PSEUDOSPECTRAL EFFECTS AND BASINS OF ATTRACTION

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Abstract. Stability of equilibria of complex mechanical systems is often investigated using spectral properties of the linearization around the equilibrium. The linearized operator, A, is called *stable* if all of its eigenvalues have negative real parts. The simplest quantitative measure of stability is then the distance of the spectrum to the imaginary axis, $1/\mu(A)$. When $\mu(A)$ is large, that is when eigenvalues close to the real axis, we expect "less" stability.

We propose a different measure of instability of the linearization, which is *pseudospectral* rather than spectral in nature:

$$\psi(A) \stackrel{\text{def}}{=} \max_{\operatorname{Re} z > 0} 4 \operatorname{Re} z \|(A - zI)^{-1}\|^2 .$$

The more standard pseudospectral measure of instability $\|H(A)\|_\infty \stackrel{\text{def}}{=} \max_{y\in\mathbb{R}} \|(A - iyI)^{-1}\|$, satisfies $\|H(A)\|_\infty \leq \psi(A)$ but we can have families of matrices for which $\|H(A)\|_\infty/\psi(A) \to 0$. In all honesty, we have not been able to show any real advantages of $\psi(A)$ so perhaps it is just a red herring.

1. Introduction

The importance of resolvent estimates for evolution problems has a long tradition going back at least to Kreiss and his celebrated matrix theorem. More recently it was emphasized by Trefethen who introduced the concept of *pseudospectrum* (see (1.7) below) and investigated its importance in nonlinear problems of fluid mechanics. We refer to [3],[4] [9] for an introduction and pointers to the already substantial literature. In broad terms, while eigenvalues are part of the *lingua franca* of modern engineering, the naturally relevant concept of pseudospectra is neither well known nor understood.

In linear stability theory, we say that the equilibrium of

$$x'(t) = Ax(t) , \quad x(t) \in \mathbb{R}^n , \tag{1.1}$$

where A is an $n \times n$ real matrix, is *stable* if all the eigenvalues of A, λ, satisfy, $\operatorname{Re}\lambda < 0$. That is motivated by the basic estimate for solutions of the linear system (1.1) for A with simple eigenvalues:

$$\|x(t)\| \leq C(A) \exp(t \max_{\lambda\in\operatorname{Spec}(A)} \operatorname{Re}\lambda) , \tag{1.2}$$

with a small correction when multiple eigevalues occur (see (3.2)). The constant $C(A)$ may however be very large for matrices which are *non-normal*, that is matrices for which

$A^*A \neq AA^*$, where A^* is the adjoint of A. Nevertheless, it is useful to associate to A the following spectral measure of stability:

$$\mu(A) \stackrel{\text{def}}{=} \max_{\lambda \in \operatorname{Spec}(A)} (-\operatorname{Re}\lambda)_+^{-1}\,, \tag{1.3}$$

where $x_+ = x$ if $x \geq 0$ and 0 otherwise. We remark here that since we are dealing with finite matrices with spectra in the open left half plane the use of maximum rather than supremum, here and later, is justified.

If (1.1) is replaced by a nonlinear equation

$$x'(t) = F(x(t))x(t)\,, \quad F(0) = A\,, \quad x(0) = x_0\,, \tag{1.4}$$

then the standard nonlinear stability result (see §2) says that if $\mu(A) < \infty$ then there exist $\delta, \epsilon > 0$ such that

$$\|x_0\| < \delta \implies \|x(t)\| \leq e^{-\epsilon t}\,.$$

If A is normal then $\mu(A)$ provides a good measure of stability. The size of δ in the condition above can be directly related to μ and we can say that the smaller μ is the more stable the equilibrium $x = 0$ of the nonlinear system is – see the remark at the end of §2.

For an arbitrary A we propose a different constant which is pseudospectral rather than spectral in nature:

$$\psi(A) \stackrel{\text{def}}{=} \max_{\operatorname{Re} z > 0} 4\,(\operatorname{Re} z)\,\|(A - zI)^{-1}\|^2\,. \tag{1.5}$$

This new function of A is clearly harder to compute than the spectrally given $\mu(A)$ but it is accessible. Moreover, because $(A - zI)^{-1}$ can be identified with the Laplace transform of e^{At}, it is also amendable to analysis.

We note that by testing $(A - zI)^{-1}$ on an eigenvector of A,

$$\mu(A) \leq \psi(A)\,.$$

Also,

$$AA^* - A^*A = 0 \implies \mu(A) = \psi(A)\,, \tag{1.6}$$

see the remarks following Lemma 3. The opposite implication is not true in general[†] but for non-normal matrices we can easily have $\psi(A) \gg \mu(A)$ – see §3.

The parameter $\psi(A)$ can be directly related to Trefethen's ϵ-pseudospectrum, which is defined as follows

$$\begin{aligned} \Lambda_\epsilon(A) &= \{z \in \mathbb{C} \,:\, \|(A - zI)^{-1}\| > 1/\epsilon\} \\ &= \{\lambda \in \mathbb{C} \,:\, \exists\, \delta A\,,\ \|\delta A\| < \epsilon\,,\ \lambda \in \operatorname{Spec}(A + \delta A)\}\,, \end{aligned} \tag{1.7}$$

[†]Consider the the following non normal matrix, $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}$, for which, $\mu(A) = \psi(A) = 10$.

The second (equivalent) definition highlights the relation to spectral instability. In terms of $\Lambda_\epsilon(A)$ we have

$$\psi(A) = 4 \max_{\substack{z\in\Lambda_\epsilon(A)\\ \epsilon>0}} \frac{(\operatorname{Re} z)_+}{\epsilon^2} = 4 \max_{\epsilon>0} \frac{\sigma_\epsilon}{\epsilon^2} \,,$$

where σ_ϵ is the ϵ-pseudospectral abscissa of A ivestigated by Burke, Lewis, and Overton [2]. That paper provides an algorithm, based on [1], for computing σ_ϵ. we will use ideas presented there to compute $\psi(A)$.

We also remark that $\psi(A)$ is also closely related to the function $c(a)$ recently used by Davies [3] to estimate the size of the propagator $\exp(tA)$:

$$c(a) \stackrel{\text{def}}{=} \max_{b\in\mathbb{R}} a\|(A-(a+ib))^{-1}\| \,, \quad \mu(A) = 4 \max_{a>0} \frac{c(a)^2}{a} \,.$$

For simplicity we quote a related estimate due to Trefethen and Wright (see [3]):

$$\max_{0\leq s\leq t} \|e^{sA}\| \leq \max_{a>0} \frac{c(a)}{1+(1-e^{-at})c(a)} \,.$$

Finally we mention a relation of $\psi(A)$ to the well known H_∞ norm, $H(z) = (A - zI)^{-1}$,

$$\|H\|_\infty \stackrel{\text{def}}{=} \max_{y\in\mathbb{R}} \|(A-iyI)^{-1}\| \,.$$

The importance of this norm in the context of nonlinear evolution comes from the *small gain theorem* [10]. To state it we consider the following system:

$$x'(t) = Ax(t) + G(x(t)) \,, \quad \|G(x)\| \leq C_G\|x\| \,, \quad x(0) = x_0 \,.$$

Then for any $x_0 \in \mathbb{R}^n$,

$$K_G\|H\|_\infty < 1 \implies \|x(t)\| \longrightarrow 0 \,, \ t \to \infty \,. \tag{1.8}$$

As we will see in §3 and 4 $\|H\|_\infty$ seems to provide *a more accurate* estimate on the basin of attraction than $\psi(A)$. Not surprisingly we also have

$$\|H\|_\infty \leq \psi(A) \,, \tag{1.9}$$

see Lemma 5 below. However there is no constant depending on dimension only for which $C_n\|H\|_\infty \geq \psi(A)$ – see §3.

