbb{C}/\Lambda;\mathbb{C}^2) \to L^2_{k,-p}(\mathbb{C}/\Lambda;\mathbb{C}^2)$ with $\mathscr{Q} : L^2_{k,p}(\mathbb{C}/\Lambda;\mathbb{C}^4) \to L^2_{k,-p+1}(\mathbb{C}/\Lambda;\mathbb{C}^4)$ satisfying $H(\alpha)\mathscr{Q} = \mathscr{Q}H(\alpha)$ and

$$\mathscr{E}v(z) := Jv(-z), \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with $\mathscr{E} : L^2_{\pm K,\ell}(\mathbb{C}/\Lambda;\mathbb{C}^2) \to L^2_{\mp K,\ell}(\mathbb{C}/\Lambda;\mathbb{C}^2)$ and

$$\mathscr{E} : L^2_{0,\ell}(\mathbb{C}/\Lambda;\mathbb{C}^2) \to L^2_{0,\ell}(\mathbb{C}/\Lambda;\mathbb{C}^2) \text{ satisfying } \mathscr{E}D(\alpha)\mathscr{E}^* = -D(\alpha). \tag{2.12}$$

Finally, we also introduce their composition $\mathscr{A} : L^2_{k,p}(\mathbb{C}/\Lambda;\mathbb{C}^2) \to L^2_{k,-p}(\mathbb{C}/\Lambda;\mathbb{C}^2)$

$$\mathscr{A} := \mathscr{E}Q, \text{ with } \mathscr{A}v(z) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{v(z)} \tag{2.13}$$

with $\mathscr{A}D(\alpha)\mathscr{A} = -D(\alpha)^*$.

Using the above symmetries, we observe that

Proposition 2.2. *The spectrum of* T_0 *satisfies* $\mathrm{Spec}_{L^2_{0,p}}(T_0) = \mathrm{Spec}_{L^2_{0,-p+1}}(T_0)$. *In particular, for* $m \geq 2$ *we find* $\mathrm{tr}_{L^2_{0,p}} T_0^{2m} = \mathrm{tr}_{L^2_{0,-p+1}} T_0^{2m}$.

Proof. Let $v \in L^2_{0,p}$ satisfy $T_0 v = -\lambda v$ then by multiplying by $2D_{\bar z}$ we find $D(1/\lambda)v = 0$. Thus, $D(1/\lambda)^* Qv = 0$ with $Qv \in L^2_{0,-p}$. We thus have

$$0 = D(1/\lambda)^* Qv = D(1/\lambda)^* R(0)^* (2D_z) Qv.$$

We conclude that $(2D_z)Qv \in L^2_{0,-p+1}$ is an eigenvector to T_0^* with eigenvalue $-\bar\lambda$. □

3. Zeros, spectral gap, and rigidity

The zeros always fall into three point characterized by high symmetry: $\omega z \equiv z \mod \Lambda$. That determines them (up to Λ) as 0, $\pm z_S$, where

$$z_S = i/\sqrt{3}, \quad \omega z_S = z_S - (1+\omega),$$

is known as the *stacking point*.

3.1. Theta function argument.

We use the following notation

$$\theta(z) = \theta_1(\zeta|\omega) := -\sum_{n\in\mathbb{Z}} \exp(\pi i (n+\tfrac{1}{2})^2 \omega + 2\pi i (n+\tfrac{1}{2})(\zeta+\tfrac{1}{2})), \tag{3.1}$$

$$\theta(\zeta+m) = (-1)^m \theta(\zeta), \quad \theta(\zeta+n\omega) = (-1)^n e^{-\pi i n^2 \omega - 2\pi i \zeta n} \theta(\zeta),$$

and the fact that θ has simple zeros at Λ (and not other zeros) – see [Mu83]. We can then define

$$F_k(z) = e^{\frac{i}{2}(z-\bar z)k} \frac{\theta(z - z(k))}{\theta(z)}, \quad z(k) := \frac{\sqrt{3}k}{4\pi i}, \quad z : \Lambda^* \to \Lambda. \tag{3.2}$$

In particular, we have then

$$\begin{gathered} F_k(z+m+n\omega) = e^{-nk\,\mathrm{Im}\,\omega} e^{2\pi i n z(k)} F_k(z) = F_k(z), \\ (2D_{\bar z} + k)F_k(z) = c(k)\delta_0(z), \quad c(k) := 2\pi i \theta(z(k))/\theta'(0). \end{gathered} \tag{3.3}$$

One then has that for $u \in \ker_{L^2_0}(D(\alpha))$ vanishing at a point w one has

$$(D(\alpha)+k)F_k(z-w)u(z) = 0. \tag{3.4}$$

In particular this means that vanishing of the vector valued function u at some point, implies existence of a flat band at 0: every k is an eigenvalue of $D(\alpha)$. Presented in a slightly different way, this observation was the basis of the analysis in [TKV19].

3.2. **Zeros.** We first formalise the theta function argument of [TKV19] in a slightly different way than in [Be*22]. (Where we used the fact that a non-zero Wronskian between shifted protected states implies existence of $(D(\alpha)+k)^{-1}$ – see [Be*22, Proposition 3.3].)

Proposition 3.1. *Suppose that* $\mathbf{u}_K(\alpha) \in \ker_{H_0^1}(D(\alpha) + K)$ *is a family of protected stated of the chiral model (see* [Zw24, Theorem 1] *and references given there). Then*

$$\alpha \in \mathcal{A} \iff \exists\, z_0 \in \mathbb{C}/\Lambda \text{ such that } \mathbf{u}_K(\alpha, z_0) = 0. \tag{3.5}$$

Proof. The proof of $\Longleftarrow$ is given in §3.1, see also [Zw24, §6]. On the other hand if $\alpha \in \mathcal{A}$ then for every k (or equivalently some $k \notin \{K, -K\} + \Lambda^*$) there exists $\mathbf{v}_k \neq 0$ such that $(D(\alpha) + k)\mathbf{v}_k = 0$, where we drop the subscript of u and v in the following. Then the Wronskian,

$$W = W(\mathbf{u}, \mathbf{v}) = u_1 v_2 - u_2 v_1, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{3.6}$$

satisfies $2D_{\bar z} W = -(K + k)W$:

$$\begin{aligned} 2D_{\bar z}W &= (2D_{\bar z}u_1)v_2 + u_1(2D_{\bar z}v_2) - (2D_{\bar z}u_2)v_1 - u_2(2D_{\bar z}v_1) \\ &= (-\alpha U(z)u_2 - Ku_1)v_2 + u_1(-\alpha U(-z)v_1 - kv_2) \\ &\quad - v_1(-\alpha U(-z)u_1 - Ku_2) - u_2(-\alpha U(z)v_2 - kv_1) \\ &= -(k + K)(u_1v_2 - u_2v_1) = -(k + K)W. \end{aligned}$$

Putting $k_0 := -(k + K) \notin \Lambda^*$, the general solution of this equation is given by

$$W(z, \bar z) = e^{\frac{i}{2}(k_0 \bar z + \bar k_0 z)} w(z), \quad w \in \mathscr{O}(\mathbf{C}).$$

Since W is periodic and $z \mapsto e^{-\frac{i}{2}(k_0 \bar z + \bar k_0 z)}$ is a bounded function, w has to be constant, and for $k_0 \notin \Lambda^*$, that constant has to vanish. If follows that for $z \in \Omega := \complement \mathbf{u}^{-1}(0)$, an open dense set as $\mathbf{u} \not\equiv 0$ is real analytic (this follows from the ellipticity of the equation and analyticity of U – see [HöI, Theorem 8.6.1]), $\mathbf{v}(z, \bar z) = F(z, \bar z)\mathbf{u}(z, \bar z)$, $F \in C^\infty(\Omega)$. By applying $2D_{\bar z}$ to both sides we see that $D_{\bar z}F = (K - k)F$. Hence for some $f \in C^\infty(\Omega)$,

$$\mathbf{v}(z, \bar z) = e^{-\frac{i}{2}(k_1 \bar z + \bar k_1 z)} f(z)\mathbf{u}(z, \bar z), \quad \partial_{\bar z} f|_\Omega = 0, \quad k_1 := K - k. \tag{3.7}$$

As in the proof of [BHZ23, (4.4)] we see that f is in fact meromorphic. In fact, for a fixed z_1 we put

$$G_0(\zeta, \bar\zeta) := v_1(z, \bar z)|_{z = z_1 + \zeta}, \quad G_1(\zeta, \bar\zeta) := e^{-\frac{i}{2}(k_1 \bar z + \bar k_1 z)} u_1(z, \bar z)|_{z = z_1 + \zeta}.$$

As already remarked $\mathbf{u}$ and $\mathbf{v}$ are real analytic and hence $G_j \in \mathscr{O}(B_{\mathbb{C}^2}(0, \delta))$. With $g(\zeta) := f(z_1 + \zeta)$, we have

$$G_0(\zeta, \xi) = g(\zeta)G_1(\zeta, \xi), \quad z_1 + \zeta \in \Omega.$$

Now choose $\xi_0 \in B_{\mathbb{C}}(0, \delta/2)$ such that $\zeta \mapsto G_1(\zeta, \xi_0)$ is not identically zero (if no such ξ_0 existed, $G_1 \equiv 0$, and hence, from the equation, $\mathbf{u} \equiv 0$). But then $\zeta \mapsto g(\zeta) := G_1(\zeta, \xi_0)/G_2(\zeta, \xi_0)$ is meromorphic near $\zeta = 0$ and, as z_1 was arbitrary f is meromorphic everywhere.

For v to be periodic f cannot be constant and hence it has to have poles. But that means that $\mathbf{u}$ has to have zeros. □

We are now ready to proof Theorem 1 which is a refinement of Proposition 1.

Proof of Theo. 1. Fix $k \in \mathbb{C}/\Lambda^*$ and assume α is magic. Then there exists a nonzero function $\mathbf{u}_k(\alpha) \in \ker(D(\alpha)+k)$ that vanishes somewhere. Let $\mathbf{u} := \mathbf{u}_k(\alpha)$, and suppose that $\mathbf{u}$ has m distinct zeros $z_0, \ldots, z_{m-1}$, each of multiplicity one. This can be assumed without loss of generality as we can multiply $\mathbf{u}$ by a periodic meromorphic function with poles the zeros of $\mathbf{u}$ and simple zeros.

We first show that $\dim\ker(D(\alpha) + k) \geq m$. Assume $m \geq 2$ (the case $m = 1$ is trivial). Choose points $w_j \notin \{z_0, \ldots, z_{m-1}\}$ such that

$$w_0 + w_j \equiv z_0 + z_j \quad \text{for } j = 1, \ldots, m-1,$$

and define, for $j = 1, \ldots, m-1$,

$$\mathbf{u}_j(z, \bar{z}) := \frac{\theta(z - w_0)\theta(z - w_j)}{\theta(z - z_0)\theta(z - z_j)} \cdot \mathbf{u}(z, \bar{z}).$$

The prefactor is a meromorphic function with simple poles at z_0 and z_j (see [BHZ24, §3.1] for properties of θ), but the structure of the zeros of $\mathbf{u}$ ensures that $\mathbf{u}_j \in \ker(D(\alpha) + k)$. These functions are linearly independent: if

$$c_0 + \sum_{j=1}^{m-1} c_j \frac{\theta(z - w_0)\theta(z - w_j)}{\theta(z - z_0)\theta(z - z_j)} \equiv 0,$$

then evaluating at $z = z_j$ for $j > 0$ shows $c_j = 0$, hence all $c_j = 0$. Thus, we obtain m linearly independent functions in $\ker(D(\alpha) + k)$.

Now for the reverse inequality. Suppose $\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_{M-1}$ span $\ker(D(\alpha) + k)$, where $M := \dim\ker(D(\alpha)+k)$. Then the Wronskians $W(\mathbf{u}, \mathbf{v}_j)$ vanish identically, since they are constant and vanish at the zeros of $\mathbf{u}$. As in [BHZ24, Eq. (4.1)], this implies that each $\mathbf{v}_j(z, \bar{z}) = f_j(z)\mathbf{u}(z, \bar{z})$ for some meromorphic function $f_j(z)$, with $f_0 \equiv 1$, and the set $\{f_j\}_{j=0}^{M-1}$ linearly independent.

The functions f_j, $0 \leq j \leq M - 1$ lie in the space $L(D)$ of meromorphic functions[1] where D is the divisor defined by the zeros on $\mathbf{u}$. Therefore,

$$M = \dim\ker(D(\alpha) + k) \leq \deg(D) = m.$$

[1]See Terry Tao's blog for a quick introduction; formula (7) there for the version of the Riemann–Roch Theorem used here.

Combining with the previous inequality, we conclude

$$\dim \ker(D(\alpha) + k) = m.$$

Moreover, since every nonzero element of the kernel has exactly m zeros (counted with multiplicity), and the multiplicity is independent of k by the theta function argument (see §3.1 and [BHZ24, Lemma 4.1]), the dimension of the kernel is constant over $k \in \mathbb{C}/\Lambda^*$. By continuity of the spectrum of $H_k(\alpha)$ and compactness of $\mathbb{C}/\Lambda^*$, it follows that the flat bands at energy zero are isolated from the rest of the spectrum by a nonzero gap. □

Remarks. 1. This quickly settles [Zw24, Problem 14].

2. Although we invoked a basic version of the Riemann–Roch theorem, the proof that the number of poles of f_j's has to be greater than M is explicit. If the poles are all simple then

$$f_j(z) = \sum_{k=1}^{K(j)} \lambda_k(j) \frac{\theta'(z - a_k(j))}{\theta(z - a_k(j))} + c(j), \qquad \sum_{k=1}^{K(j)} \lambda_k(j) = 0.$$

For $f_0 \equiv 1,\ f_1, \cdots, f_{M-1}$ to be linearly independent we need $\sum_{j=1}^{M-1} K(j) \geq M$. It is not difficult to modify this construction to the case of poles of higher order.

3.3. **Rigidity.** In this subsection we provide

Proof of Theorem 2. If $\Omega u(z) := u(\omega z)$ then

$$\Omega D(\alpha) = \omega D(\alpha) \Omega, \quad \ker_{L^2_0}(D(\alpha)) = \bigoplus_{p \in \mathbb{Z}_3} \ker_{L^2_{0,p}}(D(\alpha)). \tag{3.8}$$

Theorem 1 shows that all non-zero elements of $\ker_{L^2_0}(D(\alpha))$ have $m(\alpha)$ zeros counting multiplicities. That allows us to consider the invariant subspaces $L^2_{0,p}$, separately.

