What Is So Difficult About the Preparation of Mathematics Teachers?

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1 An overview of the problem

Our universities do not adequately prepare mathematics teachers for their mathematical needs in the school classroom.[1] Most teachers cannot bridge the gap between what we teach them in the undergraduate curriculum and

[0] March 6, 2002.
After the completion of this article, I consulted a related one by Al Cuoco ([Cuoco]) and was surprised that, while his views on pre-service professional development are consistent with mine, the two articles have almost no overlap.
I have benefited greatly from the comments of M. Burmester, A. Cuoco, J. Dancis, A. Ralston, and A. Toom, and especially from the insightful critiques of P. Braunfeld, T.H. Parker, and R. Raimi. I take this opportunity to thank them all.
[1] Any such statement is understood to admit a small number of exceptions.

what they teach students in schools. Surprisingly the realization that such a gap exists seems to be of recent vintage ([Braunfeld], [Wu 1996b]–[Wu 1999a]). A main impetus behind the writing of the MET volume ([MET]) was in fact to address the problem of closing this gap. Further progress in this direction would depend on a more refined analysis of exactly where the usual preparation of teachers fails. I believe this failure lies in at least two areas. First, we have not done nearly enough to help teachers understand the essential characteristics of mathematics: its precision, the ubiquity of logical reasoning, and its coherence as a discipline. Second, the teaching of fractions and geometry in schools has very specific mathematical requirements, but our undergraduate mathematics curriculum has consistently pretended that such requirements do not exist. This neglect is partly responsible for the well-known high student dropout rates in algebra and geometry.

Although most of what I have to say is equally valid for grades K–4, especially regarding the need of precise definitions, a few of the following comments, such as those concerning logical explanations, could be misinterpreted in the context of children in grades K–3 (grade 4 is a borderline case which I have intentionally left out). For simplicity, I will only address the preparation of teachers in grades 5–12 in this article.

Mathematics is by its very nature a subject of transcendental clarity. In context, there is never any doubt as to what a concept means, why something is true, or where a certain concept or theorem is situated in the overall mathematical structure. Yet mathematics is often presented to school students as a mystifying mess. No doubt the textbooks are at fault, but many of the teachers certainly contributed their share to the obfuscation. For this reason, we would want teachers to have a firm grasp of the following characteristics of mathematics, namely,

> (1) that precise definitions form the basis of any mathematical explanation, and without explanations mathematics becomes difficult to learn,
>
> (2) that logical reasoning is the lifeblood of mathematics, and one must always ask *why* as well as find out the answer, and finally,
>
> (3) that concepts and facts in mathematics are tightly organized as part of a coherent whole so that the understanding of any fact or concept requires also the understanding of its interconnections

with other facts and concepts.

Examples are all around us to serve as reminders that we are very far from doing a good job of making our teachers aware of any of these characteristics. For example, many teachers do not take definitions seriously (see e.g., [Burmester-Wu]). Some consider it a lack of conceptual understanding when a single definition is used for a concept because they insist on using several at the same time. At present, pre-service professional development seems oblivious to the basic mathematical requirement, in case several meanings of a concept are used, that only one of them be adopted as the definition and then used to *explain* why the others are also valid. This procedure enhances our understanding of the concept in question by revealing the logical interrelatedness of the several meanings. Many teachers seem to be unaware that learning is difficult for students if they are never sure of what their teachers have in mind when precise terminology is not employed. To these teachers, making students learn a precise definition is the same as teaching-by-rote. A colleague in education once remarked to me, reacting to my perception of the need of a precise definition of a fraction starting with grades 5,[2] that getting teachers to memorize one definition of fraction is a good way to kill the good intuitions they have about fractions because, acccording to her, fractions are such a basic concept that they "sit beyond definitions". The notion that learning something basic and correct in mathematics could be detrimental to the learning of mathematics itself is strange. This is akin to the belief that promoting technical fluency in computations is the same as promoting "drill and kill". It is a true measure of how badly mathematics education has failed when we allow this kind of thinking to perpetuate the non-learning of mathematics.