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2. Elementary nonlinear stability theory

Here is a simple version of the hyperbolic stability theorem with the proof based on standard Lyapunov theory.

We denote by $\langle \bullet, \bullet \rangle$, and $\| \bullet \|$, the usual inner product and norm on $\mathbb{R}^n$.

Lemma 1. *Suppose that all the eigenvalues of A, λ, satisfy* $\operatorname{Re} \lambda < 0$. *Then for* any *positive definite matrix Q, there exists a unique positive definite matrix P such that*

$$A^* P + PA = -Q \,. \tag{2.1}$$

Proof. We define

$$P \stackrel{\text{def}}{=} \int_0^\infty e^{A^* t} Q e^{At} dt \,. \tag{2.2}$$

Since the eigenvalues of A all satisfy $\operatorname{Re} \lambda < 0$ we have

$$\| e^{tA} \| \leq C e^{-ct} \,, \quad c > 0 \,,$$

and hence the integral defining P converges. Now,

$$\langle P e^{-sA} x, e^{-sA} x \rangle = \int_{-s}^\infty \| Q^{\frac12} e^{tA} x \|^2 dt = \langle Px, x \rangle + \int_{-s}^0 \| Q^{\frac12} e^{tA} x \|^2 dt \,,$$

and,

$$\frac{d}{ds} \langle P e^{-sA} x, e^{-sA} x \rangle = \frac{d}{ds} \int_{-s}^0 \| Q^{\frac12} e^{tA} x \|^2 dt \,.$$

Hence,

$$\frac{d}{ds} \int_{-s}^0 \| Q^{\frac12} e^{tA} x \|^2 dt |_{s=0} = - \langle (A^* P + PA) x, x \rangle \,.$$

On the other hand the left hand side is equal to $\langle Qx, x \rangle$, proving the lemma. □

In the notation of the lemma we define,

$$\gamma(A) \stackrel{\text{def}}{=} \min_Q \frac{1}{2 \|P\| \|Q^{-1}\|} \,, \tag{2.3}$$

where the maximum is taken over all quadratic form.

Theorem 2. *Consider the following differential equation:*

$$x' = A + F(x) \,, \quad x(0) = x_0 \,, \quad x \in \mathbb{R}^n \,, \tag{2.4}$$

where

$$\| F(x) \| \leq K_F(\delta) \|x\|^2 \,, \quad \textit{for } \|x\| \leq \delta. \tag{2.5}$$

Suppose that all the eigenvalues, λ, of the matrix A satisfy $\operatorname{Re}\lambda < 0$, *and let* $P = P(A, Q)$ *be as in Lemma 1. Then, using definition* (2.3),

$$\|x_0\| < \min\left(\delta, \gamma(A)/K_F(\delta)\right), \implies \exists\, \epsilon, \; \|x(t)\| \le \frac{\exp(-\epsilon t)}{\epsilon}. \tag{2.6}$$

Proof. If $x(t)$ solves (2.4) then

$$\begin{aligned}
\frac{d}{dt}\langle Px, x\rangle &= \langle Px', x\rangle + \langle Px, x'\rangle \\
&= \langle (A^*P + PA)x, x\rangle + 2\langle PF(x), x\rangle \\
&\le -\|Q^{-1}\|^{-1}\|x\|^2 + 2\langle PF(x), F(x)\rangle^{\frac{1}{2}}\langle Px, x\rangle^{\frac{1}{2}} \\
&\le -\|Q^{-1}\|^{-1}\|x\|^2 + 2\|P\|^{\frac{1}{2}}K_F\|x\|^2\langle Px, x\rangle^{\frac{1}{2}} \\
&\le -\|x\|^2(\|Q^{-1}\|^{-1} - 2\|P\|^{\frac{1}{2}}K_F\langle Px, x\rangle^{\frac{1}{2}}).
\end{aligned}$$

Hence if $\langle Px_0, x_0\rangle^{\frac{1}{2}} \le (2K_F\|P\|^{\frac{1}{2}}\|Q^{-1}\|)^{-1}$ then, $\langle Px, x\rangle$ is decreasing, and hence exponentially decreasing. Since

$$\langle Px_0, x_0\rangle^{\frac{1}{2}} \le \|P\|^{\frac{1}{2}}\|x_0\|,$$

condition (2.6) follows. □

To relate this to the resolvent we make the following observation

Lemma 3. *Let P and Q be given by Lemma 1. Then for any z with* $\operatorname{Re} z > 0$ *we have*

$$2\operatorname{Re} z\|(A - zI)^{-1}\|^2 \le \|P\|\|Q^{-1}\|.$$

Proof. We have

$$Q^{\frac{1}{2}}(A - zI)^{-1} = \int_0^\infty Q^{\frac{1}{2}}e^{At}e^{-zt}dt,$$

and , using the Cauchy-Schwartz inequality,

$$\begin{aligned}
\|Q^{\frac{1}{2}}(A - zI)^{-1}x\| &\le \int_0^\infty \|Q^{\frac{1}{2}}e^{At}x\|e^{-\operatorname{Re} zt}dt \\
&\le \left(\int_0^\infty \|Q^{\frac{1}{2}}e^{At}x\|^2 dt\right)^{\frac{1}{2}}\left(\int_0^\infty e^{-2\operatorname{Re} zt}dt\right)^{\frac{1}{2}} = (2\operatorname{Re} z)^{-\frac{1}{2}}\langle Px, x\rangle^{\frac{1}{2}}.
\end{aligned}$$

Since

$$\|(A - zI)^{-1}\| \le \|Q^{-1}\|^{\frac{1}{2}}\|Q^{\frac{1}{2}}(A - zI)^{-1}\|,$$

the lemma follows. □

If $z = -\bar{\lambda}_0$ and $Q = I$, where $Ax_0 = \lambda_0 x_0$, then the estimate in the previous lemma becomes

$$\sqrt{2\operatorname{Re} z}\|(A - zI)^{-1}x_0\| = \langle Px_0, x_0\rangle,$$

showing that it is optimal. We also note that

$$\max_{\operatorname{Re} z>0} 4 \operatorname{Re} z |z-\lambda_0|^{-2} = \max_{t>0,\ s\in\mathbb{R}} \frac{4t}{(t-\operatorname{Re}\lambda_0)^2+(s-\operatorname{Im}\lambda_0)^2}$$
$$= \begin{cases} 1/|\operatorname{Re}\lambda_0| & \operatorname{Re}\lambda_0<0 \\ \infty & \operatorname{Re}\lambda_0 \geq 0 \end{cases},$$

that is

$$\max_{\operatorname{Re} z>0} 4 \operatorname{Re} z |z-\lambda_0|^{-2} = \frac{1}{(-\operatorname{Re}\lambda_0)_+}.$$

Hence, in the notation of §1, we have $\psi(A) = \mu(A)$ if A is normal as then we can choose an orthonormal basis of eigenvectors.