Let $u \in \ker_{L^2_{0,0}}(D(\alpha))$. Since $\omega z_S = z_S - (1 + \omega)$, we find

$$\begin{aligned} u(\pm z_S + \zeta) &= u(\pm \omega z_S + \omega \zeta) = u(\pm z_S \mp (1 + \omega) + \omega \zeta) \\ &= \operatorname{diag}(e^{-i\langle \mp(1+\omega), K\rangle}, e^{i\langle \mp(1+\omega), K\rangle}) \mathscr{L}_{\mp(1+\omega)} u(\pm z_S + \omega \zeta), \\ &= \operatorname{diag}(\omega^{\pm 1}, \omega^{\mp 1}) u(\pm z_S + \omega \zeta) \end{aligned}$$

that, is $u(\pm z_S + \omega \zeta) = \operatorname{diag}(\omega^{\mp 1}, \omega^{\pm 1}) u(\pm z_S + \zeta)$. This shows that

$$M_u(\pm z_S) \not\equiv 0 \mod 3.$$

We now recall the symmetry $\mathscr{E} : L^2_{0,p}(\mathbb{C}/\Lambda; \mathbb{C}^2) \to L^2_{0,p}(\mathbb{C}/\Lambda; \mathbb{C}^2)$ (valid under the assumption (1.3)):

$$\mathscr{E} D(\alpha) \mathscr{E}^* = -D(\alpha), \quad \mathscr{E} v(z) := J v(-z), \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.9}$$

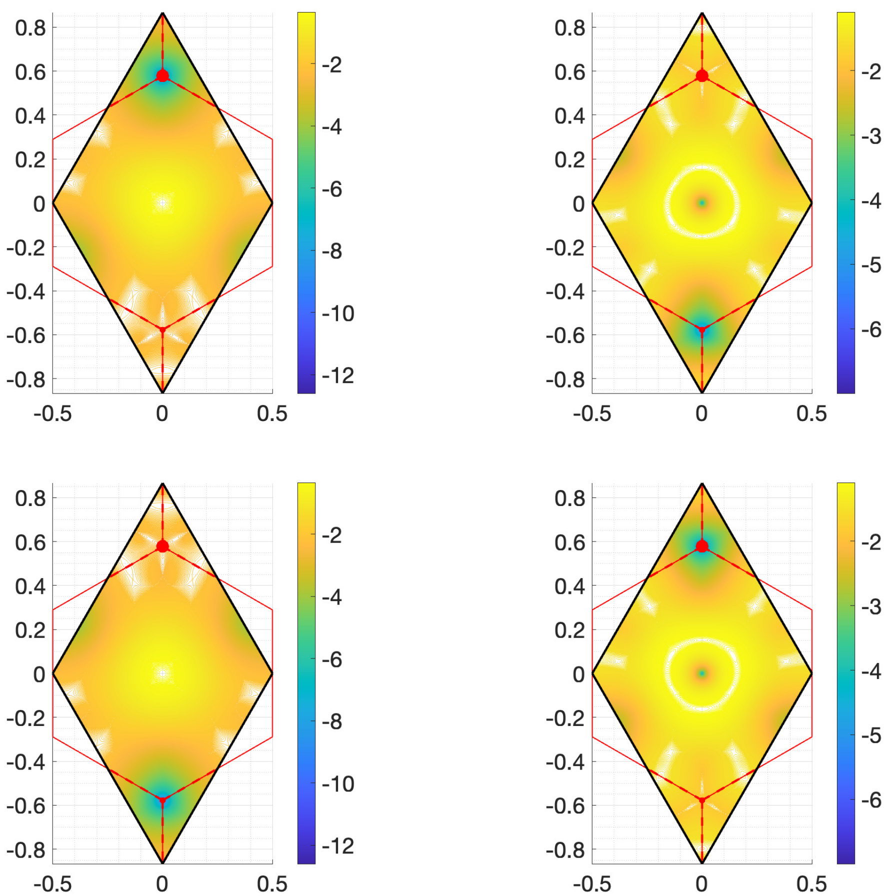

FIGURE 4. Modulus of flat band wavefunctions of $\ker_X(D(\alpha))$ at first magic angle $\alpha = 0.853799$ with $X = L^2_{i,j}$ with $i = K$(top), $i = -K$ (bottom), $j = 0$ (left), $j = 1$ (right) for potential $U_\pm = U_2(\pm\bullet)$ in (1.12).

It follows that $M_u(z_S) = M_u(-z_S)$. In particular, any element of $\ker_{L^2_{0,0}}(D(\alpha))$ has zeros at $z = \pm z_S$ (for both signs). Zeros at any other point are three-fold degenerate by rotational symmetry (3.8) and that shows that

$$\sum_{z \in \mathbb{C}/\Lambda} M_u(z) \bmod 3 \in \{1, 2\}. \tag{3.10}$$

The same conclusion holds for the subspace $\ker_{L^2_{0,1}}(D(\alpha))$ as can been seen using the properties of the Weierstraß $\wp$-function, $\wp(z) := \wp(z; \omega, 1)$ [Mu83, §I.6]:

$$\wp(\omega z) = \omega \wp(z) \quad \text{and} \quad \wp(z) = 0 \implies z = \pm z_S + \Lambda, \quad \wp'(\pm z_S) \neq 0.$$

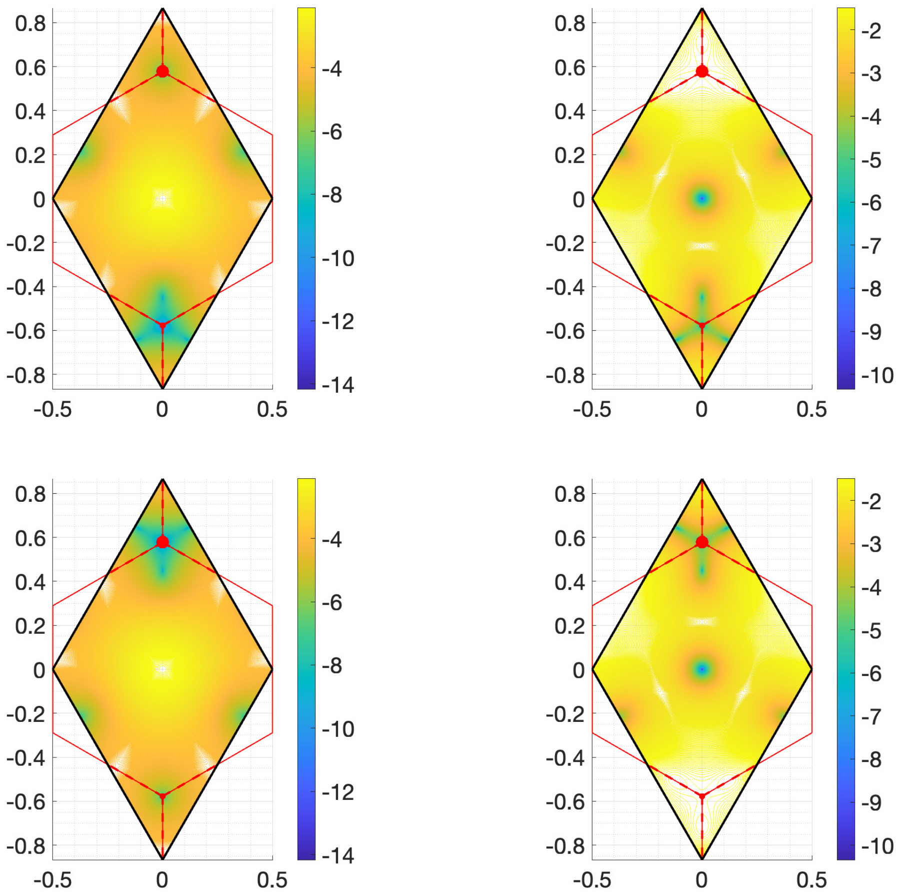

FIGURE 5. Flat band wavefunctions of $\ker_X(D(\alpha))$ at first magic angle $\alpha = 0.853799$ with $X = L^2_{0,0}$ (left) and $X = L^2_{0,1}$ (right) for potential $U_\pm = U_2(\pm\bullet)$ in (1.12) upper component, top and lower component, bottom.

This shows that

$$\wp(z)\ker_{L^2_{0,1}(\mathbb{C}/\Lambda;\mathbb{C}^2)}(D(\alpha)) = \ker_{L^2_{0,0}(\mathbb{C}/\Lambda;\mathbb{C}^2)}(D(\alpha)) \tag{3.11}$$

and hence (3.10) holds for elements of $\ker_{L^2_{0,1}}(D(\alpha))$ as well.

Finally, suppose that $u \in \ker_{L^2_{0,2}}(D(\alpha))$. Since then $u(\omega z) = \omega u(z)$ (see (3.8)) we see that $M_u(0) \equiv 1 \mod 3$. As above, we find

$$\begin{aligned} u(\pm z_S + \zeta) &= \bar\omega u(\pm\omega z_S + \omega\zeta) = \bar\omega u(\pm z_S \mp (1+\omega) + \omega\zeta) \\ &= \bar\omega \operatorname{diag}(e^{-i\langle\mp(1+\omega),K\rangle}, e^{i\langle\mp(1+\omega),K\rangle})\mathscr{L}_{\mp(1+\omega)}u(\pm z_S + \omega\zeta) \\ &= \bar\omega \operatorname{diag}(\omega^{\pm1}, \omega^{\mp1})u(\pm z_S + \omega\zeta) \\ &= \operatorname{diag}(\omega^{\frac12(1\mp1)}, \omega^{\frac12(1\pm1)})u(\pm z_S + \omega\zeta), \end{aligned}$$

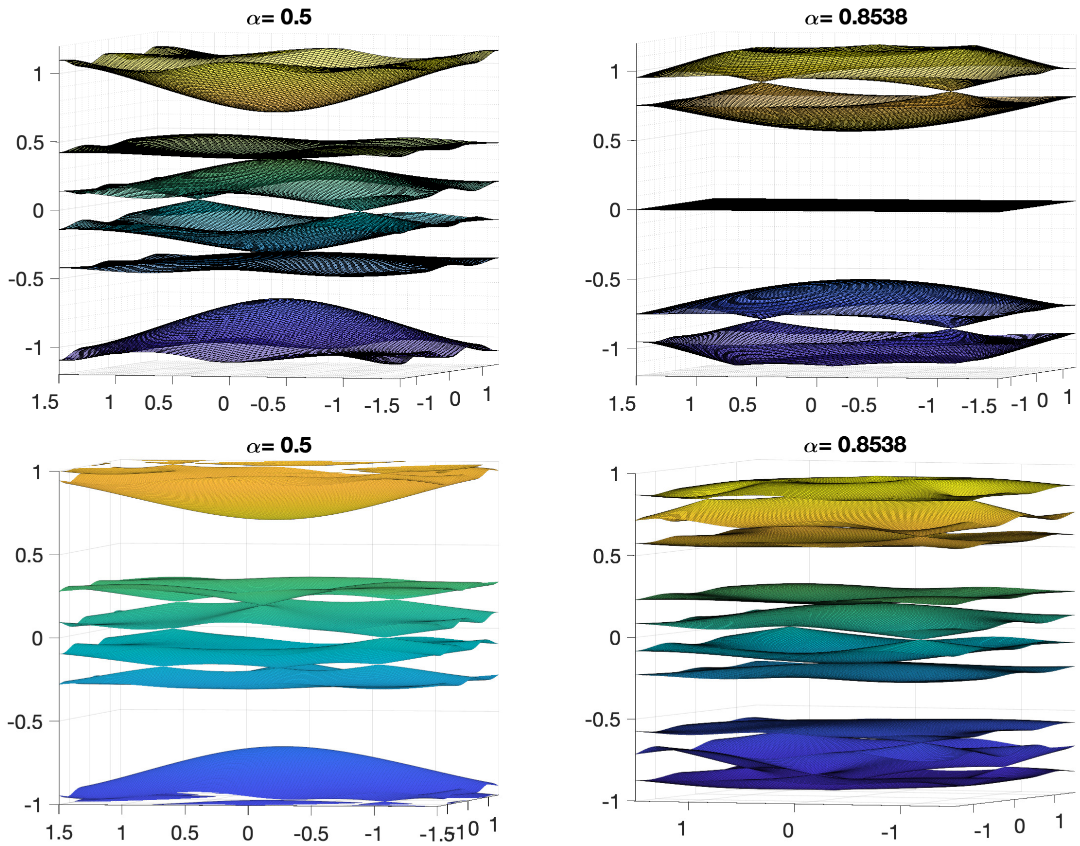

FIGURE 6. Bands of Hamiltonian (2.3) at $\alpha = 0.5$ (left) and $\alpha = 0.8538$ (right) with potential $U_\pm := U_2(\pm\bullet)$ (1.12). Bands of full continuum Bistritzer-MacDonald Hamiltonian [Be*21, (1)] with same α and $\beta = 0.7\alpha$ with potential $U_\pm = U_2(\pm\bullet)$ (1.12) and anti-chiral potential $V(z) := 2\partial_z U_2(z)$.

that is $u(\pm z_S + \omega\zeta) = \operatorname{diag}(\omega^{\frac{1}{2}(\pm 1-1)}, \omega^{\frac{1}{2}(\mp 1-1)})u(\pm z_S + \zeta)$. Hence if $u(z_S + \zeta) = (u_1(\zeta), u_2(\zeta))^t$ then $u_1(\omega\zeta) = u_1(\zeta)$ and $u_2(\omega\zeta) = \bar\omega u_2(\zeta)$ and the order of the zero of u_1 is 0 mod 3 and that of u_2 is 2 mod 3. This shows that $M_u(\pm z_S) \bmod 3 \in \{0, 2\}$. Using the $\mathscr{E}$ symmetry (3.9), we conclude that

$$M_u(z_S) + M_u(-z_S) \not\equiv 2 \mod 3. \tag{3.12}$$

Other points $z \notin \{0, \pm z_S\}$ satisfy $M_u(z) = M_u(\omega z) = M_u(\omega^2 z)$ by rotational symmetry. Combining this observation with the fact that the multiplicity as 0 is 1 mod 3 and (3.12) we obtain

$$\sum_{z\in\mathbb{C}/\Lambda} M_u(z) \not\equiv 0 \mod 3.$$

In all three cases, the total multiplicity of zeros is not divisible by 3, ruling out the case $m(\alpha) \equiv 0 \bmod 3$.