I would like to draw a line between my insistence on precise definitions and certain well-known excesses of the New Math, such as defining functions as a set of ordered pairs with special properties, or making a meticulous distinction between "numbers" and "numerals" even in elementary school. The purpose of having clearly stated definitions is to let readers know exactly where they stand with respect to the mathematics. The mathematical accuracy of the definition therefore should be consonant with what is appropriate in context. By no means should fourth graders be exposed to the definition

[2] This is in accordance with the Mathematics Content Standards of California. It could be grade 6 in other states.

of area as a Lebesgue measure or fifth graders be introduced to fractions as equivalence classes of ordered pairs of whole numbers. Nevertheless, it is possible to give a precise definition of area or fraction that satisfies all the given constraints. See §3 and §5 for further discussions. The art of "saying enough but not too much" is of course a main concern in the preparation of mathematics teachers, and this concern will be a recurrent theme in the present article.

It is generally recognized that The absence of logical reasoning from mathematics classrooms is a main culprit in bringing about the present mathematics education crisis. It is well-known that within mathematics, logical reasoning is synonymous with theorem proving. What is less known is that logical reasoning underlies all forms of problem solving. Each time one solves a problem, one in fact proves a theorem. Mathematics is nothing but problem discovering and problem solving. For example, Fermat discovered the phenomenon that what we call *Fermat's Last Theorem* might be true, and Andrew Wiles solved it some 350 years later. Thus the emphasis on logical reasoning includes the imperative for teachers to be problem solvers. However, "problem solving" has a special (and misleading) connotation in the context of school mathematics that I wish to avoid, because it suggests only that teachers "solve problems that come out of the curriculum". What teachers must be able to do is not only to solve problem in that sense, but also to make logical deductions within all the theorems in the school curriculum. Short of that, they would not be able to provide logical explanations to students.

As to teachers' need to know the coherence of mathematics, one example would suffice to indicate why it matters. The way proportional reasoning is taught in middle school as a rule does not reflect any awareness of the intimate connection between proportional reasoning and linear functions. The failure to teach proportional reasoning well is supposed to be a major issue in mathematics education. If so, then one can clearly point the finger to this missed connection. See §4 for further discussion of this point.

Consider now the second area of failure in our preparation of teachers. Fractions are so problematic a topic in school mathematics teaching because serious *mathematical* instruction on fractions in school takes place in grades 5–7 (or in some districts, 6–8). The delicacy of the situation stems from the fact that for children at this age, one must teach fractions with a minumum of abstraction, and yet the mathematical foundation of fractions must be firmly laid in these early grades because the school mathematics curriculum does not revisit this topic beyond grade 7, and also because fractions form the bridge between the arithmetic of whole numbers and algebra ([Wu 2001c]). Balancing the elementary character of the exposition against the need to make it so mathematically robust as to be usable for the next six or seven years is a task too taxing for any school teacher (cf. [Wu 2001b]). Teachers need help, but the universities are not providing any. If we want fractions to be better taught, we should pinpoint the mathematical difficulties and provide prospective teachers with sound *mathematical* alternatives rather than just put forth the wishful thinking that teachers should be able to think flexibly about rational numbers and reason about proportion.

As to the subject of school geometry, the problem is that if universities do not teach it, or do not teach it well, then the only exposure to school geometry that geometry teachers ever have will be their own high school experience in geometry. The latter of course has been scandalously unsatisfactory for a long time, to the point where many school geometry courses cease to prove any theorems. Perhaps there is no other way to break this vicious circle than to actually teach the teachers this material and show them how to do better. Of primary concern in such a course on geometry, therefore, would be to firm up teachers' geometric intuition and to increase their understanding of the purpose of geometric proofs and axiomatic systems. This discussion will be further pursued in §6.

As topics in the school mathematics curriculum, fractions and geometry have one thing in common: the failure to teach them properly is responsible for precipitous student dropouts. The failure to teach fractions sensibly to students is directly related to the inability to prepare students for algebra (cf. [Wu 2001c]), and the high dropout rate in algebra is of course a national *cause célèbre*. As to the dropout of students from high school geometry, the absence of sensible geometry texts is one reason, but it would not be excessive to speculate that teachers' discomfort with the geometry curriculum, especially with geometric proofs and axiomatic development, is another. At a time of change in mathematics education, teachers need support from their professional development in order to face the new challenge, but the universities have not risen to the occasion.