We note that the constant in the estimate of the *basin of attraction* satisfies

$$\frac{1}{2}\|P\|^{-1}\|Q^{-1}\|^{-1}K_F^{-1}(\delta) \leq (\psi(A)K_F(\delta))^{-1}, \tag{2.7}$$

and hence

$$\min(\delta, 1/(\psi(A)K_F(\delta)),$$

can be used as indication of its smallness. If the resolvent is large in the right half plane, this constant is very small. Clearly, this does not necessarily provide a sharp estimate on the size of the basis of attraction but nevertheless it provides a useful guide to its size and motivates definition (1.5). The larger $\psi(A)$ is the smaller the basin of attraction *might* be.

In the case of normal operators it is easy to see that for some choices of non-linearities the estimates (2.7) are optimal. We can see this in a simple nonlinear one dimensional problem: $x' = (a+fx)x$, $a<0$, $f>0$. Then $\mu=\psi=1/|a|$ and for $x(0)>1/(f\mu)$ the solution blows up in finite time.

Better estimates on the size of x_0 in Theorem 2 can be obtained using the *Bounded Real Lemma* [10] – see Lemma A.2 below. More specifically we have the *small gain theorem:*

Theorem 4. *Suppose that the assumptions of Theorem 2 hold and that*

$$1/\gamma > \|H(A)\|_\infty \stackrel{\text{def}}{=} \max_{y\in\mathbb{R}} \|(A-iy)^{-1}\|.$$

Let P be a positive definite solution of $PA + A^*P + P^2 + \gamma^2 I = 0$ guaranteed by Lemma A.2. Then

$$\|x_0\| < \min\left(\delta, k(P)^{-\frac{1}{2}}\gamma/K_F(\delta)\right), \implies \exists\, \epsilon,\ \|x(t)\| \leq \frac{\exp(-\epsilon t)}{\epsilon}, \tag{2.8}$$

where $k(P) \stackrel{\text{def}}{=} \|P\|\|P^{-1}\|$ is the condition number of P.

Proof. We follow the same procedure as in the proof of Theorem 2:

$$\begin{aligned}\frac{d}{dt}\langle Px, x\rangle &= \langle Px', x\rangle + \langle Px, x'\rangle \\ &= \langle (A^*P + PA)x, x\rangle + 2\langle PF(x), x\rangle \\ &\leq \langle (A^*P + PA + P^2)x, x\rangle + \|F(x)\|^2 \\ &\leq -\gamma^2\|x\|^2(1 - \gamma^{-2}K_F^2\|x\|^2) \\ &\leq -\gamma^2\|x\|^2(1 - \gamma^{-2}\|P^{-1}\|\|x\|_P^2K_F^2)\,.\end{aligned}$$

Hence if $\|x_0\|_P \leq \|P^{-1}\|^{\frac{1}{2}}\gamma/\gamma$ then, $\|x\|_P$ is decreasing, and hence exponentially decreasing. Since $\|x_0\|_P \leq \|P\|^{\frac{1}{2}}\|x_0\|$, (2.8) follows. □

We observe that the estimate of the basin of attraction given in Theorem 4 satisfies

$$\|x_0\| < \max_{\delta>0}\left(\delta, k(P)^{-\frac{1}{2}}\|H(A)\|_\infty^{-1}K_F(\delta)^{-1}\right) \leq \max_{\delta>0}\left(\delta, \|H(A)\|_\infty^{-1}K_F(\delta)^{-1}\right)\,. \tag{2.9}$$

For completeness we include the proof of the relation between $\|H\|_\infty$ and ψ:

Lemma 5. *For any matrix A we have*

$$\max_{y\in\mathbb{R}}\|(A - iyI)^{-1}\| \leq \psi(A)\,.$$

Proof. If we write $A(y) = A - iyI$ then the statement will follow from showing that

$$\|A(y)^{-1}\| \leq \max_{x>0} 4x\|(A(y) - xI)^{-1}\|\,.$$

Hence if for any invertible matrix B we show that

$$\|B^{-1}\| \leq \max_{x>0} 4x\|(B - xI)^{-1}\|\,,$$

we will prove the lemma. By simple scaling $x \mapsto x/\|B^{-1}\|$, we can assume that $\|B^{-1}\| = 1$, and that means that we have to show that

$$\min_{x>0}\frac{\sigma_{\min}(B - xI)}{2\sqrt{x}} \leq 1\,,$$

where $\sigma_{\min}(C) = \min_{\|x\|=1}\|Cx\|$ is the minimal singular value of C. If $u \in \mathbb{R}^n$ satisfies

$$\sigma_{\min}(B) = \|Bu\| = \|u\| = 1\,,$$

then

$$\begin{aligned}\min_{r>0}\sigma_{\min}\left(\frac{B}{2r} - \frac{r}{2}I\right) &= \min_{r>0}\min_{\|x\|=1}\left\|\left(\frac{B}{2r} - \frac{r}{2}I\right)x\right\| \\ &\leq \min_{r>0}\frac{\|Bu\|}{2r} + \frac{r\|u\|}{2} = \min_{r>0}\left(\frac{1}{2r} + \frac{r}{2}\right) = 1\,,\end{aligned}$$

which proves the lemma. □

3. A simple example

Here is a simple example which illustrates the pseudospectral instability for non-linear systems.

Let $0 < \gamma < 1$ and consider the following system of equations:

$$y'(t) = -\gamma y(t) + J_n y(t) + F(y(t)) \,, \quad y(0) = y_0 \,, \quad y \in \mathbb{R}^n \,, \tag{3.1}$$

where J_n is the Jordan matrix:

$$J_n = (J_{ij})_{1\leq i,j\leq n} \,, \quad J_{ij} = \begin{cases} 1 & \text{for } k = j+1 \\ 0 & \text{otherwise.} \end{cases} \,,$$

and

$$F\left([y_1, \cdots, y_n]^t\right) = [y_1^2, 0, \cdots, 0]^t \,.$$

This means that we are considering a system of Riccati equations.