We also see that if $m(\alpha) = 1$ then $\dim \ker_{L^2_{0,2}} D(\alpha) = 1$, as otherwise (3.11) would imply $m(\alpha) > 1$. If $m(\alpha) = 2$ the only possibility is $\dim \ker_{L^2_{0,0}} D(\alpha) = \dim \ker_{L^2_{0,1}} D(\alpha) = 1$. □

The location of zeros for flat band Bloch functions at a degenerate magic angle are illustrated in Figures 4 and 5.

4. Trace computations

To prove the existence of degenerate magic angles (Theorem 3) we argue by contradiction using the Birman–Schwinger operator T_k defined in (2.5). From Theorem 2, we see that in the case if all the α's were all simple then the traces of T_k^{2p} restricted to $L^2_{0,0}$ or $L^2_{0,1}$ would have to vanish. For a general k, the operator T_k does *not* preserve the rotational invariant subspaces $L^2_{0,j}$. To achieve that we set $k = 0$ so that the proof reduces to showing that $\operatorname{tr}((T_0)^{2\ell}_{L^2_{0,0}}) \neq 0$ for some value of ℓ. That is done using the previous rationality condition $\operatorname{tr}((T_0)^{2\ell}_{L^2_{0,0}}) = q_\ell \pi/\sqrt{3}$ for $q_\ell \in \mathbb{Q}$ obtained before by the authors [BHZ23, Theorem 1] and some elementary arguments involving transcendental numbers.

4.1. Traces on rotationally invariant subspaces.

We recall that an orthonormal basis of $L^2_0(\mathbb{C}/3\Lambda; \mathbb{C})$ is given by setting

$$e_\nu(z) := e^{i\langle \nu, z\rangle}/\sqrt{\operatorname{Vol}(\mathbb{C}/3\Lambda)}, \quad \nu \in \Lambda^* + K, \quad \langle \nu, z\rangle := \operatorname{Re}(\bar{z}\nu).$$

We see that $\Omega e_\nu = e_{\bar{\omega}\nu}$. This means that an orthonormal basis of $L^2_{0,j}$ is given by

$$e_{[\nu]}(z) = \frac{1}{\sqrt{3}}\big(e_\nu(z) + \omega^j e_{\omega\nu}(z) + \bar{\omega}^j e_{\bar{\omega}\nu}(z)\big), \ \ \nu \in \Lambda^* + K, \ \ [\nu] = \{\nu, \omega\nu, \bar{\omega}\nu\}.$$

Following our approach developed in [BHZ23], we compute the sum of powers of magic angles by computing traces of the operator T_k defined in (2.8). Since odd powers of T_k have vanishing traces it suffices to compute the traces of powers of the Hilbert-Schmidt operator

$$A_k := R(k)U(z)R(k)U(-z) : L^2_0 \to L^2_0, \quad k \notin (K + \Lambda^*) \cup (-K + \Lambda^*) = \mathcal{K}_0. \tag{4.1}$$

Due to the relation

$$\forall k \notin \mathcal{K}_0, \quad \Omega^{-1} A_k \Omega = A_{\omega k},$$

we see that subspaces $L^2_{0,j}$ are *not* in general invariant by A_k. This makes a direct application of the strategy of [BHZ23] impossible. However, we see that the operator

A_0 does preserve this smaller subspace. From now on, we therefore specialize to $k=0$. For $\ell \geq 2$, one can compute the trace on $L^2_{0,j}$:

$$\operatorname{tr}\left((A_0)^{\ell}_{|L^2_{0,j}}\right) = \sum_{[\nu],\nu\in\Lambda^*+K} \langle A_0^{\ell} e_{[\nu]}, e_{[\nu]}\rangle.$$

Now, we write that, using bilinearity of the scalar product

$$3\langle A_0^{\ell} e_{[\nu]}, e_{[\nu]}\rangle = \sum_{h=0}^{2} \langle A_0^{\ell} e_{\omega^h\nu}, e_{\omega^h\nu}\rangle + \sum_{k\neq h} \omega^{j(k-h)} \langle A_0^{\ell} e_{\omega^h\nu}, e_{\omega^k\nu}\rangle.$$

Thus, when summing on $[\nu]$, the first term gives a third of the trace on L^2_0, (which was computed in [Be*22] for $\ell = 2$ and $U_0 = U_1$ and shown to be equal to $4\pi/\sqrt{3}$)

$$\begin{aligned}\operatorname{tr}((A_0)^{\ell}_{|L^2_{0,j}}) &= \tfrac{1}{3}\operatorname{tr}(A_0^{\ell}) + \tfrac{1}{3}\sum_{[\nu],\nu\in\Lambda^*+K}\sum_{k\neq h} \omega^{j(k-h)} \langle A_0^{\ell} e_{\omega^h\nu}, e_{\omega^k\nu}\rangle \\ &=: \tfrac{1}{3}\operatorname{tr}(A_0^2) + \tfrac{1}{3}\mathcal{R}_{\ell,j}.\end{aligned} \tag{4.2}$$

4.2. **Existence of degenerate magic angles.** Our strategy now consists in using [BHZ23, Theorem1] and the fact that $\pi/\sqrt{3}$ is transcendental to contradict the conclusion of Theorem 2. More explicitly, we will prove the following statement:

Theorem 6. *We consider the Hamiltonian (2.3) with a potential $U \in C^\infty(\mathbb{C}/3\Lambda)$ and $U_+ := U$ and $U_- := U(-\bullet)$ satisfying the first two symmetries of (2.1) with only finitely many non-zero Fourier modes $a_p \in \pi\mathbb{Q}(\omega/\sqrt{3})$, appearing in the decomposition (2.11). Then, if we denote $\mathcal{A}(U)$ the set of (complex) magic angles for the potential U and if $\mathcal{A}(U) \neq \emptyset$, there exists $\alpha \in \mathcal{A}(U)$ which is not simple. This also applies to the Bistritzer–MacDonald potential $U_+ := U_1$ defined in (1.12).*

Proof. Step 1 (Existence of non-zero trace): We start by noticing that the existence of a magic angle is equivalent to the existence of a non-vanishing trace

$$\exists \ell \geq 2, \quad \operatorname{tr}((A_0)^{\ell}_{|L^2_0}) \neq 0.$$

This follows from the properties of the regularized Fredholm determinant, cf. [BHZ23].

Step 2 (Trace is transcendental): We fix such an ℓ for which $\operatorname{tr}((A_0)^{\ell}_{|L^2_0}) \neq 0$. Using [BHZ23, Theorem 5], and the hypothesis on the potential, this implies that $\operatorname{tr}(A_0^{\ell}) \in \pi\mathbb{Q}(\omega)$. Since the trace is non-zero by assumption, this proves that $\operatorname{tr}(A_0^{\ell})$ is transcendental.

Step 3 ($\mathcal{R}_{\ell,j}$ is a finite sum): We now show that the sum defining the remainder $\mathcal{R}_{\ell,j}$ in (4.2) is always a finite sum, under the assumption that the potential has only finitely many non-zero Fourier mode. We start with the formula defining the remainder

$$\mathcal{R}_{\ell,j} := \sum_{[\nu],\nu\in\Lambda^*+K}\sum_{k\neq h} \omega^{j(k-h)} \langle A_0^{\ell} e_{\omega^h\nu}, e_{\omega^k\nu}\rangle.$$

The summand $\langle A_0^\ell e_{\omega^h\nu}, e_{\omega^k\nu}\rangle$ is non-zero only if $A_0^\ell e_{\omega^h\nu}$ has a non-vanishing Fourier mode corresponding to $e_{\omega^k\nu}$. Now, if we look at the definition of A_0 (see 4.1), we see that the $R(k)$ part acts diagonally (with coefficients in $(i\pi)^{-1}\mathbb{Q}(\omega)$ as we chose $k=0$) on the Fourier basis, on the other hand, the $U(z)$ and $U(-z)$ parts act as a finite sum of weighted shifts on this basis (it is here where we use the assumption of having finitely many non-vanishing Fourier modes). Moreover, by assumption, the weights are elements of $(i\pi)\mathbb{Q}(\omega)$.This means that there exists a finite subset $\mathcal{F}_U^\ell \subset 3\Gamma^*$ such that

$$\forall \nu \in \Lambda^* + K, \quad A_0^\ell e_\nu = \sum_{\eta \in \mathcal{F}_U^\ell} a_\eta e_{\nu+\eta}, \quad a_\eta \in \mathbb{Q}(\omega). \tag{4.3}$$

But this means that there exists a constant $R > 0$ such that for any $\eta \in \mathcal{F}_U^\ell$, we have $|\eta| \le R$. In particular, if $\langle A_0^\ell e_{\omega^h\nu}, e_{\omega^k\nu}\rangle$ is non-zero, then $|\omega^h\nu - \omega^k\nu| \le R$. Now, because $h \ne j$, this inequality is false outside a compact set for ν. But because ν is on a lattice, which is discrete, we conclude that the above inequality is true for at most a finite number of ν. Thus, the sum defining $\mathcal{R}_{\ell,j}$ is finite.

Step 4 $\mathcal{R}_{\ell,j} \in \mathbb{Q}(\omega)$: Finally, for the non-zero terms of the sum, we use (4.3) again to conclude that $\langle A_0^\ell e_{\omega^h\nu}, e_{\omega^k\nu}\rangle = a_\eta \in \mathbb{Q}(\omega)$. This proves that $\mathcal{R}_{\ell,j} \in \mathbb{Q}(\omega)$.

Step 5 (Proof by contradiction): Since $\mathcal{R}_{\ell,j} \in \mathbb{Q}(\omega)$ is algebraic and thus $\operatorname{tr}((A_0)^\ell_{|L^2_{0,j}}) \ne 0$ by (4.2). This contradicts the conclusion of Theorem 2; thus proving the existence of non-simple magic angle for the potential U. □

5. Infinite number of degenerate magic angles

We now adapt the argument, already used in [BHZ23, Theorem 6], to prove that the number of non-simple magic angles is actually infinite. This actually refines the previous theorem by showing there is an infinite number of non-simple magic angles.

In the next theorem we use the same notation and assumptions as in Theorem 6. The definition of multiplicity is given in (1.13).

Theorem 7. *Let*

$$\mathcal{A}_m(U) := \{\alpha \in \mathcal{A}(U) : m_U(\alpha) \ge 2\}$$

be the set of non-simple magic angles. Then

$$|\mathcal{A}(U)| > 0 \implies |\mathcal{A}_m(U)| = +\infty. \tag{5.1}$$

In particular, the set of magic angles for the Bistritzer–MacDonald potential $U = U_1$ (see (1.12)) is infinite.

In addition, if for $N \ge 0$, and $a = (a_p)_{\{p\in\Lambda^*;\|p\|_\infty \le N\}}$, U_a is given by (2.11) with coefficients a, then (5.1) holds for a generic (in the sense of Baire) set of coefficients $a = (a_p)_{\{p\in\Lambda^*;\|p\|_\infty \le N\}} \in \mathbb{C}^{(2N+1)^2}$ which contains $(\pi\mathbb{Q}(\omega/\sqrt{3}))^{(2N+1)^2}$. Here, we used the notation $\|p\|_\infty = \|\frac{4\pi i}{\sqrt{3}}(p_1 + p_2\omega)\|_\infty := \max(p_1, p_2)$.

Proof. We start by observing that since π is transcendental on $\mathbb{Q}$, it is also transcendental in $\mathbb{Q}(\omega/\sqrt{3})$. Now, we shall assume that there exist only finitely many non-simple eigenvalues of A_0^2 on L_0^2. This implies, by Theorem 2 that $(A_0)^\ell_{|L^2_{0,1}}$ has only finitely many eigenvalues, we denote them by $\lambda_i \in \mathbb{C}$ for $i = 1, .., N$. Then we define the n-th symmetric polynomial

$$e_n(\lambda_1, \dots, \lambda_N) = \sum_{1 \le j_1 < j_2 < \dots < j_n \le N} \lambda_{j_1} \cdots \lambda_{j_n}.$$

Newton identities show that this polynomial can be expressed as

$$e_n(\lambda_1, \dots, \lambda_N) = (-1)^n \sum_{\substack{m_1 + 2m_2 + \dots + nm_n = n \\ m_1 \ge 0, \dots, m_n \ge 0}} \prod_{i=1}^n \frac{(-\operatorname{tr}(A_0)^{2i}_{|L^2_{0,1}})^{m_i}}{m_i! i^{m_i}} \tag{5.2}$$

where $e_n = 0$ for $n > N$. The fact that $\mathcal{A}(U) \neq \emptyset$ implies, by Theorem 6 that $\mathcal{A}_m(U) \neq \emptyset$. Now, this means that there is a non-vanishing trace of $(A_0)^\ell_{|L^2_{0,2}}$. Choose m_0 to be the minimal power for which the trace is non-zero. Choose $n = m_0 \times K$ where K is a large integer, and using the fact that $e_n = 0$, we deduce that π is the root of the polynomial of degree K with coefficients in $\mathbb{Q}\left(\frac{\omega}{\sqrt{3}}\right)$ given by

$$\sum_{\substack{m_1 + 2m_2 + \dots + nm_n = n \\ m_1 \ge 0, \dots, m_n \ge 0}} \prod_{i=1}^{m_0 \times K} (\operatorname{tr}(A_0)^{2i}_{|L^2_{0,1}})^{m_i} = \sum_{\substack{m_1 + 2m_2 + \dots + nm_n = n \\ m_1 \ge 0, \dots, m_n \ge 0}} \prod_{i=1}^{m_0 \times K} (\underbrace{\frac{1}{3} \operatorname{tr}(A_0)^{2i}_{|L^2_0}}_{\in \mathbb{Q}\left(\frac{\omega}{\sqrt{3}}\right)\pi} - \underbrace{\mathcal{R}_{i,1}}_{\in \mathbb{Q}\left(\frac{\omega}{\sqrt{3}}\right)})^{m_i}$$
$$= 0.$$

The power $m_1 \cdots m_n$ of π is maximized, among the tuples we sum by the unique choice $m_i = \delta_{i,m_0} K$. By choice of m_0, this gives that the above polynomial has a non-zero leading coefficient and is therefore non-zero. This contradicts the fact that π is transcendental and concludes the proof.