In terms of these two areas of failure in our preparation of teachers, we can now briefly speculate on why this particular endeavor is so difficult. Asking prospective teachers to learn precise definitions, proofs, and interconnections among mathematical topics is tantamount to asking them to change their fundamental belief systems. To most of them, these basic characteristics of mathematics run contrary to their experience in K–12 education. They have rarely, if ever, been exposed to any clearly stated definitions, seen how to use definitions to explain any mathematical facts, or been told that the many disjointed facts they learned fit into an underlying structure. Short of getting some professors to perform heroic acts beyond the call of duty — year in and year out — to show teachers what mathematics is really like when it is done properly, it is impossible to effect such radical change in teachers' perception of mathematics within the time of a few required university mathematics courses.[3] This is not even taking into account of the amount of will power and effort that the teachers themselves must exert in order to bring about this change. As to the teaching of fractions and school geometry from a higher perspective, most university mathematics departments simply would not consider such courses to be of "college level".[4] Moreover, teaching such courses well requires a sensitive understanding of the K–12 curriculum as well as mathematical and pedagogical competence, and few on the university level can meet both sets of requirements.

I can only hope that such a bleak situation would improve as more attention is focussed on the mathematics education of teachers. The fanfare attending the publication of MET ([MET]) is an auspicious beginning.

A general disclaimer would be appropriate at this point. The reason I have described in such detail some of the ills of mathematics teaching in schools is not to lay the blame of the current mathematics education crisis squarely on teachers. Rather, without such a precise description, it would be impossible to make specific recommendations on how to better prepare teachers. The reader will take note of the constant emphasis throughout this article on the causal relationship between what the universities teach prospective teachers and what teachers teach in school classrooms. Therefore, it would be no more accurate to say that this article castigates the teaching profession than to say that a physician's precise diagnosis of an illness criticizes the patient. In both cases, the factual analysis is the means to an end, — improvement of teaching or improvement of health — but by no means the end itself.

[3] From this standpoint, the recommended mathematics coursework of 9 semester-hours for elementary teachers seems to be far from sufficient for producing competent teachers.

[4] Again, there are rare exceptions to the rule.

2 Definitions and proofs

There are two kinds of detective novels. One kind concentrates more on the dramatic ebb and flow of the plot and less on the internal logic of its development. It witholds some critical information until the very end when, in the form of *deus ex machina*, it is suddenly sprung on the reader to bring the murderer to justice. The other kind gives the reader all the facts needed to solve the murder in a straightforward manner, so that when at the end the murder case is solved by deductive reasoning on the basis of the available facts,[5] the reader feels foolish in not having seen the obvious and vows to do better the next time. If the teaching of mathematics can be compared to a detective novel, then it should be the second kind and not the first. Whatever information is given to students, it should be one hundred percent sufficient to provide the basis for logical deductions in everything else that follows. There should be no surprises, and no hidden facts up a teacher's sleeve. Otherwise, students would always harbor the suspicion that, any time they cannot solve a problem, it is because the teacher has not been up front with them about all the facts they need. It would not take long before they stop thinking, and then would stop trying to make sense of the mathematics in front of them. At the end, mathematics becomes a kind of magic that they cannot hope to comprehend. This is more or less the situation we have on our hands.

Mathematics is an open book. It is accessible to anyone who is willing to abide by the explicitly stated rules of the game. Students must buy into this accessibility before they can learn. For that to happen, mathematics teachers must have the same conviction about the accessibility in the first place. But they will not if they do not consistently get this message from their own K–12 education, from their university courses, and from the textbooks they use to teach. Because we cannot count on K–12 education or school textbooks for the help we need at this juncture, extra burden is then put on the way we prepare teachers in the universities.