The linearization of this system is given by

$$y'(t) = A_n y(t) \,, \quad A_n = -\gamma I + J_n \,,$$

which is *linearly stable*. It is easy to solve:

$$y_j(t) = \sum_{k=j}^{n} y_k(0) t^{k-j} e^{-\gamma t} \,. \tag{3.2}$$

The nilpotent structure produces a "hump" before the eigenvalues take over. That hump can be understood in terms of the resolvent of A_n which is large in $D(-\gamma, 1)$:

$$\|(A_n - \lambda)^{-1}\| \simeq |\lambda + \gamma|^{-n} \,, \tag{3.3}$$

see [8, §2.2] for the discussion of the Jordan block and [3] for general bounds on $\exp(tA)$. Here we indicate directly the reason for (3.2) and (3.3). Since $J_n^n = 0$ we have

$$(z - J_n)^{-1} = \frac{1}{z}(I - J_n/z)^{-1} = \sum_{k=1}^{n} \frac{J_n^{k-1}}{z^k} \,.$$

If we denote $\{e_j\}_{j=1}^n$ the standard basis of $\mathbb{R}^n$, then

$$J_n^{k-1} e_j = \begin{cases} e_{j-k+1} & j \leq k \\ 0 & j > k \end{cases}$$

and, for $|z| \leq 1$,

$$\|(z - J_n)^{-1} e_j\|^2 = \left\| \sum_{k=1}^{j} z^{-k} e_{j-k+1} \right\|^2 = \sum_{k=1}^{j} |z|^{-2k} \leq j|z|^{-2j} \,.$$

Hence,

$$\|(z-J_n)^{-1}x\| \le \|x\| \left(\sum_{j=1}^{n} j|z|^{-2j}\right)^{\frac{1}{2}} \le n|z|^{-n}\|x\|\,, \quad |z| \le 1\,.$$

By putting $x = e_n$ we obtain, for all $z \in \mathbb{C}$,

$$|z|^{-n} \le \|(z-J_n)^{-1}\| \le n \max(1, |z|^{-n})\,.$$

For $A_n = J_n - \gamma$, $0 < \gamma < 1$, we obtain

$$\begin{aligned} \psi(A_n) &= \max_{x>0} 4x\|(J_n - (\gamma + x))^{-1}\|^2 \ge \max_{x>0} 4x(x+\gamma)^{-2n} \\ &= 4\frac{\gamma^{-2n+1}}{2n-1}\left(1 + \frac{1}{2n-1}\right)^{-2n} > \frac{2\gamma^{-2n+1}}{ne}\,. \end{aligned}$$

On the other hand

$$\|H(A_n)\|_\infty = \max_{y\in\mathbb{R}} \|(A_n - iy)^{-1}\| \le n \max_{y\in\mathbb{R}} \max(|\gamma + iy|^{-n}, 1) = n\gamma^{-n}\,.$$

Clearly, $\mu(A_n) = 1/\gamma$, and for $n > 1$,

$$\mu(A_n) \ll \|H(A_n)\|_\infty \ll \psi(A_n)\,, \quad \gamma \to 0\,.$$

In particular, if we put

$$B_n(\gamma) \stackrel{\text{def}}{=} \frac{2\gamma^{-2n+1}}{ne}(J_n - \gamma)\,,$$

then

$$1 \le \psi(B_n(\gamma) \le C_n\,, \quad \|H(B_n(\gamma))\|_\infty \longrightarrow 0\,, \quad \gamma \to 0\,.$$

The non-linear term in (3.1) amplifies the effect of the hump. We first need the following

Lemma 6. *Suppose that $\beta_j(t)$ and $\alpha_j(t)$ are piecewise continuous bounded functions. If*

$$w_j'(t) = \beta_j(t)w_j(t) + \alpha_j(t) + w_j(t)^2\,, \quad j = 1,2\,, \quad w_1(0) \le w_2(0)\,,$$

(3.4)
$$\beta_1(t) \le \beta_2(t)\,, \quad \alpha_1(t)\exp(\int_0^t \beta_2(s)ds) \le \alpha_2(t)\exp(\int_0^t \beta_1(s)ds)\,, \quad 0 \le t \le T\,,$$

then, provided the solutions exist,

(3.5)
$$w_1(t) \le w_2(t)\,, \quad 0 \le t \le T\,.$$

In particular the conclusion holds if $\beta_1(t) = \beta_2(t)$ and $\alpha_1(t) \le \alpha_2(t)$.

Proof. We put $B_j(t) = \int_0^t \beta_j(s)ds$ so that $v_j \stackrel{\text{def}}{=} w_j e^{-B_j}$ satisfy $v_j' = \alpha_j e^{-B_j} + e^{B_j}v_j^2$. Then

$$\begin{aligned} (v_1^{-1} - v_2^{-1})' &= -\alpha_1 e^{-B_1}v_1^{-2} + \alpha_2 e^{-B_2}v_2^{-2} - e^{B_1} + e^{B_2} = F(v_1^{-1} - v_2^{-1}) + G\,, \\ G &\stackrel{\text{def}}{=} e^{B_2} - e^{B_1} + (\alpha_2 e^{-B_2} - \alpha_1 e^{-B_1})v_2^{-2} \ge 0\,, \quad F \stackrel{\text{def}}{=} (v_1^{-1} + v_2^{-1})\alpha_1 e^{-B_1}\,. \end{aligned}$$

Since $v_1^{-1}(0) - v_2^{-1}(0) \ge 0$ it follows that $v_1^{-1}(t) - v_2^{-1}(t) \ge 0$ for $0 \le t \le T$, that is $w_1(t) \le e^{B_2 - B_1}w_1(t) \le w_2(t)$, concluding the proof. ☐

Let us now consider (3.1) with the initial condition given by

$$y_j(0) = 0\,,\ 1 \le j < n\,,\quad y_n(0) = \epsilon > 0\,,$$

so that

$$y_j(t) = \epsilon t^{n-j} e^{-\gamma t}\,,\quad 1 < j \le n\,,$$
$$y_1'(t) = -\gamma y_1(t) + y_1(t)^2 + \epsilon t^{n-1} e^{-\gamma t}\,,\quad y_1(0) = 0\,.$$

We can now use Lemma 6 with $w_2 = y_1$, $w_1(0) = w_2(0) = 0$, $\beta_1 = \beta_2 = -\gamma$, $\alpha_2(t) = \epsilon t^{n-1} e^{-\gamma t}$, and

$$\alpha_1(t) = \begin{cases} 0 & t \le T_0 \\ \epsilon T_0^{n-1} e^{-\gamma T} & t > T_0\,, \end{cases}$$

where T_0 and T will be chosen. The equation

$$w' = -2bw + a^2 + w^2\,,\quad w(0) = 0\,,\quad a^2 - b^2 > 0\,,$$

has an explicit solution,

$$w(t) = \frac{a^2 \sin((a^2-b^2)^{\frac12} t)}{(a^2-b^2)^{\frac12}\cos((a^2-b^2)^{\frac12} t) - b\sin((a^2-b^2)^{\frac12} t)}\,,$$

which blows up at $0 < t = \tan^{-1}((a^2-b^2)^{\frac12}/b)/(a^2-b^2)^{\frac12} < \pi/(2(a^2-b^2)^{\frac12})$. Hence using Lemma 6 the blow up of y_1 is guaranteed if the blow up of w_1 occurs at time less than T. That means that the blow up of w above, with $b = \gamma/2$ and $a^2 = \alpha_2(t)$, $t > T_0$, occurs in time less than $T - T_0$. That follows from

$$4\epsilon T_0^{n-1} e^{-\gamma T} - \gamma^2 > (T - T_0)^{-2}\pi^2\,,$$

or

$$\epsilon > (\gamma^2 + (T-T_0)^{-2}\pi^2) T_0^{1-n} e^{\gamma T}\,.$$

Hence the solution blows up for

$$\epsilon > C_n \gamma^{n+1}\,,\quad C_n \stackrel{\text{def}}{=} \min_{0<\rho_0<\rho} (1 + (\rho-\rho_0)^{-2}\pi^2)\rho_0^{-n+1} e^{\rho}\,.$$

A similar argument shows that for $\epsilon < c_n \gamma^{n+1}$ the solution decays exponetially.