Now, let $a = (a_p)_{\{p \in \Lambda^*; \|p\|_\infty \le N\}} \in \mathbb{C}^{(2N+1)^2} \in \mathbb{C}^{(2N+1)^2}$ and assume that $\mathcal{A}(U_a) \neq \emptyset$. Then, we can find an open neighborhood of a, $\Omega_a \ni a$, such that for coefficients $b = (b_p)_{\{p \in \Lambda^*; \|p\|_\infty \le N\}} \in \Omega_a$ we have $\mathcal{A}(U_b) \neq \emptyset$. Take $q = (q_p)_{\{p \in \Lambda^*; \|p\|_\infty \le N\}} \in (\pi \mathbb{Q}(\omega/\sqrt{3}))^{(2N+1)^2} \cap \Omega_a$ for which we then have $|\mathcal{A}(U_q)| = \infty$. Continuity of eigenvalues of T_k as the potential U changes shows that the $V_{m,a} := \{b \in \Omega_a : |\mathcal{A}(U_b)| \ge m\}$ is open and dense in Ω_a. Hence, the set coefficients for which $0 < |\mathcal{A}(U_b)| < \infty$ is given by $\bigcup_{m \in \mathbb{N}} \bigcup_{q \in (\mathbb{Q} + i\mathbb{Q})^{2N+1}} \Omega_q \setminus V_{m,q}$. It is then meagre and does not contain $(\pi \mathbb{Q}(\omega/\sqrt{3}))^{(2N+1)^2}$. $\square$

6. Numerical evaluation of the trace and existence of non-real magic angle

In this section the potential $U_\pm$ will be taken to be equal to $U_1(\pm\bullet)$ defined in (1.12). In (4.2), we have proven that the traces on the rotational-invariant subspaces can be written as

$$\operatorname{tr}((A_0)^\ell_{|L^2_{0,j}}) = \tfrac{1}{3}\operatorname{tr}(A_0^\ell) + \tfrac{1}{3}\mathcal{R}_{\ell,j}, \tag{6.1}$$

where the remainder was shown to be a finite sum. Although the first term $\operatorname{tr}(A_0^\ell)$ is a priori an infinite sum, the authors provided in [BHZ23, Theo. 7] a semi-explicit formula which can be evaluated rigorously with computer assistance for $U = U_1$ and small values of ℓ. From [BHZ23, Table 1][2], we see that

$$\operatorname{tr}((A_0^2)_{|L^2_0}) = \frac{4\pi}{\sqrt{3}}, \quad \operatorname{tr}((A_0^3)_{|L^2_0}) = \frac{96\pi}{7\sqrt{3}}, \quad \operatorname{tr}((A_0^4)_{|L^2_0}) = \frac{40\pi}{\sqrt{3}}.$$

We can read off from the above $\operatorname{tr}(A_0^2)\operatorname{tr}(A_0^4) < \operatorname{tr}(A_0^3)^2$. If all magic angles were real, then by ℓ^p-interpolation $\operatorname{tr}(A_0^2)\operatorname{tr}(A_0^4) \geq \operatorname{tr}(A_0^3)^2$, which is a contradiction. In other words, we have proven that

Proposition 6.1. *Let $U = U_1$ be the potential defined in (1.12), then $\mathcal{A} \cap \mathbb{C} \setminus \mathbb{R} \neq \emptyset$.*

Our goal here is to mimic this argument on rotational-invariant subspace by computing the finite remainders $\mathcal{R}_{\ell,j}$ using computer assistance to find the exact results.

From doing so, we obtain the following result.

Proposition 6.2. *For the Bistritzer-MacDonald potential U_1 defined in (1.12), we have*

$$\operatorname{tr}((A_0^2)_{|L^2_{0,1}}) = \operatorname{tr}((A_0^2)_{|L^2_{0,0}}) = \frac{4\pi}{3\sqrt{3}} - 3 \approx -0.581601 < 0$$

and $\operatorname{tr}((A_0^2)_{|L^2_{0,2}}) = \frac{4\pi}{\sqrt{3}} + 6 \approx 8.4184$. *For the higher powers, we find*

$$\operatorname{tr}((A_0^3)_{|L^2_{0,2}}) = \frac{32\pi}{7\sqrt{3}} + \frac{810}{49} \approx 24.8223 \text{ and } \operatorname{tr}((A_0^4)_{|L^2_{0,2}}) = \frac{40\pi}{3\sqrt{3}} + \frac{4374}{91} \approx 72.2499.$$

This implies the inequality

$$\operatorname{tr}((A_0^2)_{|L^2_{0,2}})\operatorname{tr}((A_0^4)_{|L^2_{0,2}}) < \operatorname{tr}((A_0^3)_{|L^2_{0,2}})^2.$$

We conclude that for any $j \in \mathbb{Z}_3$, there is a non-real magic angle $\alpha_j \in \mathbb{C} \setminus \mathbb{R}$ with corresponding eigenfunction $u \in L^2_{0,j}$ of T_k. By Theorem 2, we conclude the existence of non-real and non-simple magic angles.

[2]The traces $\operatorname{tr}(A_0^2)$ and $\operatorname{tr}(A_0^4)$ were explicitly computed "by hand" in [Be*22] and strictly speaking, the following argument relies on computer assistance only for obtaining the exact value of $\operatorname{tr}(A_0^3)$.

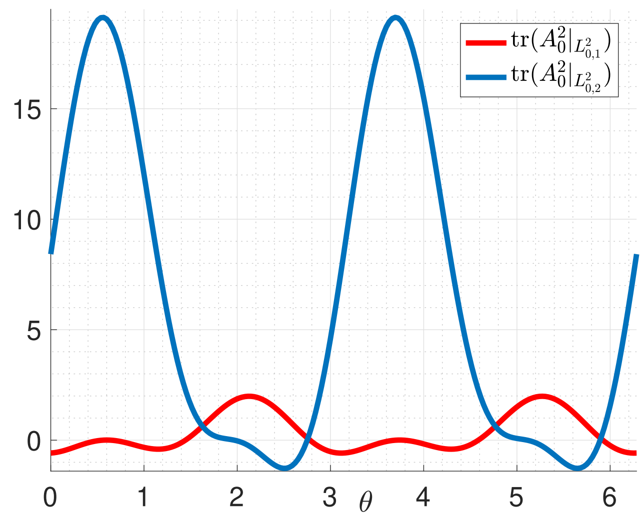

FIGURE 7. $\mathrm{tr}((A_0^2)_{|L^2_{0,1}})$ and $\mathrm{tr}((A_0^2)_{|L^2_{0,2}})$ for potentials $U_\pm(z) := U_\theta(\pm z)$ in (6.2). While for $\theta = 0$, $U_{\theta=0} = U_1$ we see that $\mathrm{tr}((A_0^2)_{|L^2_{0,2}}) > 0$ and $\mathrm{tr}((A_0^2)_{|L^2_{0,1}}) < 0$. For $\theta = 2\pi 7/8 \approx 5.5$ and $U_{\theta=2\pi 7/8} = U_2$ we have $\mathrm{tr}((A_0^2)_{|L^2_{0,2}}) < 0$ and $\mathrm{tr}((A_0^2)_{|L^2_{0,1}}) > 0$, instead.

We note that as the traces depend continuously on the potential U, the inequalities

$$\mathrm{tr}((A_0^2)_{|L^2_{0,1}}) = \mathrm{tr}((A_0^2)_{|L^2_{0,0}}) < 0 \text{ and } \mathrm{tr}((A_0^2)_{|L^2_{0,2}})\,\mathrm{tr}((A_0^4)_{|L^2_{0,2}}) < \mathrm{tr}((A_0^3)_{|L^2_{0,2}})^2$$

remain true for small perturbations of U and so does the existence of a non-real and non-simple magic angle. As stated in the introduction, the potential U_2, defined in (1.12), leads to real and doubly-degenerate magic angles. We then see numerically that $\mathrm{tr}((A_0^2)_{|L^2_{0,1}}) = \mathrm{tr}((A_0^2)_{|L^2_{0,0}}) > 0$, see Figure 7. To interpolate between these two opposite behaviors, we introduce the potentials

$$U_\theta(z) := U(z) = (\cos\theta - \sin\theta)U_1(z) + \sin\theta U_2(z), \tag{6.2}$$

see https://math.berkeley.edu/~zworski/Interpolation.mp4 for a movie showing the dependence of the set of magic angle when θ varies.

In Figure 7 we show $\mathrm{tr}((A_0^2)_{|L^2_{0,0}}), \mathrm{tr}((A_0^2)_{|L^2_{0,2}})$ as a function of θ, verifying that the inequality $\mathrm{tr}((A_0^2)_{|L^2_{0,0}}) < 0$ holds for a large range of values θ.

Remark. This previous computation could be made rigorous at the cost of adapting the algorithm used in [BHZ23, Theo. 7] to the potential U_θ in order to compute the first term in (6.1).

7. Generic simplicity in each representation

7.1. **Generalized potentials.** We now consider the general class of potentials $U_\pm(z)$ satisfying

$$U_\pm(\omega z) = \omega U_\pm(z), U_\pm(z+\gamma) = e^{\mp 2i\langle \gamma, K\rangle} U_\pm(z), \quad \gamma \in \Gamma. \tag{7.1}$$

We do *not* however assume $\overline{U_\pm(\bar z)} = -U_\pm(z)$ and then define

$$V(z) := \begin{pmatrix} 0 & U_+(z) \\ U_-(z) & 0 \end{pmatrix} \text{ such that } D_V(\alpha) = 2D_{\bar z} + \alpha V(z).$$

It is convenient to use the following Hilbert space of *real analytic* potentials defined using the following norm: for fixed $\delta > 0$,

$$\|V\|_\delta^2 := \sum_{\pm} \sum_{k \in \Lambda^*/3} |a_k^\pm|^2 e^{2|k|\delta}, \qquad U_\pm(z) = \sum_{k \in K + \Lambda^*} a_k^\pm e^{\pm i\langle z, k\rangle}. \tag{7.2}$$

Then we define $\mathscr{V} = \mathscr{V}_\delta$ by

$$V \in \mathscr{V} \iff V \text{ satisfies (7.1)}, \ \|V\|_\delta < \infty. \tag{7.3}$$

We note that we have as before,

$$\mathscr{L}_{\mathbf{a}} D_V(\alpha) = D_V(\alpha) \mathscr{L}_{\mathbf{a}}, \quad \Omega D_V(\alpha) = D_V(\alpha)\Omega.$$

We also recall the antilinear symmetry $\mathscr{A} : L^2_{k,j} \to L^2_{k,-j}$ defined by

$$\mathscr{A} := \begin{pmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{pmatrix}, \quad \Gamma v(z) = \overline{v(z)}, \qquad \mathscr{A} D_V(\alpha) \mathscr{A} = -D_V(\alpha)^*. \tag{7.4}$$

7.2. **Proof of generic simplicity.** Our proof of Theorem 4 is an adaptation of the argument for generic simplicity of resonances by Klopp–Zworski [KZ95] – see also [DyZw19, §4.5.5].

We then use the decomposition

$$L^2_0 = \bigoplus_{j=0}^{2} L^2_{0,j}, \quad L^2_{0,j} \simeq L^2(F),$$

where F is a fixed fundamental domain of G_3. For $V \in \mathscr{V}$ and $R = (2D_{\bar z})^{-1}$

$$V : L^2_{0,j} \to L^2_{0,j-1}, \quad R : L^2_{0,j-1} \to L^2_{0,j} \implies RV : L^2_{0,j} \to L^2_{0,j}.$$

Before proceeding we record the following regularity result:

Lemma 7.1. *Suppose that for some $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$ and $w \in L^2(\mathbb{C}/3\Lambda; \mathbb{C})$, $(RV - \lambda)^k w = 0$. Then $w \in C^\omega(\mathbb{C}/3\Lambda; \mathbb{C})$, that is, w is* real analytic. *The same conclusion holds if $(V^* R^* - \lambda)^k w = 0$.*

Proof. We prove a slightly more general statement that $(RV-\lambda)^k w = f \in C^\omega(\mathbb{C}/3\Lambda;\mathbb{C}^2)$ implies that $w \in C^\omega(\mathbb{C}/3\Lambda;\mathbb{C}^2)$. We proceed by induction on k. For $k=0$, $w=f$. If $k>0$, we put $\widetilde{w} := (RV-\lambda)^{k-1}w$ and note that (the case of $\lambda=0$ is even simpler)

$$D_V(-1/\lambda)\widetilde{w} = 2\lambda^{-1}D_{\bar{z}}(RV-\lambda)\widetilde{w} = 2\lambda^{-1}D_{\bar{z}}f \in C^\omega.$$

This means that $\widetilde{w}$ is a solution of an elliptic equation with analytic coefficients, hence it is analytic [HöI, Theorem 9.5.1]. The inductive hypothesis now shows that w is analytic.

In the case of $(V^*R^*-\lambda)^k w = 0$, we proceed similarly but put $\widetilde{w} := R^*(V^*R^* - \lambda)^{k-1}w$, so that

$$D_V(-1/\bar{\lambda})^*\widetilde{w} = 2\bar{\lambda}^{-1}D_z R^*(V^*R^*-\lambda)(V^*R^*-\lambda)^{k-1}w = 2\bar{\lambda}^{-1}D_z R^* f \in C^\omega.$$

Since $(V^*R^*-\lambda)^{k-1}w = 2D_z\widetilde{w}$ the inductive argument proceeds as before. □

The next lemma shows that we have generic simplicity for operators restricted to the three representations:

Lemma 7.2. *There exists a generic subset of $\mathscr{V}_j$ of $\mathscr{V}$ such that for $V \in \mathscr{V}_j$, the eigenvalues of $RV|_{L^2_{0,j}}$ are* simple.