Imagine then that you are a prospective teacher and you are trying to understand fractions. Suppose you come across the following explanation of the concept of a fraction:

[5] The Nero Wolfe novels of Rex Stout are goood examples of this kind.

> *Sometimes the word fraction refers to a certain form of writing numbers. In this sense of the word, a fraction is a pair of numbers written in the form $\frac{a}{b}$, usually with the stipulation that the bottom number should be nonzero. This use of the word fraction refers to a form of writing a number, a notational system, a symbol, two numbers with a bar written between them.*
>
> *Sometimes the word fraction is used to refer to one of several interpretations of rational numbers, traditionally called the* part-whole interpretation. *That is, a fraction represents one or more parts of a unit that has been divided into some number of equal-sized pieces. In this case, the word fraction refers not to the notation, but to a particular interpretation or meaning or conceptual understanding underlying the fraction symbol.*
>
> *When we speak of a fraction as a number, we are really referring to the underlying rational number, the number the fraction represents.* [What then is a rational number?] *The counting numbers (1, 2, 3, 4, ...) are used to answer the question "How many?" in situations when it is implicit that we mean "How many whole things?" ... The rational numbers are use for answering the question "How much?" They enable us to talk about wholes as well as pieces of a whole. ... A rational number may be viewed as a quotient, that is, as the result of division.*
>
> *Fraction symbolism may be used to represent many different interpretations or personalities of a rational number, but the first that children meet, at least in the present curriculum, is the part-whole comparison. It is a very important interpretation of rational numbers because it provides the language and the symbolism for the other rational number personalities.*

These passages are taken directly from a volume on professional development in the subject of fractions, but I am less interested in criticizing a specific case than in criticizing a generic phenomenon in education. Reading through something like this, you would be justified in thinking that you have not been told what a fraction really is. Somewhere down the line, you expect yet another "personality" of a rational number in addition to "part-whole", "quotient", etc. to be sprung on you, in the same way that readers of the first kind of detective novels would expect sudden revelations of hitherto unheard of developments near the end. Perhaps a fraction such as $\frac{3}{5}$ *is* three separate objects after all: 3, 5, and a bar between them. In despair, you give up on ever learning what fractions are all about, and in due course this despair would be transmitted to your students, willy-nilly. And this is how the vicious circle begins.

A similar fate awaits anyone who tries to understand what "rate" means. For example:

> Rates are a common way of stating a relationship between two quantities. Fractional parts of a whole are one kind of relationship; the rate at which some quantity is repeated or generated or used is another. A rate is a fraction because it also expresses a relationship between two numbers.

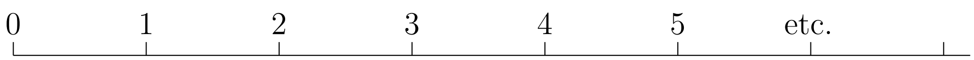
Now this is supposed to explain what "rate" is in a *mathematical* context, so that it can henceforth be used in computations, formulas, and equations. No one knows how to do any of these for a "relationship between two quantities', when "relationship" itself has yet to be defined. Furthermore, how can the way "some quantity is repeated or generated or used" suddenly become a fraction when fraction up to this point is nothing but part-of-a-whole, as the passage iself admits? This is no different from explaining to children that a carpet is a soft and thick piece of material used to protect the floor, and then adding after a brief pause that a carpet is also used to fly humans from royal palaces to far away places. Without denying that such a fantasy makes effective fairy tales, I do not believe it has any place in a correctly presented mathematical exposition.

To provide a contrast, let us see what a precise definition of fraction as a number can do in the way of clarifying why a fractions may be "viewed" as a "quotient". Let a unit be fixed; then for whole numbers m and n, the fraction $\frac{m}{n}$ is usually defined by "divide the unit into n equal parts and take m of them". There are at least two problems with such a definition.[6] The first one is that a fraction according to this definition is an "activity", and it is difficult, if not impossible, to talk about adding or dividing "activities".