That indicates that as $\gamma \to 0$ (2.7) (that is $\psi(A)$) gives the correct order of magnitude of the basin of attraction only for $n = 2$. In higher dimension the estimate (2.9) (that is $\|H(A)\|_\infty$) is much better.

4. Numerical examples

In this section we present numerical results for the following class of non-linear equations:

$$\dot{x}(t) = Ax(t) + F(x(t))x(t)\,,\quad x(0) = x_0\,, \tag{4.1}$$

where $F(x) \in M_{nn}(\mathbb{R})$ is *linear* in $x \in \mathbb{R}^n$.

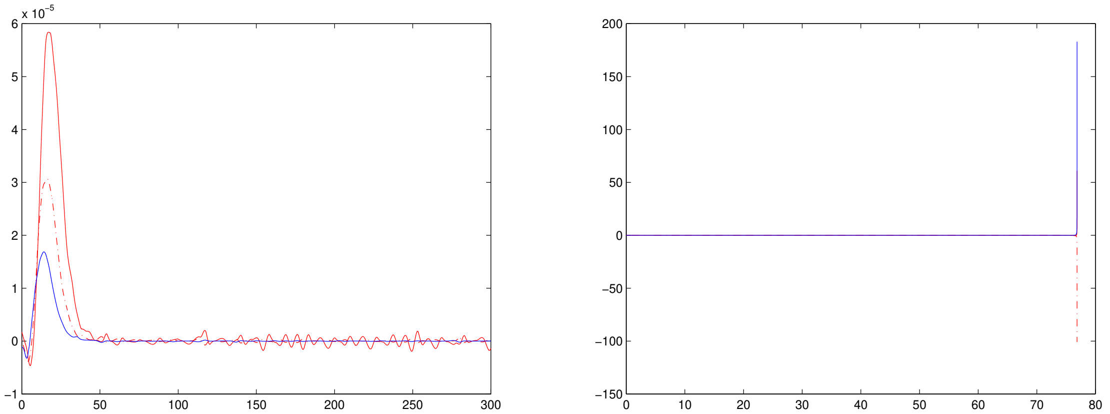

FIGURE 1. Plots of solutions for $A = J_{10}$ and $r = 1$ on the left, and $r = 5$ on the right. We have $\psi(A) = 7.59 \times 10^4$, $\|H(A)\|_\infty = 1.36 \times 10^3$.

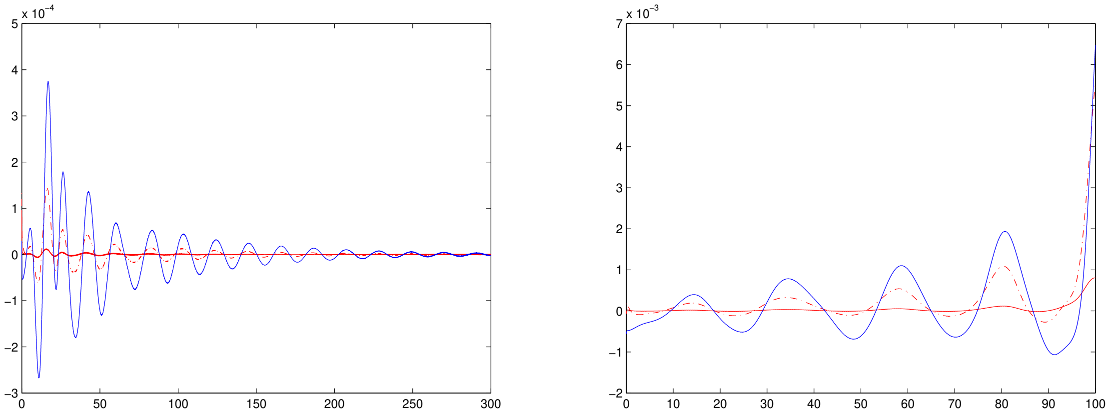

FIGURE 2. Plots of solutions for $A \in M_{20,20}$ equal to the discretization of the Orr-Sommerfeld operator with $R = 2500$, and $r = 120$ on the left, and $r = 200$ on the right. Here $\psi(A) = 3.73 \times 10^4$ and $\|H(A)\|_\infty = 1.76 \times 10^3$.

We to compare the size of the basin of attraction around the trivial solution solution with $x_0 = 0$ with the upper bound using $\psi(A)$. In §2 we saw that

$$\min(\delta, 1/(\psi(A)K_F(\delta))\,,$$

where $\|F(x)\| \leq K_F(\delta)\|x\|^2$, for $\|x\| \leq \delta$. For a quadratic perturbation, we have :

$$K_F(\delta) = K_F(1)$$

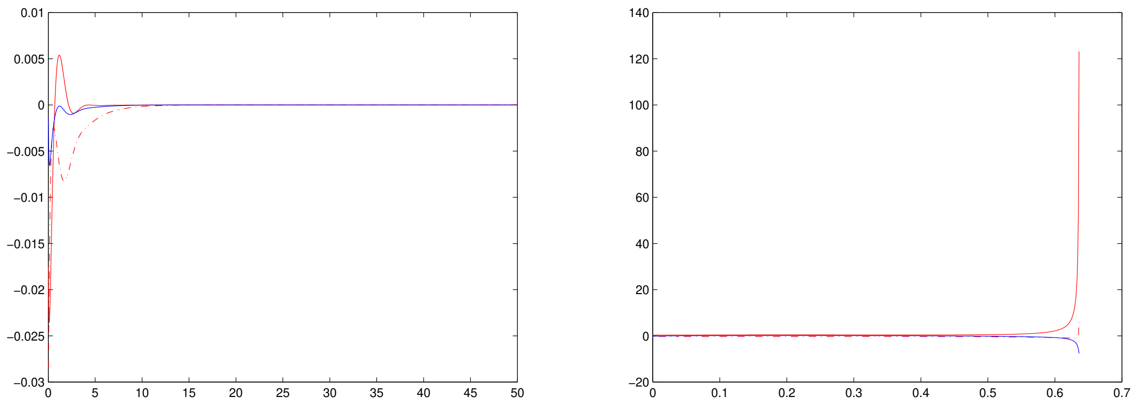

FIGURE 3. Plots of solutions for $A = C - B^*B \in M_{10,10}$ where C is a skew Hermitian random matrix (GOE), and B is random matrix with i.i.d. normal entries; $\psi(A) = 2.66$, $\|H(A)\|_\infty = 2.46$; $r = 1$, $K_F = 3.79$ on the left, and $r = 12$, $K_F = 3.47$ on the right.