Proof. We follow the presentation in the proof of [DyZw19, Theorem 4.39] with modifications needed for our case. We fix j and consider all operators as acting on $\mathscr{H} := L^2_{0,j}$. The eigenvalue multiplicity is defined using the resolvent:

$$m_V(\lambda) := \frac{1}{2\pi i}\operatorname{tr}\oint_\lambda (\zeta - RV)^{-1}d\zeta,$$

where the integral is over a sufficiently small positively oriented circle around λ. We then define

$$\mathscr{E}_r := \{W \in \mathscr{V} : m_W(\lambda) \le 1,\ \lambda \in \mathbb{C}\setminus D(0,r)\}. \tag{7.5}$$

We want to show that for $r>0$, $\mathscr{E}_r$ is open and dense. That will show that the set

$$\mathscr{E} := \{W \in \mathscr{V} :\ \forall\,\lambda,\ m_W(\lambda) \le 1\} = \bigcap_{n\in\mathbb{N}} \mathscr{E}_{\frac{1}{n}}$$

is generic (and in particular, by the Baire category theorem, it has a nowhere dense complement).

Suppose that RW has exactly one eigenvalue λ_0 in $D(\lambda,r)$ and $\operatorname{Spec}(RW)\cap D(\lambda,2r) = \{\lambda_0\}$. Putting $\Omega := D(\lambda,r)$ we then define

$$\Pi_W(\Omega) := \frac{1}{2\pi i}\int_{\partial\Omega}(\zeta - RW)^{-1}d\zeta, \quad m_W(\Omega) := \operatorname{tr}\Pi_W(\Omega). \tag{7.6}$$

If $V \in \mathscr{V}$ and $\|V\|_\delta$ is sufficiently small then for $\zeta \in \partial\Omega$,

$$(R(W+V)-\zeta)^{-1} = (RW-\zeta)^{-1}(I + RV(RW-\zeta)^{-1})^{-1},$$

exists and we can define $\Pi_{W+V}(\Omega)$ as in (7.6). This also shows that if $\|V\|_\delta < \varepsilon$ for sufficiently small ε then for $\zeta \in \partial\Omega$,

$$(RW - \zeta)^{-1} - (R(W+V) - \zeta)^{-1} = \mathcal{O}_\varepsilon(\|V\|_\delta)_{\mathscr{H}\to\mathscr{H}}.$$

It follows that $\|\Pi_W(\Omega) - \Pi_{W+V}(\Omega)\|_{\mathscr{H}\to\mathscr{H}} \leq C_\varepsilon\|V\|_\delta$. In particular, if we take $\|V\|_\delta < 1/C_\varepsilon$, then $\Pi_W(\Omega)$ and $\Pi_{W+V}(\Omega)$ have the same rank

$$m_{W+V}(\Omega) \text{ is constant for } \|V\|_\delta \text{ sufficiently small.} \tag{7.7}$$

This immediately implies that $\mathscr{E}_r$ is open: if λ is a simple eigenvalue of RW then $m_W(\Omega) = 1$ this values does not change under small perturbations.

Now we want to show that $\mathscr{E}_r$ is dense. This follows from the following statement

$$\forall\, W \in \mathscr{V},\ \varepsilon > 0\ \ \exists\, V \in \mathscr{V} \quad W + V \in \mathscr{E}_r, \quad \|V\|_\delta < \varepsilon. \tag{7.8}$$

As the number of eigenvalues of RW outside $D(0,r)$ is finite, it is enough to prove a local statement as it can be applied successively to obtain (7.8) (once an eigenvalue is simple it stays simple for sufficiently small perturbations). That is, it is enough to show that

$$\begin{gathered} \forall\, W \in \mathscr{V},\ \varepsilon > 0\ \ \exists\, V \in \mathscr{V}\ \forall\, \lambda \in \Omega \\ m_{W+V}(\lambda) \leq 1, \quad \|V\|_\delta < \varepsilon. \end{gathered} \tag{7.9}$$

As in [KZ95] we proceed by induction and start by noting that one of two cases has to occur:

$$\forall\, \varepsilon > 0\ \ \exists\, V \in \mathscr{V},\ \lambda \in \Omega \quad 1 \leq m_{W+V}(\lambda) < m_{W+V}(\Omega), \quad \|V\|_\delta < \varepsilon, \tag{7.10}$$

or

$$\exists\, \varepsilon > 0\ \forall\, V \in \mathscr{V}, \|V\|_\delta < \varepsilon\ \exists\, \lambda = \lambda(V) \in \Omega \quad m_{W+V}(\lambda) = m_{W+V}(\Omega). \tag{7.11}$$

The first case implies that adding an arbitrarily small V to W produces at least two distinct eigenvalues of $R(V+W)$. The second case implies that for any small perturbation preserves maximal multiplicity.

We will now show that (7.11) *cannot occur*. For that assume that $m_W(\lambda) = M$ and that (7.11) holds. For $V \in \mathscr{V}$, $\|V\|_\delta < \varepsilon$, put, in the notation of (7.6),

$$k(V) := \min\{k : (R(W+V) - \lambda(V))^k \Pi_{W+V}(\Omega) = 0\}.$$

Then $1 \leq k(V) \leq M$ and $\mathscr{V} \ni V \mapsto k(V)$ is a *lower semi-continuous* function. In fact, if $\|V_j - V\|_{\mathscr{V}} \to 0$ and then, from (7.6), we see that $(R(W+V_j) - \lambda(V_j))^k \Pi_{W+V_j}(\Omega) = 0$, then $(R(W+V) - \lambda(V))^k \Pi_{W+V}(\Omega) = 0$.

We also define

$$k_0 := \max\{k(V) : V \in \mathscr{V}, \|V\|_\delta < \varepsilon/2\}.$$

It follows that if $k(V') = k_0$ then $k(V + V') = k_0$ for $\|V\|_\delta < \rho$, with a sufficiently small ρ. Hence we can replace W by $W + V'$, decrease ε and assume that

$$\begin{aligned}(R(W+V) - \lambda(V))^{k_0}\Pi_{V+W}(\Omega) &= 0,\\ (R(W+V) - \lambda(V))^{k_0-1}\Pi_{V+W}(\Omega) &\neq 0,\\ m_{W+V}(\lambda(V)) = \operatorname{tr}\Pi_{V+W} &= M > 1, \quad \forall V, \ \ \|V\|_\delta < \varepsilon.\end{aligned} \tag{7.12}$$

To see that (7.12) is impossible we first assume that $k_0 > 1$. Take $V = V(t) = W + tV$, $\|V\|_{C^M} < \varepsilon$, $t \in [-1, 1]$. For $h, g \in \mathscr{H}$ we define (dropping Ω in $\Pi_\bullet(\Omega)$)

$$\begin{aligned} w(t) &:= (R(W+tV) - \lambda(t))^{k_0-1}\Pi_{W+tV}h,\\ \widetilde{w}(t) &:= ((W^* + tV^*)R^* - \overline{\lambda(t)})^{k_0-1}\Pi^*_{W+tV}g.\end{aligned}$$

By our assumption (7.12) we can choose g and h so that $w := w(0) \not\equiv 0$ and $\widetilde{w} := \widetilde{w}(0) \not\equiv 0$. Lemma 7.1 then implies that

$$\operatorname{supp} w = \operatorname{supp}\widetilde{w} = \mathbb{C}/3\Lambda. \tag{7.13}$$

Since $\lambda(t)$ is assumed to be the only eigenvalue of $RV(t)$ in Ω and since it has fixed algebraic and geometric multiplicity, the functions $t \mapsto \lambda(tV), \Pi_{W+tV}, w(t)$ depend smoothly on t. Hence, we can differentiate:

$$\begin{aligned} 0 &= \frac{d}{dt}(R(W+tV) - \lambda(t))^{k_0}\Pi_{W+tV}h\\ &= \sum_{\ell=0}^{k_0-1}(R(W+tV) - \lambda(t))^{\ell} RV (R(W+tV) - \lambda(t))^{k_0-1-\ell}\Pi_{W+tV}h\\ &\quad + (R(W+tV) - \lambda(t))H(t)\end{aligned}$$

where $H(t) \in \mathscr{H}$. We now put $t = 0$ and take the $\mathscr{H}$ inner product with $\widetilde{w}$: the term with $H(0)$ disappears as $(RW - \lambda(0))^{k_0}\Pi^*_W \equiv 0$ as do all the terms with $\ell > 0$. Consequently, we obtain

$$\forall V \in \mathscr{V} \ \ \langle Vw, R^*\widetilde{w}\rangle = 0.$$

Since $V \in L^2_{0,1}$, $w \in L^2_{0,j}$, $R^*\widetilde{w} \in L^2_{0,j+1}$, we conclude that (with $\circ_j$ denoting components of $\bullet = w, \widetilde{w}$)

$$\langle U_+ w_2, R^*\widetilde{w}_1\rangle_{L^2(F)} + \langle U_- w_1, R^*\widetilde{w}_2\rangle_{L^2(F)} = 0, \tag{7.14}$$

where F is a fundamental domain of the joint group action defined by $\mathscr{L}$ and $\mathscr{C}$. Since V is arbitrary on F, this implies that $\bar{w}(z)(R^*\widetilde{w})(z) \equiv 0$, which in turn contradicts (7.13).

It remains to consider the case of $k_0 = 1$ in (7.12). In that case the finite rank projection Π_W can be written as (with the notation, $(f \otimes g)(u) := f\langle u, g\rangle$)

$$\Pi_W = \sum_{j=1}^{M} w_j \otimes \widetilde{w}_j, \quad \langle w_j, \widetilde{w}_k\rangle = \delta_{jk}, \quad (RW - \lambda_0)w_j = 0, \quad (W^*R^* - \bar{\lambda}_0)\widetilde{w}_k = 0. \tag{7.15}$$

Then,

$$\begin{aligned} 0 &= \frac{d}{dt}\left[(\lambda(t) - R(W+tV))\Pi_{W+tV}\right] \\ &= \lambda'(t)\Pi_{W+tV} - RV\Pi_{W+tV} + (\lambda(t) - R(W+tV))\frac{d}{dt}\Pi_{W+tV} \end{aligned}$$

Applied to w_j and paired with $\widetilde{w}_k$ we get at $t=0$,

$$0 = \lambda'(0)\delta_{jk} - \langle RVw_j, \widetilde{w}_k\rangle.$$

Hence we need to show that for $j \neq k$

$$\langle RVw_j, \widetilde{w}_k\rangle = 0, \quad \forall V \in \mathscr{V} \implies w_j = \widetilde{w}_k = 0. \tag{7.16}$$

But that is done as in the discussion after (7.14).

We have now proved that (7.10) holds and we use it now to prove (7.9) by induction on $m_W(\lambda_0)$ where λ_0 is the unique eigenvalues of RW in $D(\lambda_0, 2r)$, $\Omega := D(\lambda_0, r)$. If $m_W(\lambda_0) = 1$ there is nothing to prove. Assuming that we proved (7.9) for $m_W(\lambda_0) < M$ assume that $m_W(\lambda_0) = M$. From (7.10) we see that we can find V, $\|V_0\|_\delta < \varepsilon/2$ such that $m_{W+V_0}(\Omega) = m_W(\Omega)$ (see (7.7)) and such that all eigenvalues in Ω, $\lambda_1, \cdots, \lambda_k$, satisfy $m_{W+V_0}(\lambda_j) < M$. We now find r_j such that,

$$\begin{gathered} D(\lambda_j, 2r_j) \subset \Omega, \quad D(\lambda_j, 2r_j) \cap D(\lambda_k, 2r_k) = \emptyset, \quad j \neq k, \\ \{\lambda_j\} = D(\lambda_j, 2r_j) \cap \operatorname{Spec}(R(W+V_0)). \end{gathered}$$

We put $\Omega_j := D(\lambda_j, r_j)$ and apply (7.9) successively to $W + V_0 + \cdots V_{j-1}$, $j = 1, \cdots, k$, in Ω_j with $\|V_j\|_\delta < \varepsilon/2^{j+1}$. That gives the desired $V = \sum_{j=0}^k V_j$. ☐

8. Generic simplicity

In this section we complete the proof of Theorem 4.

We already showed in Proposition 2.2[3] that $\operatorname{Spec}_{L^2_{0,0}}(RW) = \operatorname{Spec}_{L^2_{0,1}}(RW)$ and know from the previous Lemma that we can ensure simplicity of spectra of RW in each representation $L^2_{0,j}$. We shall now see that we can split spectra of RW in $L^2_{0,0}, L^2_{0,1}$ from the one in $L^2_{0,2}$.

Lemma 8.1. *Suppose that*

$$\operatorname{Spec}_{L^2_{0,j}}(RW) \cap D(\lambda_0, 2r) = \{\lambda_0\}, \quad j \in \mathbb{Z}_3, \ r > 0,$$

[3]We stated Proposition 2.2 for a smaller class of potentials than the generalized tunnelling potentials considered here, see (7.1), but the proof only uses only translational and rotational symmetries which are still satisfied for generalized tunnelling potentials

and λ_0 *is a simple eigenvalue of* $RW|_{L^2_{0,j}}$*. Then, for every* $\varepsilon > 0$ *there exists* $V \in \mathscr{V}$*,* $\|V\|_\delta < \varepsilon$*, such that for some* $\lambda_1 \neq \lambda_2$

$$\begin{aligned} &\mathrm{Spec}_{L^2_{0,2}}(R(W+V)) \cap D(\lambda_0, r) = \{\lambda_2\}, \\ &\mathrm{Spec}_{L^2_{0,j}}(R(W+V)) \cap D(\lambda_0, r) = \{\lambda_1\}, j \in \{0,1\}. \end{aligned} \tag{8.1}$$

Proof. As in (7.15) we have $w_k, \widetilde{w}_k \in L^2_{0,j_k}$, such that $\langle w_k, \widetilde{w}_k\rangle = 1$, and

$$(2\lambda_0 D_{\bar z} - W) w_k = 0, \quad (2\bar\lambda_0 D_z - W^*) R^* \widetilde{w}_k = 0.$$

Since the eigenvalue λ_0 is assumed to be simple, (7.4) gives

$$R^* \widetilde{w}_p = \gamma_{1-p} \mathscr{A} w_{1-p} = \gamma_{1-p} \begin{pmatrix} \bar w_{(1-p)2} \\ -\bar w_{(1-p)1} \end{pmatrix}, \quad w_p = \begin{pmatrix} w_{p1} \\ w_{p2} \end{pmatrix}, \quad \gamma_p \in \mathbb{C}^*. \tag{8.2}$$

We can split an eigenvalue with eigenvectors w_k, if we can find V such that (see (7.14) for the notation)

$$\begin{gathered} \langle V w_2, R^* \widetilde{w}_2 \rangle_{L^2(F)} \neq \langle V w_0, R^* \widetilde{w}_0 \rangle_{L^2(F)}, \text{ with} \\ \langle V w_2, R^* \widetilde{w}_2 \rangle = \bar\gamma_2 \int_F \big(U_+(z) w_{22}^2(z) - U_-(z) w_{21}^2(z)\big)\, dm(z) \text{ and} \\ \langle V w_0, R^* \widetilde{w}_0 \rangle = \bar\gamma_1 \langle V w_0, \mathscr{A} w_1 \rangle = \bar\gamma_1 \int_F (U_+(z) w_{02}(z) w_{12}(z) - U_-(z) w_{01}(z) w_{11}(z))\, dm(z) \end{gathered}$$

where we used (8.2) to obtain the last equality. If for all (analytic) $U_\pm$ the terms were equal it would follow that $\bar\gamma_2 w_{2\ell}^2 = \bar\gamma_1 w_{0\ell} w_{1\ell}$ for $\ell \in \{1,2\}$. This implies that $w_{2\ell}$ vanishes at $0, \pm z_S$. However, the zeros at $\pm z_S$ have to be at least of order 2 since by rotational and translational symmetry

$$\begin{aligned} w_2(z \pm z_S) &= \bar\omega w_2(\omega(z \pm z_S)) = \bar\omega w_2(\omega z \pm z_S \mp (1+\omega)) \\ &= \bar\omega \,\mathrm{diag}(\omega^{\pm 1}, \omega^{\mp 1}) w(\omega z \pm z_S). \end{aligned} \tag{8.3}$$

This means that for instance at z_S we have $w_2(z \pm z_S) = \mathrm{diag}(1, \omega) w(\omega z \pm z_S)$ which means that the first component has to vanish at least to third order and the second component at least to second order. This implies that w_2 has at least 5 zeros counting multiplicities and this is impossible by the usual theta function argument [BHZ24, Lemma 4.1]. ∎

We can now finish

Proof of Theorem 4. Lemma 7.2 (strictly speaking its proof) and Lemma 8.1 now show that for every $r > 0$, the set

$$\mathscr{V}_r := \{V : \ RV|_{L^2_{0,1} \oplus L^2_{0,2}} \text{ has simple eigenvalues in } \mathbb{C} \setminus D(0,r)\}$$

is open and dense. We then obtain $\mathscr{V}_0$ by taking the intersection of $\mathscr{V}_{1/n}$. ∎

9. The Chern number of a two-fold degenerate flat band

In this section we compute the Chern number of the flat band in the case of 2-fold degeneracy. We start by a general discussion of of the Chern connection and the Berry connection in the case holomorphic vector bundles. Although we stress our case of the two torus, §§9.1 and 9.2 apply to vector bundles over more general manifolds.

9.1. **The Chern connection.** Suppose that $\pi : E \mapsto X$ is a holomorphic vector bundle over a torus $X = \mathbb{C}/\Lambda^*$ (see [TaZw23, §2.7] for a quick introduction sufficient for our purposes or [We07] for an in-depth treatment), and that E is a sub-bundle of a trivial Hilbert bundle over X, $X \times \mathscr{H}$, where $\mathscr{H}$ is a Hilbert space. This gives a hermitian structure on E: for $k \in X$, we introduce an inner product on the fibers $E_k := \pi^{-1}(k)$, using $E_k \subset \mathscr{H}$:

$$\langle \zeta, \zeta' \rangle_k := \langle \zeta, \zeta' \rangle_{\mathscr{H}}, \quad \zeta, \zeta' \in E_k.$$

We then have two natural connections on E, the *Chern connection*, available when the bundle is holomorphic and equipped with hermitian structure, and a hermitian connection[4], available for any smooth vector bundle embedded in a Hilbert bundle. In the context of vector bundles of eigenfunctions, the latter is called the *Berry connection* and we adopt this terminology for the general case as well.

We first define the Chern connection. For that we choose a local holomorphic trivialization $U \subset X$, $\pi^{-1}(U) \simeq U \times \mathbb{C}^n$, for which the hermitian metric is given by

$$\langle \zeta, \zeta \rangle_k = \langle G(k)\zeta, \zeta \rangle = \sum_{i,j=1}^{n} G_{ij}(k)\zeta_i\bar{\zeta}_j \quad \zeta \in \mathbb{C}^n, \quad k \in U. \tag{9.1}$$

We note that if $\{u_1(k), \dots, u_n(k)\} \subset \mathscr{H}$ is a basis of E_k for $k \in U$, and $U \ni k \to u_j(k)$ are holomorphic, then $G(k)$ is the Gramian matrix:

$$G(k) := \left(\langle u_i(k), u_j(k) \rangle_{\mathscr{H}}\right)_{1 \leq i,j \leq n}. \tag{9.2}$$

If $s : X \to E$ is a section, then the Chern connection $D_C : C^\infty(X; E) \to C^\infty(X; E \otimes T^*X)$, over U is given by (using only the local trivialization and (9.1))

$$\begin{gathered} D_C s(k) := ds(k) + \eta_C(k)s(k), \\ \eta_C(k) := G(k)^{-1}\partial_k G(k)\, dk \in C^\infty(U, \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \otimes (T^*U)^{1,0}). \end{gathered} \tag{9.3}$$

Here ∂_k denotes the holomorphic derivative and the notation $(T^*U)^{1,0}$ indicates that only dk and not $d\bar{k}$ appear in the matrix valued 1-form η_C, $\eta_C = \eta_C^{1,0}$. We also recall that D_C is the unique hermitian connection with this property – see [We07, Theorem 2.1].

[4] $D : C^\infty(X, E) \to C^\infty(X, E \otimes T^*X)$ is a connection if for any $f \in C^\infty(X)$, $D(fs) = fDs + sdf$. A connection D is hermitian if $d\langle s(k), s'(k) \rangle_k = \langle Ds(k), s'(k) \rangle_k + \langle s(k), Ds'(k) \rangle$.

For the definition of the Berry connection we only require that $E \to X$ is a smooth vector bundle which is a subbundle of $X \times \mathscr{H}$, where $\mathscr{H}$ is a Hilbert space. That means for $k \in X$ we have a well defined orthogonal projection $\Pi(k) : \mathscr{H} \to E_k := \pi^{-1}(k)$ and an inclusion map $\iota : E \hookrightarrow X \times \mathscr{H}$. The formula for the Berry connection is then given by

$$D_B s(k) := \Pi(k)(d(\iota \circ s)(k)). \tag{9.4}$$

To find a local expression similar to (9.3) we use the Gramian (9.2). If $s(k) = \sum_{j=1}^n s_j^U(k) u_j(k) =: A(k) s^U(k)$, $A(k) : \mathbb{C}^n \to \mathscr{H}$ (so that $A(k)$ provides a local trivialization) then $\Pi(k) = A(k) G(k)^{-1} A(k)^*$ and

$$\begin{aligned} D_B s(k) &= \Pi(k) \sum_{j=1}^n \big(ds_j^U(k) u_j(k) + s_j^U(k) du_j(k) \big) \\ &= A(k)(ds^U(k) + \eta_B(k) s^U(k)), \\ \eta_B(k) &= G(k)^{-1} B(k) \in C^\infty(U, \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \otimes T^*U), \\ B(k)_{\ell j} &:= \langle du_j(k), u_\ell(k) \rangle_{\mathscr{H}} \in C^\infty(U, T^*U). \end{aligned} \tag{9.5}$$

These formulas hold for choices of u_j which are not necessarily holomorphic. However if, as in (9.2), $k \mapsto u_j(k)$ are holomorphic, then

$$\begin{aligned} (\partial_k G(k))_{ij} dk &= \langle \partial_k u_i(k), u_j(k) \rangle_{\mathscr{H}} dk + \langle u_i(k), \partial_{\bar{k}} u_j(k) \rangle_{\mathscr{H}} dk \\ &= \langle \partial_k u_i(k), u_j(k) \rangle_{\mathscr{H}} dk \\ &= \langle du_i(k), u_j(k) \rangle = B(k)_{ij}, \end{aligned} \tag{9.6}$$

since $\partial_{\bar{k}} u_j(k) = 0$ and $dw = \partial_k w \, dk + \partial_{\bar{k}} w \, d\bar{k}$. In particular, that means that in the notation of (9.3) and (9.4)

$$\begin{aligned} U \ni k \mapsto u_\ell(k) \text{ holomorphic} \quad &\Longrightarrow \quad \eta_C(k) = \eta_B(k), \ k \in U \\ &\Longrightarrow \quad D_C = D_B, \end{aligned} \tag{9.7}$$

We record this standard fact as

Proposition 9.1. *Suppose that X is a complex manifold and $E \mapsto X$ is a holomorphic vector bundle with a holomorphic embedding $\iota : E \to X \times \mathscr{H}$ into a trivial Hilbert bundle. Then the Berry connection (9.4) and the Chern connection (9.3) defined using the hermitian structure on $\mathscr{H}$ are equal.*

Remark. As was pointed out to us by Michael Singer, the conclusion (9.7) could be deduced directly from the uniqueness of the Chern connection mentioned after (9.3): using (9.4) we have $D_B^{(0,1)} s(k) = \Pi(k)(d^{(0,1)}(\iota \circ s)(k))$. But as the embedding ι (an inclusion, in our case) is holomorphic this implies that $D_B^{(0,1)} s(k) = 0$ for holomorphic sections. This and being hermitian characterize the Chern connection. We should also stress that the discussion above does not depend on the fact that X has complex dimension one.

The curvature of a connection D is given by

$$\Theta := D \circ D, \tag{9.8}$$

which is a globally defined two form with values in $\mathrm{Hom}(E, E)$. In a local trivialization in which $D = d + \eta$, we have $\Theta = d\eta + \eta \wedge \eta$. For the Chern connection, for X of any dimension $\Theta = \bar\partial \partial \eta_C$ since (9.3) shows that $\partial \eta_C = -\eta_C \wedge \eta_C$ (when X has a complex dimension one, this is obvious as $dk \wedge dk = 0$). It is then immediate from (9.7) that

$$\Theta := D_C \circ D_C = D_B \circ D_B, \tag{9.9}$$

that is, in the holomorphic case, the curvatures defined using the Chern curvature or the Berry curvature agree for holomorphic vector bundles embedded in trivial Hilbert bundles.

The Chern class (a Chern number in the case of $\mathbb{C}/\Lambda^*$) is given by

$$c_1(E) := \frac{i}{2\pi} \int_{\mathbb{C}/\Lambda^*} \operatorname{tr} \Theta \in \mathbb{Z},$$

where we note that over $U \subset \mathbb{C}/\Lambda^*$ for which we defined (9.2),

$$\begin{aligned} \operatorname{tr} \Theta &= \partial_{\bar k} \operatorname{tr} G(k)^{-1} \partial_k G(k) \, d\bar k \wedge dk \\ &= \partial_{\bar k} \partial_k \log g(k) \, d\bar k \wedge dk, \qquad g(k) := \det G(k), \end{aligned} \tag{9.10}$$

where we used Jacobi's formula [DyZw19, (B.5.14)]. In particular,

$$H(k) := \partial_{\bar k} \partial_k \log g(k) = g(k)^{-2} (g(k) \partial_{\bar k} \partial_k g(k) - |\partial_k g(k)|^2).$$

For any holomorphic hermitian vector bundle the trace of the curvature of the Chern connection, $\operatorname{tr} \Theta$ can be interpreted as a curvature of a line bundle. If $\pi : E \to X$ has rank n, we obtain a line bundle $\pi : L := \wedge^n E \to X$. It inherits hermitian structure from E. If we define the Chern connection on $\wedge^n E$ as in (9.3) (using only holomorphy and the hermitian structure) we obtain a new curvature Θ_L which is a differential two form on X, and

$$\Theta_L = \operatorname{tr} \Theta.$$

In case when E embeds holomorphically in $X \times \mathscr{H}$ we can then take, as in (9.2), $k \mapsto u_j(k) \in \mathscr{H}$, $j = 1, \cdots, n$, a local holomorphic basis of E. Then for

$$\Phi(k) := \wedge_{j=1}^n u_j(k) \in \wedge^n E_k \subset \wedge^n \mathscr{H}, \tag{9.11}$$

we have

$$\|\Phi(k)\|^2_{\wedge^n \mathscr{H}} = \det \left((\langle u_j(k), u_\ell(k) \rangle_{\mathscr{H}})_{1 \le j, \ell \le n} \right) = \det G(k) = g(k).$$

In particular when $X = \mathbb{C}/\Lambda^*$, we obtain, as in [BHZ24, (5.10)],

$$\Theta_L = H(k) \; d\bar k \wedge dk, \tag{9.12}$$

where H is given by

$$H(k) = \|\Phi(k)\|^{-4} \left(\|\Phi(k)\|^2 \|\partial_k \Phi(k)\|^2 - |\langle \partial_k \Phi(k), \Phi(k) \rangle|^2 \right) \geq 0, \tag{9.13}$$

where $\| \bullet \| = \| \bullet \|_{\wedge^n \mathscr{H}}$.

Remark. From a physics perspective the construction of the line bundle $\wedge^n E$, in the case of $E \subset X \times \mathscr{H}$ can be interpreted as the Slater determinant of the individual Bloch functions on the fermionic n-particle Hilbert space. We thus find that the trace of the curvature of the rank n vector bundle coincides with the curvature of the line bundle described by the n-particle wavefunction.

9.2. **The Berry curvature.** For completeness we derive the standard formula for the curvature of the Berry connection (9.4):

Proposition 9.2. *Suppose that $\pi : E \to X$ is a complex vector bundle over a manifold X and that there exists an embedding $\iota : E \to X \times \mathscr{H}$ into a trivial Hilbert bundle. Then the curvature of the connection (9.4) is given in terms of the orthogonal projection $\Pi(k) : \mathscr{H} \to E_k := \pi^{-1}(k)$, as*

$$\Theta = \Pi \, d\Pi \wedge d\Pi|_E, \tag{9.14}$$

and is a differential two form with values in $\mathrm{Hom}(E, E)$.