Hence the upper bounds on the rough estimates of Theorems 2 and 4, (2.7) and (2.9), respectively, are

$$(\psi(A)K_F(1))^{-1}, \quad (\|H(A)\|_\infty K_F(1))^{-1}.$$

Our aim is to compare this estimation with what we find with the experiments, to see if this size is sharp or not.

First we need to compute $K_F(1)$. We generate a large sample of random normal vectors of size N, for which the norm is scaled to 1. Then we compute $\|F(x)x\|$ for these vectors, and we take the maximum. We need to take a large sample to evaluate this $K_F(1)$, but as we only do it once for each quadratic perturbation, the computation time will not be critical.

We consider now a random quadratic perturbation:

$$F(x)x = \langle Bx, x\rangle \tag{4.2}$$

with $B = \texttt{randn}(n, n, n)$ random normal tensor of size n^3. Once we have computed $K_F(1)$ and $\psi(A)$, we do the following experiment: for each different random quadratic perturbations, we take random initial conditions inside($r < 1$) or outside($r > 1$) the supposed radius $\|x_0\| = r/(\psi(A)K_F(1))$ and we observe the evolution of the system to see if it is stable or not. To study the evolution of the system with respect to the time, we use the solver of MATLAB called `ode23`. If we observe that the system is unstable, we can conclude that we are outside the radius of the basin of attraction. If we observe that it is stable for a large sample of experiments, we cannot conclude anything but we will say that experimentally, the system remains stable inside this radius.

We obviously cannot show all the experiments we did, so we decide to show figures of instability if it happens for some random choice of random quadratic perturbation or of initial conditions, and if we did not see any case of instability, we will show figures of stability.
We have done these experiments for different kind of matrices A:

- First we take random matrices of the form:

$$A = C - C^* - BB^*\,, \quad C = \texttt{randn}(n)\,, \quad B = \texttt{randn}(n)\,. \tag{4.3}$$

  These matrices have eigenvalues with negative real part, which we need to compute $\Psi(A)$. We take $n = 10$ and we observe the evolution of the system for different values of r. These matrices are weakly non normal, which means that even if $\Psi(A) > \mu(A)$, the ratio $\Psi(A)/\mu(A)$ is experimentally always smaller than 10.
  For these matrices, the estimation of the basin of attraction we made seems good, because for all the experiments we made, the system was stable inside this radius and unstable outside.
- Let us consider now the case of the jordan blocks already studied:

$$A_n = -0.5 * I + J_n$$

  . For this family of matrices, $\Psi(A_n)$ is big and the pseudospectral effects are strong. We also take $n = 10$. For this matrix, $\mu(A_10) = 2$ and $\Psi(A_10) \simeq 7.5910^4$ we observe the following behaviour:
  Here the behaviour is really different: the solution blows up in finite time inside the radius $1/\psi(A)K_F(1)$, which means that the estimation is not sharp for this family of matrices.
- Now we want to take for $A_n(R)$ the discretization of the Orr-Sommerfeld operator. It appears in the linearization of a planar fluid flow and is available at `http://web.comlab.ox.ac.uk/projects/pseudospectra` – see [4]. The parameter R represents the Reynolds number, and we will take $R = 2500$ in all the following experiments. The dimension of the system is here $n = 20$. For this matrix, $\mu(A_20(2500)) \simeq 69$ and $\Psi(A_20(2500)) \simeq 3.7410^4$
  What we conclude about these experiments is that the basin of attraction is obviously smaller than $1/\psi(A)K_F(1)$ for a quadratically random perturbation. This estimation seems more accurate for weakly non normal matrices (as the first example seems to indicate), whereas it is much looser for strongly non normal matrices.

Appendix A: Algebraic Riccati equations and quadratic forms on symplectic vector spaces.

In this appendix we present a self-contained mathematical discussion of the *algebraic Riccati equation*. The connection between the solutions of the algebraic Riccati equation and Lagrangian subspaces is well known (see [6] and references given there) but our presentation is related to the classical work on the symplectic classification of quadratic forms – see [7, §21.5] and references given there.

The algebraic Riccati equation is the following matrix equation for $P = P^*$,

$$A^*P + PA + PCP + D = 0\,, \quad A, D, C \in M_{nn}\,,\ D = D^*\,,\ C = C^*\,. \tag{A.1}$$

It appears naturally in the construction of more general Lyapunov norms $\langle Px, x\rangle$, as we have seen in a special case of $D = Q \gg 0$, $C = 0$, in Lemma 1. In Theorem 4 we used a positive definite solution in the case where $C = D = I$, and $\operatorname{Spec}(A) \subset \{\operatorname{Re} z < 0\}$. Its existence will be guaranteed by Lemma A.2 below.

Let V be a real vector space (which we can think of as $\mathbb{R}^n$), and V' its dual. Then $W = V \otimes V'$ is a symplectic vector space with the symplectic quadratic form defined as

$$\sigma((x, \xi), (y, \eta)) = \langle \xi, y\rangle - \langle \eta, x\rangle\,. \tag{A.2}$$

Its basic properties are

$$\sigma(X, Y) = -\sigma(Y, X)\,, \quad \forall\, Y \in W\,,\ \sigma(X, Y) = 0\ \Rightarrow X = 0\,,$$

and in fact any quadratic form on an even dimensional space W can be put in the form (A.2).

Now let $Q(X, Y)$ be a symmetric quadratic form on W obtained by polarization of $p(X) = Q(X, X)$,

$$\begin{gathered} p(x, \xi) = \langle Dx, x\rangle + 2\langle Ax, \xi\rangle + \langle C\xi, \xi\rangle\,, \\ Q((x, \xi), (y, \eta)) = \langle Dx, y\rangle + \langle Ax, \eta\rangle + \langle Ay, \xi\rangle + \langle C\xi, \eta\rangle\,. \end{gathered} \tag{A.3}$$

The Hamiltonian matrix of p is defined as

$$F = \frac{1}{2}\begin{pmatrix} p''_{\xi x} & p''_{\xi\xi} \\ -p''_{xx} & -p''_{x\xi} \end{pmatrix} = \begin{pmatrix} A & C \\ -D & -A^* \end{pmatrix},$$

but its invariant definition comes from using the symplectic form:

$$\sigma(X, FY) = Q(X, Y)\,, \quad X, Y \in W\,. \tag{A.4}$$

The matrix F is in the symplectic Lie algebra, in the sense that $\sigma(X, FY) + \sigma(FX, Y) = 0$ which is clear from the symmetry to Q and the antisymmetry of σ.

We denote $V_\lambda \subset W_{\mathbb{C}}$ the space of generalized eigenvectors of F belonging to the eigenvalue λ. Here $W_{\mathbb{C}}$ denotes the complexification of W (which can be thought of as $\mathbb{C}^{2n}$). The symplectic form, σ, becomes a non-degenerate antisymmetric form on $W_{\mathbb{C}}$, which can be defined using (A.2), with the understanding that $\langle \bullet, \bullet\rangle$ is a *complex* bilinear form. The basic fact about V_λ's is

Lemma A.1. *We have*

$$\lambda + \mu \neq 0 \implies \sigma(V_\lambda, V_\mu) = 0\,, \tag{A.5}$$

and V_λ is the dual space of $V_{-\lambda}$,

$$V_\lambda \overset{\alpha}{\simeq} V'_{-\lambda}\,, \quad \alpha(X)(Y) = \sigma(X,Y)\,, \quad X \in V_\lambda\,, \quad Y \in V_{-\lambda}\,. \tag{A.6}$$

Consequently, V_0, $V_\lambda \oplus V_{-\lambda}$ are symplectic vector spaces.

Proof. In fact, for $\mu \neq -\lambda$, $(F+\mu)V_\lambda = V_\lambda$, and

$$\sigma(V_\lambda, V_\mu) = \sigma((F+\mu)^N V_\lambda, V_\mu) = \sigma(V_\lambda, (-F+\mu)^N V_\mu) = 0\,,$$

if N is large enough. The rest of the lemma follows from this. □

In particular, the eigenvalues of F have the property the

$$\lambda \in \mathrm{Spec}(F) \implies \bar\lambda, -\lambda, -\bar\lambda \in \mathrm{Spec}(F)\,.$$

We recall that subspace $\Lambda \subset W$ is called Lagrangian if σ vanishes identically on Λ and $\dim \Lambda = \dim W/2$. In other words Λ is Lagrangian if and only if

$$\Lambda = \Lambda^\sigma \overset{\text{def}}{=} \{X \;:\; \forall\, Y \in \Lambda\,,\ \sigma(X,Y) = 0\}\,.$$

A generalized geometric solution of (A.1) is a *Lagrangian* subspace of W on which Q vanishes identically. In view of (A.4) that is equivalent to asking for a Lagrangian plane which is invariant under F.

To explain the connection with (A.1) we proceed as follows. Let $\pi : W = V \otimes V' \to V$ be the natural projection. If $\pi : \Lambda \to V$ is a bijection then

$$\Lambda = \{(x, Px) \;:\; x \in V\}\,, \quad P = P^*\,.$$

In fact a graph of P is Lagrangian if and only if $P = P^*$. Suppose that Q vanishes identically on Λ, that is Λ is invariant under F. Then, using (A.3),

$$\begin{aligned} 0 = Q((x,Px),(y,Py)) &= \langle Dx, y\rangle + \langle Ax, Py\rangle + \langle Ay, Px\rangle + \langle CPx, Py\rangle \\ &= \langle (PA + A^*P + D + PCP)x, y\rangle\,, \end{aligned}$$

for all $x, y \in V$, which means that (A.1) hold.

The generalized problem is solved in the complexified setting using Lemma A.1: a Lagrangian subspace, $\widetilde\Lambda \subset W_{\mathbb{C}}$, is invariant under F if and only if

$$\widetilde\Lambda = \Lambda_0 \oplus \bigoplus_{\lambda \in \Sigma} \Lambda_\lambda\,, \quad F\Lambda_\lambda \subset \Lambda_\lambda \subset V_\lambda \oplus V_{-\lambda}\,, \tag{A.7}$$

where Λ_λ is a Lagrangian subspace of the symplectic space $V_\lambda \oplus V_{-\lambda}$, and Σ is any set satisfying $\Sigma \cup (-\Sigma) = \mathrm{Spec}(F) \setminus \{0\}$ and $\Sigma \cap (-\Sigma) = \emptyset$.

Rather than analyze the general problem in the real case we will consider a special case relevant to this paper. Hence, suppose that

$$\mathrm{Spec}(F) \cap i\mathbb{R} = \emptyset\,. \tag{A.8}$$

In that case we have two natural solutions to the generalized real problem:

$$\Lambda_\pm \stackrel{\text{def}}{=} \bigoplus_{\pm \operatorname{Re}\lambda > 0} \left((V_\lambda + V_{\bar\lambda}) \cap W \right) . \tag{A.9}$$

When λ is real, V_λ is spanned (over $\mathbb{C}$) by real generalized eigenvectors which also provide a basis for $V_\lambda \cap W$.

When λ is complex, the components in (A.9), $W_\lambda \stackrel{\text{def}}{=} (V_\lambda + V_{\bar\lambda}) \cap W$, are spanned (over $\mathbb{R}$) by $X + \overline{X}$ and $(X - \overline{X})/i$. Hence W_λ is invariant under F, and by Lemma A.1 σ vanishes on W_λ. Its real dimension is given by the complex dimension of $V_\lambda \oplus V_{\bar\lambda}$, and consequenty the dimension of $\Lambda_\pm$ is equal to $\dim V$.

Thus $\Lambda_\pm$ gives a Lagrangian subspace of W on which Q vanishes identically. To obtain a solution of (A.1) we need conditions which guarantee that $\pi : \Lambda_\pm \to V$ is surjective, as that will imply that $\Lambda_\pm = \{(x, P_\pm x) \; : \; x \in V\}$, $P_\pm = P_\pm^*$, and that $P_\pm$ solves (A.1).

We now claim that, for F satisfying (A.8),

$$C \gg 0 \implies \pi\restriction_{\Lambda_\pm} \text{ is injective.} \tag{A.10}$$

In fact, $\Lambda_\pm$ is invariant under F and $\sigma\restriction_{\Lambda_\pm} = 0$. Hence $p\restriction_{\Lambda_\pm} = 0$. But, since $C \geq 0$,

$$0 = p(0, \xi) = \langle C\xi, \xi \rangle \implies \xi = 0 \,,$$

and $\ker \pi\restriction_{\Lambda_\pm} = \{(0,0)\}$.

To relate this approach to Lemma 1 we note that if $\operatorname{Spec}(A) \subset \{\operatorname{Re} z < 0\}$ and $C = 0$ or $D = 0$, then (A.8) is automatically satisfied. Also, $(F - \lambda)^N((0,\xi)) = 0$ is easily seen to imply that $\xi = 0$. Thus,

$$\operatorname{Spec}(A) \subset \{\operatorname{Re} z < 0\} \,, \quad C = 0 \text{ or } D = 0 \implies \pi\restriction_{\Lambda_\pm} \text{ is injective.} \tag{A.11}$$

Hence under conditions (A.8) and (A.10) or under the condition (A.11) we can find solutions to (A.1).