Proof. This is a local computation so for some $U \subset X$ we can choose a smoothly varying orthonormal basis $\{u_j(k)\}_{j=1}^n$, $k \in U$. Then in the notation of (9.5) (we drop the dependence on k in $A(k)$, $\Pi(k)$ and E_k)

$$A : \mathbb{C}^n \to \mathscr{H}, \quad A^* : \mathscr{H} \to \mathbb{C}^n, \quad AA^* = \Pi, \quad A^*A = I_{\mathbb{C}^n}. \tag{9.15}$$

With the trivialization given by A, we have (using (9.15))

$$Ds = A^*(\Pi(d(As)) = A^*\Pi A ds + A^*\Pi dA s = ds + A^* dA s =: ds + \eta ds.$$

Hence, in this trivialization, the curvature is a differential two form with values in $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$:

$$\begin{aligned} A^*\Theta A &= d\eta + \eta \wedge \eta = d(A^* dA) + A^* dA \wedge A^* dA \\ &= dA^* \wedge dA + A^* dA \wedge A^* dA. \end{aligned}$$

The curvature $\Theta = D_B \circ D_B$ which is a differential form with values in $\mathrm{Hom}(E, E)$, is then given by

$$\begin{aligned} \Theta = \Pi\Theta\Pi &= A(dA^* \wedge dA + A^* dA \wedge A^* dA)A^* \\ &= A dA^* \wedge dA A^* + AA^* dA \wedge A^* dA A^* \\ &= A dA^* \wedge dA A^* + A dA^* A \wedge dA^* A A^*, \end{aligned} \tag{9.16}$$

where we used $d(A^*A) = 0$.

The right hand side in (9.14) is given by

$$\begin{aligned} AA^*d(AA^*) \wedge d(AA^*) &= AA^*((dAA^* + AdA^*) \wedge (dAA^* + AdA^*) \\ &= AA^* \left(dA \wedge (A^*dA)A^* + dA \wedge (A^*A)dA^* \right. \\ &\quad \left. + AdA^* \wedge dAA^* + AdA^* \wedge dAA^*\right). \end{aligned}$$

From (9.15) we see that $A^*A = I_{\mathbb{C}^n}$ and that $A^*dA = -dA^*A$. Hence,

$$\begin{aligned} \Pi d\Pi \wedge d\Pi = AA^* &\left(-dA \wedge dA^*AA^* + dA \wedge dA^* \right. \\ &\left. + AdA^* \wedge dAA^* + AdA^* \wedge AdA^*\right). \end{aligned}$$

Acting on E, $AA^* = I_E$ and hence the first two terms in the bracket cancel:

$$\Pi d\Pi \wedge d\Pi|_E = AdA^* \wedge dAA^*|_E + AdA^* \wedge AdA^*|_E.$$

But from (9.16) that is the same as the action of Θ on E.

□

9.3. **Proof of Theorem 5.** We now consider

$$V(k) := \ker_{H_0^1}(D(\alpha) + k) \subset L_0^2. \tag{9.17}$$

This defines a (trivial) vector bundle $\widetilde{E} \to \mathbb{C}$:

$$\widetilde{E} := \{(k, v) : v \in V(k)\} \subset \mathbb{C} \times L_0^2(\mathbb{C}/\Lambda; \mathbb{C}^2).$$

To define a vector bundle over the torus $\mathbb{C}/\Lambda^*$ we define an equivalence relation on $\mathbb{C} \times L_0^2(\mathbb{C}/\Lambda; \mathbb{C}^2)$:

$$\exists\, p \in \Lambda^* \ (k, u) \sim_\tau (k + p, \tau(p)^{-1}u), \quad [\tau(p)u](z) = e^{i\langle z,k\rangle}v(z), \tag{9.18}$$

and notice that $\tau(p)^{-1}V(k) = V(k + p)$. Using this (see [TaZw23, Lemma 8.4] or [BHZ24, Lemma 5.1]),

$$E := \widetilde{E} \,/ \sim_\tau \to \mathbb{C}/\Lambda^*. \tag{9.19}$$

is a holomorphic vector bundle over $\mathbb{C}/\Lambda^*$.

Since $\Pi(k + p) = \tau(p)^{-1}\Pi(k)\tau(p)$, the Berry connection defined by (9.4) on $\widetilde{E}$, satisfies

$$(D_B s)(k + p) = \Pi(k + p)(d(\iota \circ s)(k + p)) = \tau^{-1}(p)\Pi(k)d(\iota \circ \tau(p)s(k + p)).$$

Hence, if for $k \in U \subset X$, $(k, s(k)) \sim_\tau (k', s'(k'))$ then $k' = k+p$, $s'(k+p) = \tau(p)^{-1}s(k)$, for some $p \in \Lambda^*$ and

$$(k, D_B s(k)) \sim_\tau (k', D_B s'(k')).$$

This means that D_B is a well defined connection on $\widetilde{E}$. Since the Chern connection is intrinsically defined on $\widetilde{E}$ using holomorphic and hermitian structures, the two connections are equal.

If $m(\alpha) = m$ then by Theorem 1 $u \in \ker_{H_0^1} D(\alpha)$ has exactly m zeros and let us first assume that they are simple (this can always be arranged by multiplication by a meromorphic function). Let us denote them by $z_1, \cdots, z_m$. Then

$$V(k) = \left\{ \sum_{\ell=0}^{m} \zeta_\ell F_k(z - z_\ell) u_0(z), \zeta \in \mathbb{C}^m \right\}, \quad k \notin \Lambda^*, \tag{9.20}$$

where F_k's were defined in (3.2). When $p \in \Lambda^*$ we have

$$V(p) = \{ e^{i\langle p, z\rangle} u_0(z) f(z) : f \in L(D) \}, \tag{9.21}$$

where D is the divisor defined by the zeros of u_0. We can write elements of $L(D)$ as follows (see [Mu83, §I.6]):

$$f(z) = \mu_0 + \sum_{\ell=1}^{m} \mu_\ell \frac{\theta'(z - z_\ell)}{\theta(z - z_\ell)}, \qquad \sum_{\ell=1}^{m} \mu_\ell = 0. \tag{9.22}$$

We now consider

$$G_k(z) := e^{-i\langle k, z\rangle} F_k(z), \qquad G_p(z) = e_p(0)^{-1}, \ p \in \Lambda^*.$$

We recall from [BHZ24, Lemma 3.3] that for $p \in \Lambda^*$,

$$\begin{aligned} F_{k+p}(z) &= e_p(k)^{-1} \tau(p) F_k(z), \quad \tau(p) v(z) := e^{i\langle p, z\rangle} v(z) \\ e_p(k) &:= \frac{\theta(z(k))}{\theta(z(k+p))} = (-1)^n (-1)^m e^{i\pi n^2 \omega + 2\pi i n z(k)}, \quad z(p) = m + n\omega. \end{aligned}$$

Hence, for $p \in \Lambda^*$,

$$\begin{aligned} \theta(z(k+p))^{-1} G_{k+p}(z) &= \frac{\theta(z(k)}{\theta(z(k+p))} e^{-i\langle (k+p), z\rangle} \theta(z(k))^{-1} F_{k+p}(z) \\ &= e_p(k) e^{-i\langle (k+p), z} e_p(k)^{-1} e^{i\langle p, z - z_\ell\rangle} F_k(z) \\ &= e^{-i\langle k, z - z_\ell\rangle} \theta(z(k))^{-1} F_k(z) \\ &= \theta(z(k))^{-1} G_k(z), \end{aligned}$$

that is, $k \mapsto \theta(z(k))^{-1} G_k(z - z_\ell)$ is periodic with respect to Λ^*. If $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{C}^m$, $\sum_{j=1}^{m} \lambda_j = 0$, then

$$G_k(z, \lambda) := \sum_{j=1}^{m} \lambda_j \theta(z(k))^{-1} (G_k(z - z_j) - e_k(0)^{-1}), \tag{9.23}$$

is also periodic and smooth in k at Λ^*: for $p \in \Lambda^*$ we have

$$\begin{aligned} G_p(z,\lambda) = G_0(z,\lambda) &= \sum_{j=1}^m \lambda_j (z'(0)\theta'(0))^{-1} \partial_k \left(e^{-i\langle z,k\rangle} F_k(z-z_j) \right)|_{k=0} \\ &= \sum_{j=1}^m \lambda_j (z'(0)\theta'(0))^{-1} \left(\tfrac{i}{2}(z-z_j) - z'(0)\frac{\theta'(z-z_j)}{\theta(z-z_j)} \right) \\ &= \mu_0 + \sum_{j=1}^m \mu_j \frac{\theta'(z-z_j)}{\theta(z-z_j)}, \qquad \sum_{j=1}^m \mu_j = 0. \end{aligned}$$

Hence in view of (9.21) and (9.22) we can extend (9.20) to all $k \in \mathbb{C}$ as follows:

$$V(k) = \left\{ e^{i\langle z,k\rangle} G_k(z,\lambda) u_0(z) + \lambda_0 F_k(z-z_1) u_0(z) : \lambda_0 \in \mathbb{C}, \lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{C}^m, \ \sum_{\ell=1}^m \lambda_\ell = 0 \right\}$$

We now introduce

$$W(k) := \left\{ e^{i\langle z,k\rangle} G_k(z,\lambda) u_0(z) : \lambda = (\lambda_1, \cdots, \lambda_m), \ \sum_{\ell=1}^m \lambda_\ell = 0 \right\}, \quad k \in \mathbb{C}.$$

As in (9.19) this family of subspaces of L^2_0 defines a rank $m-1$ vector bundle, $F \to \mathbb{C}/\Lambda^*$. Since $k \mapsto G_k(z,\lambda)$ is periodic, F is trivial. If $E_1 \to \mathbb{C}/\Lambda^*$ is the line bundle coming from the family of subspaces of L^2_0,

$$V_1(k) := \mathbb{C} F_k(z-z_1) u_0(z), \quad k \in \mathbb{C},$$

(again, in the sense of (9.19)) we see as in [Le*20] and [BHZ24, (5.9),(B.8)] that $c_1(E_1) = -1$. Since $E = F \oplus E_1$, we obtain $c_1(E) = -1$.

Finally, we observe that for $\Omega : L^2_0(\mathbb{C}/\Lambda;\mathbb{C}) \to L^2_0(\mathbb{C}/\Lambda;\mathbb{C})$, $\Omega u(z) := u(\omega z)$, $\ker_{H^1_0}(D(\alpha) + \bar\omega k) = \Omega \ker_{H^1_0}(D(\alpha) + k)$ (see [BHZ24, §2.1]). Hence, in the notation of (9.4). $\Omega \Pi(k) \Omega^* = \Pi(\bar\omega k)$. If $Rk := \bar\omega k$, this means that $R^*\Pi = \Omega \Pi \Omega^*$. Also the pull back of Θ by R is well defined and, using (9.14) we see that

$$R^*\Theta = R^*(\Pi d\Pi \wedge d\Pi) = R^*\Pi d(R^*\Pi) \wedge d(R^*)\Pi = \Omega(\Pi d\Pi \wedge d\Pi)\Omega^* = \Omega\Theta\Omega^*.$$

In particular, in the notation of (9.12), we have

$$\operatorname{tr} R^*\Theta = \operatorname{tr}\Theta \implies H(\bar\omega k) = H(k).$$

Strictly speaking we should, just as we did at the end of (9.19), justify passing to the quotient. That is again easy by noting that $\Omega\tau(p)\Omega^* = \tau(\bar\omega p)$. This completes the proof of Theorem 5.

$X = L^2_{0,2}$	$X = L^2_{0,0}$	$X = L^2_{0,1}$
1.2400 − 0.0000i	1.6002 + 0.0000i	1.6002 + 0.0000i
1.2400 − 0.0000i	1.2583 − 1.1836i	1.2583 − 1.1836i
1.3424 + 1.6788i	1.2583 + 1.1836i	1.2583 + 1.1836i
1.3424 − 1.6788i	1.4019 − 2.2763i	1.4019 − 2.2763i
2.9543 + 0.0000i	1.4019 + 2.2763i	1.4019 + 2.2763i
1.4575 + 2.7610i	1.5001 + 3.3130i	1.5001 + 3.3130i
1.4575 − 2.7610i	1.5001 − 3.3130i	1.5001 − 3.3130i
3.5878 + 1.9298i	3.4078 + 1.3122i	3.4078 + 1.3122i
3.5878 − 1.9298i	3.4078 − 1.3122i	3.4078 − 1.3122i

TABLE 2. Magic angles for $\theta = 2.808850897$ and $U_\pm := U_0(\pm\bullet)$ with $U_0(\zeta) = \cos(\theta)U_1(\zeta) + \sin(\theta)\sum_{i=0}^{2}\omega^i e^{-(\zeta\bar\omega^i - \bar\zeta\omega^i)}$ such that $1/\alpha \in \mathrm{Spec}_X(T_0)$ (counting algebraic multiplicity). The magic angle with algebraic multiplicity 2 and geometric multiplicity 1 is highlighted in blue.

10. Numerical observations

Here we present two numerical observations related to our mathematical results. For all our numerics, we used a Fourier discretization of the operators, see [Be*21, Sec.5.1] for an explanation, with $N = 101$ Fourier coefficients per spatial dimension.

10.1. Algebraic multiplicities in the spectral characterization.

Theorem 2 implies that it is impossible to have

$$\dim\ker_{L^2_0}(D(\alpha)) = \dim\ker_{L^2_{0,2}}(D(\alpha)) = 2$$

which is equivalent to having eigenvalues of geometric multiplicity 2 for T_0, i.e.

$$\dim\ker_{L^2_0}(T_0 - 1/\alpha) = \dim\ker_{L^2_{0,2}}(T_0 - 1/\alpha) = 2,$$

we can indeed have that $1/\alpha$ is an eigenvalue of algebraic multiplicity 2 and geometric multiplicity 1 on L^2_0 and $L^2_{0,2}$. This is illustrated in Table 2 and Figure 8. In particular, it implies that T_k in general is not diagonalizable. Since the algebraic multiplicity of T_k is independent of k, it follows by Theorem 1 that the geometric multiplicity is independent of k. Examples of this are exhibited in Table 2 and Figures 8 and 9.

10.2. Behaviour of the curvature.

Since we established in Theorem 5 that $H(\omega z) = H(z)$ where H is the scalar curvature. We conclude that 0 and $\pm z_S$ are critical points

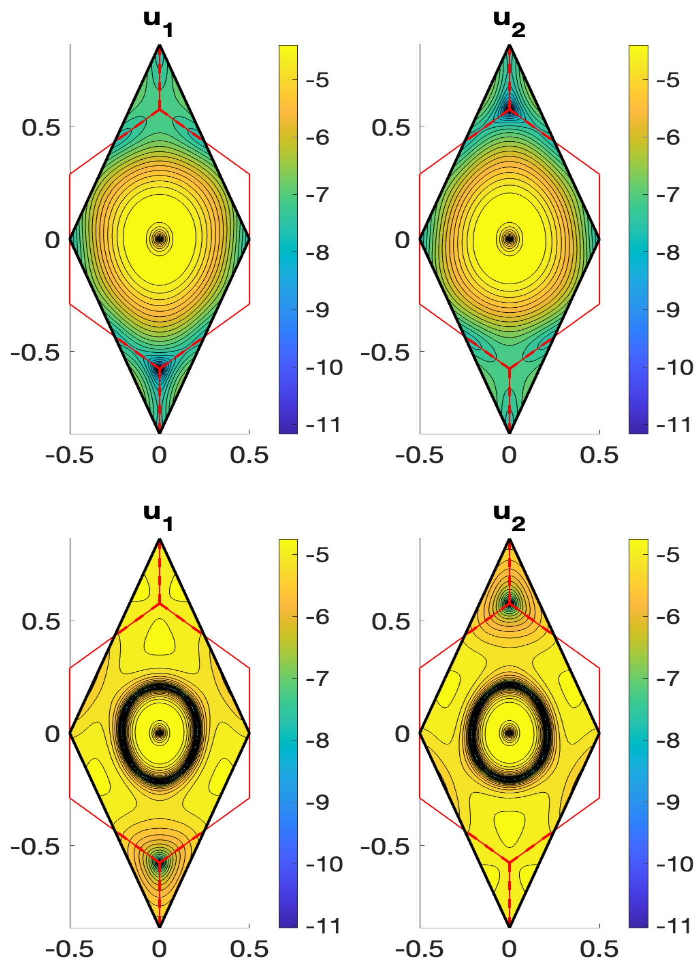

FIGURE 8. The first two singular values of $D(\alpha)$ are $2.804e-15$ and 3.990 suggesting the existence of only one flat band at $\alpha = 1.2400$ for $\theta = 2.808850$ and $U_\pm := U_0(\pm\bullet)$ with $U_0(\zeta) = \cos(\theta)U_1(\zeta) + \sin(\theta)\sum_{i=0}^{2}\omega^i e^{-(\zeta\bar\omega^i-\bar\zeta\omega^i)}$. The eigenvector of T_0 with eigenvalue $1/\alpha$ is shown on top and the generalized one at the bottom.

of H. In addition, the symmetry $\mathscr{E}$ defined in (2.12) and the formula (3.4) imply that the Gramian matrix satisfies for simple or two-fold degenerate magic angles

$$G(k) = G(-k).$$

This implies the symmetries in Figure 10.

However, while it seems that the maximum is attained at Γ and the minima at K, K', we do not have an analytical argument for this at the moment.

Figure 11 shows that the standard deviation of the Berry curvature, for the potential U_2 with only two-fold degenerate real magic angles, increases monotonically for the real

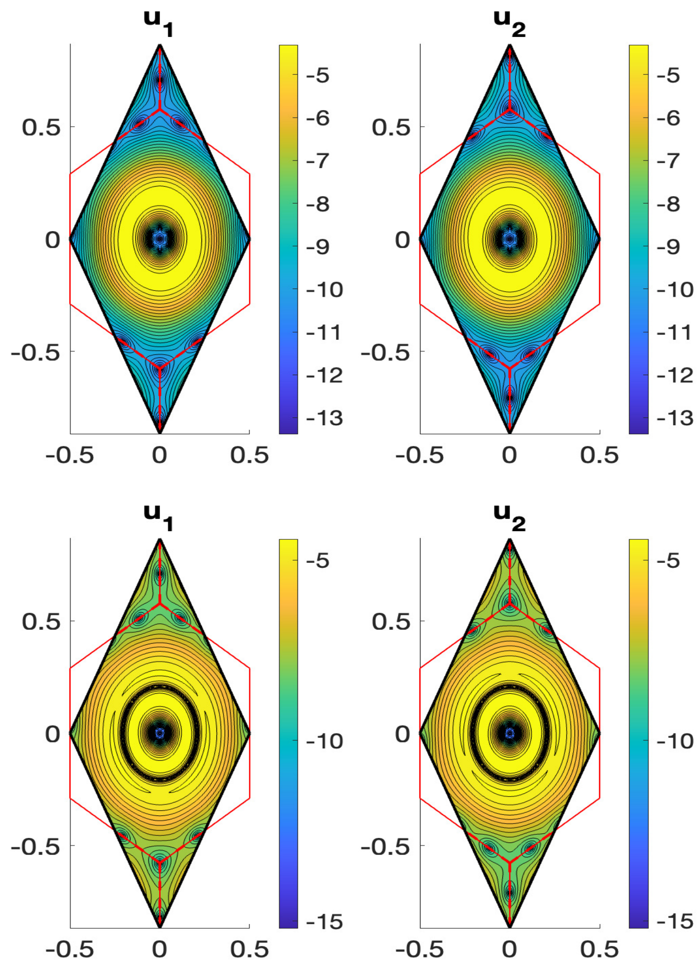

FIGURE 9. One flat band for $\alpha = 5.3811$ for $\theta = 2.7672151$ and $U_\pm := U_0(\pm\bullet)$ with $U_0(\zeta) = \cos(\theta)U_1(\zeta) + \sin(\theta)\sum_{i=0}^{2}\omega^i e^{-(\zeta\bar\omega^i - \bar\zeta\omega^i)}$. The eigenvector of T_0 with eigenvalue $1/\alpha$ is shown on top and the generalized one at the bottom.

magic angles. This is in contrast to the case of simple magic angles in [BHZ24, Figure 7].

References

[Be*21] S. Becker, M. Embree, J. Wittsten and M. Zworski, *Spectral characterization of magic angles in twisted bilayer graphene,* Phys. Rev. B **103**, 165113, (2021).

[Be*22] S. Becker, M. Embree, J. Wittsten and M. Zworski, *Mathematics of magic angles in a model of twisted bilayer graphene*, Probab. Math. Phys. **3** (2022), 69–103.

[BZ23] S. Becker and M. Zworski, *Dirac points for twisted bilayer graphene with in-plane magnetic field*, J. Spectr. Theory, 14 (2024), no. 2, 479–511.

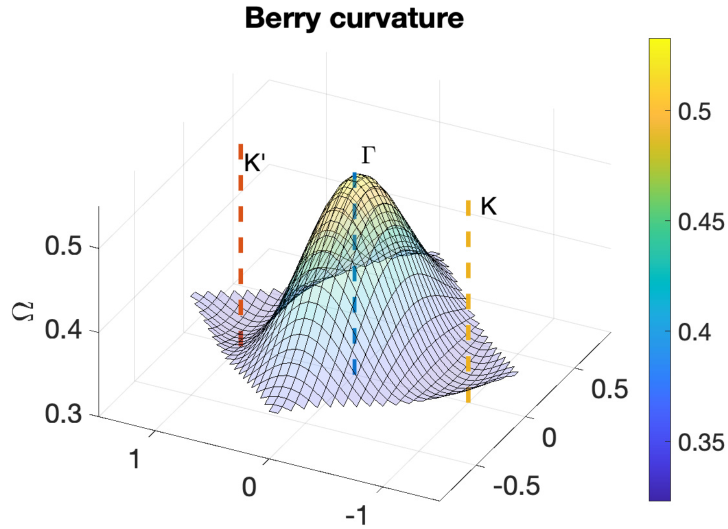

FIGURE 10. The plot of the curvature of the holomorphic line bundle corresponding to the first two-fold generate magic angle, defined in (9.12) with potential $U_\pm := U_2(\pm\bullet)$, as in (1.12). The extrema at K, Γ, K' follow from Theorem 5.

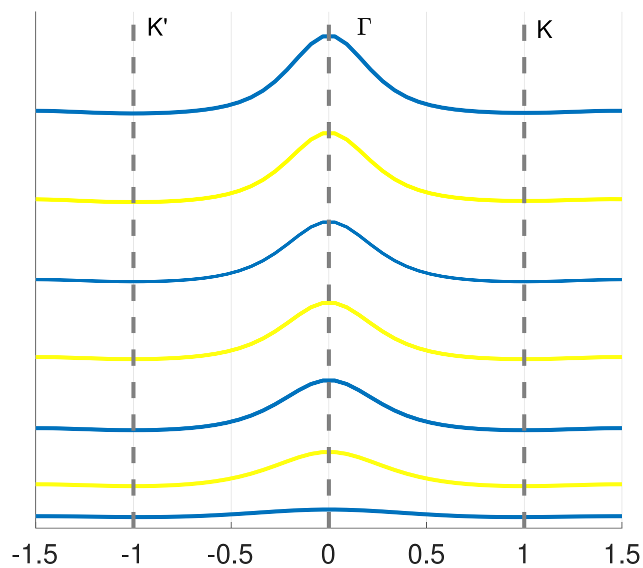

FIGURE 11. Cross-section of curvature for $k_x = 0$ for the first seven magic angles with potential $U_\pm := U_2(\pm\bullet)$, as in (1.12), in increasing order. The extrema at K, Γ, K' follow from Theorem 5.

[BHZ23] S. Becker, T. Humbert and M. Zworski, *Integrability in the chiral model of magic angles*, Comm. Math. Phys. **403**(2023), 1153–1169.

[BHZ24] S. Becker, T. Humbert and M. Zworski, *Fine structure of flat bands in a chiral model of magic angles,* Ann. Henri Poincaré (2024). https://doi.org/10.1007/s00023-024-01478-3.

[BiMa11] R. Bistritzer and A. MacDonald, *Moiré bands in twisted double-layer graphene.* PNAS, **108**, 12233–12237, (2011).

[CGG22] E. Cancès, L. Garrigue, D. Gontier, *A simple derivation of moiré-scale continuous models for twisted bilayer graphene.* arXiv:2206.05685.

[Cao18] Cao, Y., Fatemi, V., Fang, S. et al. *Unconventional superconductivity in magic-angle graphene superlattices.* Nature 556, 43-50, (2018).

[DuNo80] B.A. Dubrovin and S.P. Novikov, *Ground states in a periodic field. Magnetic Bloch functions and vector bundles.* Soviet Math. Dokl. **22**, 1, 240–244, (1980).

[De23] T. Devakul, P. J. Ledwith, L. Xia, A. Uri, S. de la Barrera, P. Jarillo-Herrero, L. Fu *Magic-angle helical trilayer graphene.*arXiv:2305.03031, 2023.

[DyZw19] S. Dyatlov and M. Zworski, *Mathematical Theory of Scattering Resonances,* AMS 2019, `http://math.mit.edu/~dyatlov/res/`

[HöI] L. Hörmander, *The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis,* Springer Verlag, 1983.

[IN25] V. Iugov, N. Nekrasov, *Yang-Mills flows for multilayered graphene*, arXiv:2504.19097, (2025).

[KZ95] F. Klopp and M. Zworski, *Generic simplicity of resonances,* Helv. Phys. Acta **68**(1995), 531–538.

[Ka80] T. Kato, *Perturbation Theory for Linear Operators*, Corrected second edition, Springer, 1980.

[KhZa15] S. Kharchev and A. Zabrodin, *Theta vocabulary I.* J. Geom. Phys. **94**(2015), 19–31.

[Le22] C. Le, Q. Zhang, C. Fan, X. Wu, C.-K.. Chiu, *Double and Quadruple Flat Bands tuned by Alternative magnetic Fluxes in Twisted Bilayer Graphene*, Phys. Rev. Lett. **132**(2024), 246401.

[Le*20] P.J. Ledwith, G. Tarnopolsky, E. Khalaf, and A. Vishwanath, *Fractional Chern insulator states in twisted bilayer graphene: An analytical approach,* Phys. Rev. Research 2, 023237, 2020.

[Mu83] D. Mumford, *Tata Lectures on Theta. I.* Progress in Mathematics, **28**, Birkhäuser, Boston, 1983.

[PT23] FK Popov, G Tarnopolsky, *Magic Angles In Equal-Twist Trilayer Graphene*, arXiv:2303.15505, 2023.

[Ser19] M. Serlin, *Intrinsic quantized anomalous Hall effect in a moiré heterostructure*, Science, Vol 367, Issue 6480, 900-903, (2019).

[SGG12] P. San–Jose, J. González, and F. Guinea, *Non-Abelian gauge potentials in graphene bilayers,* Phys. Rev. Lett. 108, 216802 (2012).

[Si77] B. Simon, *Notes on infinite determinants of Hilbert space operators*, Adv. in Math. 24 (1977), 244-273.

[TaZw23] Z. Tao and M. Zworski, *PDE methods in condensed matter physics,* Lecture Notes, 2023, `https://math.berkeley.edu/~zworski/Notes_279.pdf`.

[TKV19] G. Tarnopolsky, A.J. Kruchkov and A. Vishwanath, *Origin of magic angles in twisted bilayer graphene*, Phys. Rev. Lett. 122, 106405, (2019).

[Wa*22] A. B. Watson, T. Kong, A. H. MacDonald, and M. Luskin *Bistritzer-MacDonald dynamics in twisted bilayer graphene*, arXiv:2207.13767.

[WaLu21] A. Watson and M. Luskin, *Existence of the first magic angle for the chiral model of bilayer graphene*, J. Math. Phys. **62**, 091502 (2021).

[We07] R.O. Wells, *Differential Analysis on Complex Manifolds*, 3rd edition, Springer Verlag, (2007).

[Yan18] M. Yankowitz, *Tuning superconductivity in twisted bilayer graphene* , Science, Vol 363, Issue 6431 pp. 1059-1064, (2019).

[Zw24] M. Zworski, *Mathematical results on the chiral model of twisted bilayer graphene,* (with an appendix by Mengxuan Yang and Zhongkai Tao), J. Spectr. Theory, **14**(2024), 1063–1107.

Email address: simon.becker@math.ethz.ch

ETH Zurich, Institute for Mathematical Research, 8092 Zurich, CH.

Email address: tristan.humbert@ens.psl.eu

Department of Mathematics, University of California, Berkeley, CA 94720, USA.

Email address: zworski@math.berkeley.edu

Department of Mathematics, University of California, Berkeley, CA 94720, USA.