As a special application we can now prove the result need for the proof of Theorem 4:

Lemma A.2. *Suppose that the spectrum of A lies in $\{\operatorname{Re} z < 0\}$. Then*

$$\exists \, P \gg 0 \,, \quad PA + A^*P + P^2 + \gamma^2 I = 0 \iff \gamma \|H(A)\|_\infty < 1 \,. \tag{A.12}$$

Proof. We will only prove the sufficiency of the condition on $\|H(A)\|_\infty$. We first observe that $\|H(A/\gamma)\|_\infty = \gamma \|H(A)\|_\infty$ and that

$$(P/\gamma)(A/\gamma) + (A/\gamma)^* + (P/\gamma)^2 + I = \gamma^{-2}(PA + A^*P + P^2 + I) \,.$$

Hence it is enough to prove the lemma with $\gamma = 1$ and $\|H(A)\|_\infty < 1$. To guarantee the existence of P solving (A.12) we need, in view of (A.8) (the condition (A.10) is clearly satisfied), show that

$$\operatorname{Spec} \begin{pmatrix} A & I \\ -I & -A^* \end{pmatrix} \cap i\mathbb{R} = \emptyset \,.$$

Now,

$$\det \begin{pmatrix} A - iy & I \\ -I & -(A-iy)^* \end{pmatrix} = \det(I - (A-iy)(A-iy)^*) \geq |\|(A-iy)^{-1}\|^{-2} - 1|^n \,.$$

Hence, if $\|(A-iy)^{-1}\| < 1$ for all $y \in \mathbb{R}$ then the determinant above is never 0 and (A.8) holds. We now take $P = P+$ where $\Lambda_\pm = \{(x, P_\pm x)\}$.

It remains to show that P is positive definite. For that we can deform A to $A_0 = -2I$, say, so that

$$\operatorname{Spec} \begin{pmatrix} A_t & I \\ -\gamma_t^2 & -A_t^* \end{pmatrix} \cap i\mathbb{R} = \emptyset \,, \quad A_0 = -2I \,, \quad A_1 = A \,, \quad \gamma_0 = \gamma_1 = 1 \,, \quad \gamma_t \neq 0 \,.$$

This is always possible by the argument above if $\|(A_t - iy)^{-1}\|$ stays bounded for y real. By the construction of $\Lambda_\pm$ we obtain a continuous family of *invertible* self-adjoint operators P_t, and an explicit calculation shows that P_0 is positive definite. Hence $P = P_1$ is also positive definite (the set of positive definite operators is open and closed in the set of invertible operators). □

Appendix B: Computing $\psi(A)$

In this appendix we show how readily available algorithms give a way to compute $\psi(A)$.

The ϵ-pseudospectrum of a matrix A is described the smallest singular value of $A - zI$. Let $\sigma_{\min}(A)$ be the smallest singular value of the matrix A. Then,

$$\sigma_{\min}(A - zI) = \|(A - zI)^{-1}\|^{-1}$$

Hence,

$$\psi(A) = 4 \left[\min_{x>0,y} \frac{\sigma^2_{\min}(A - (x+iy)I)}{x} \right]^{-1}$$

Based on ideas of [1], Burke-Lewis-Overton [2] provide a code for computing the inverse of the H_∞ norm of A, iterpreted as the *continuous distance to instability* (cdi):

$$\operatorname{cdi}(A) \stackrel{\text{def}}{=} \min_y \left[\sigma_{\min}(A - xI - iyI)\right] \,.$$

The expression for $\psi(A)$ becomes

$$\psi(A) = 4 \left[\min_{x>0} \frac{\operatorname{cdi}(A - x)}{x} \right]^{-1} \,. \tag{B.1}$$

The algorithm for computing $\operatorname{cdi}(A - x)$ [1] is based on the following result:

> The matrix $B - iyI$ has α as one of its singular values if and only if the Hamiltonian matrix $H(\alpha)$ has the imaginary eigenvalue iy, where:
>
> $$H(\alpha) = \begin{pmatrix} B & -\alpha I \\ \alpha I & -B^* \end{pmatrix}$$

Therefore the problem is an eigenvalue problem for a matrix of size $2N$ instead of a singular value problem.

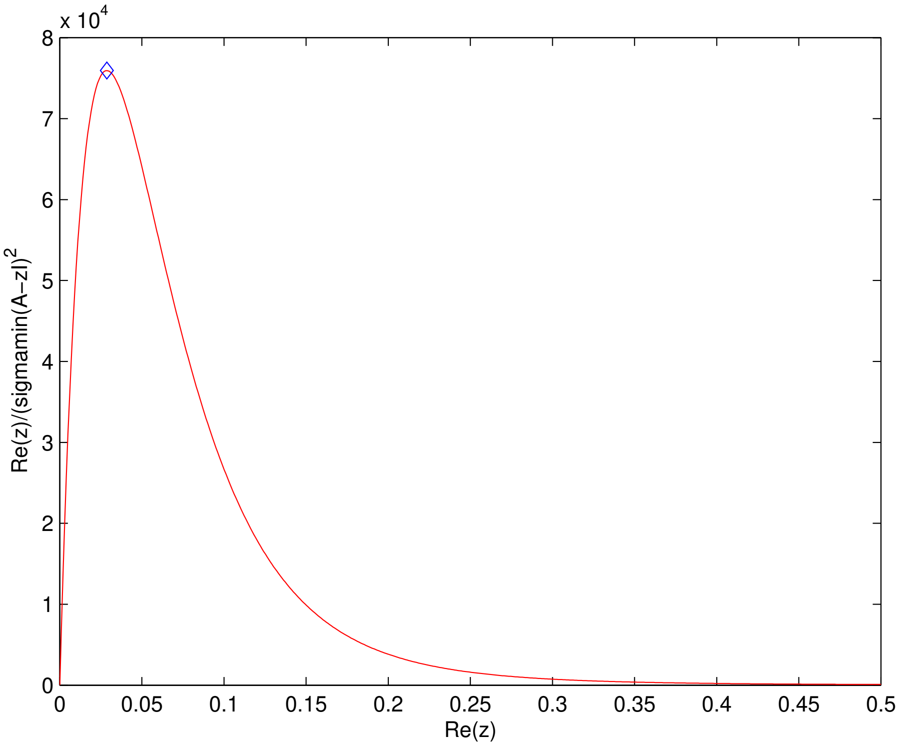

FIGURE 4. The plot of $4x\ [\mathrm{cdi}(A-x)]^{-2}$ for $A = J_10$, where J_n is the $n \times n$ Jordan block matrix.

To compute the minimum in (B.1), we use the function `fminbnd` from MATLAB's optimazation toolbox. It finds the minimum of a function using a golden section search and parabolic interpolation. In the golden section search we first choose an interval of the search, and a tolerance. Then we reduce the length of the interval until it is less than the tolerance. When a parabolic interpolation is possible, it increases the rate of convergence of the algorithm. The precise algorithm is described in detail in [5].

The input arguments for `fminbnd` are the interval of the search, the tolerance, and the function to be optimized. We have used a tolerance of $1e$-12. Then we put

(1) $x_1 = -\max \mathtt{real}(\mathtt{eig}(A))$, initial value of x
(2) $[1/\psi, x] := \mathtt{fminbnd}\ (]0, 10x_1])$ where $(]0, 10x_1])$ is the interval to be searched.

The upper bound of the interval $10x_1$ is given by the experiment, since we always find that the optimal value is reached for $x_{\mathrm{opt}} < 10x_1$. It seems intuitive that since x_1 gives the distance between the eigenvalue of A with the largest real part and the imaginary axis, it can be used as a measure to search the interval where the maximum will be reached. We observe experimnetally that the function minimized in (B.1) has a unique minimum and that makes the algorithm quite efficient.

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