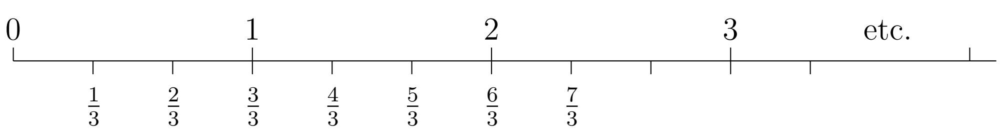
[6] A third problem, which I will let pass, is that the critical role of the *fixed* unit s often not sufficiently emphasized in textbooks or the professional development literature.

The second one is that, without a clearer description of what the unit is, the meaning of "dividing the unit into n equal parts" is ambiguous. For example, if the unit is a bag of 5 potatoes, and we wish to divide this unit into 7 equal parts (so $n = 7$), would it be "7 equal parts by weight" or "7 equal parts by volume" ? Because potatoes vary greatly in both size and weight, the distinction between the two is not trivial. In view of these difficulties, it would be mathematically more appropriate to make use of the number line to define a fraction as a point on this line, as follows.

The number line has an infinite number of equi-spaced points marked on it to the right of a fixed point. The latter is called 0, and the markers to the right of 0 are, successively, 1, 2, 3, etc.

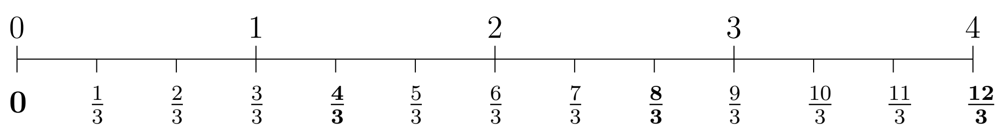


The whole numbers are now identified with their respective markers on this line. To begin the definition of a fraction, we fix a unit, which will be identified with the number 1. For whole numbers m and n with n not equal to 0, divide each line segment between successive whole numbers, $\{[0,1],\ [1,2],\ [2,3],\ldots\}$, into n *equal parts* — which will henceforth mean n sub-segments of equal length — in each of $\{[0,1],\ [1,2],\ [2,3],\ldots\}$, This then introduces further markers on the number line in addition to the original markers corresponding to the whole numbers. Note that, relative to the new collection of markers, the n-th marker to the right of 0 is just 1, the $2n$-th marker to the right of 0 is just 2, etc. Now the fraction $\frac{1}{n}$ is by definition the first marker to the right of 0, and the fraction $\frac{m}{n}$ is by definition the m-th marker to the right of 0. For definiteness, let $n = 3$, then the first few fractions of the form $\frac{m}{3}$ are given in the following picture:



(Cf. [Jensen] or [Wu 2001b]; see also the essentially equivalent presentations in [Beckmann] and [Parker-Baldridge]). We are going to prove that $\frac{m}{n}$, so defined, can be interpreted as a quotient, and will do so by merely "unpacking" the definitions without appealing to the subtle concept of *multiplying* fractions (see [Wu 2001b], §§4 and 6 for a full discussion). To proceed, I will first heed my own advice concerning definitions by *defining* what "quotient" means. For whole numbers m and n so that m is a multiple of n (i.e., $m = kn$ for some whole number k), the meaning of the *quotient* $m \div n$ in the context of whole-number arithmetic is unambiguous: it is the whole number k. However, when m ceases being a multiple of n, the meaning of $m \div n$ is usually left to the readers' imagination. Now that we have the number line, we can give a precise definition of $m \div n$ for arbitrary m and n (n not equal to 0) as follows: divide the line segment from 0 to m, $[0, m]$, into n equal parts, then $m \div n$ is the first marker to the right of 0 in this division. (Expressed differently, $m \div n$ is the length of a part when the segment $[0, m]$ is divided into n equal parts; thus we are *extending* the usual partitive definition of division between whole numbers.) Now both $m \div n$ and $\frac{m}{n}$ have been defined to be points on the number line. We will prove that these points coincide, and we express this fact by writing: $m \div n = \frac{m}{n}$. This is the precise way of saying that "the fraction $\frac{m}{n}$ may be viewed as the quotient $m \div n$".

For clarity of exposition, we let $m = 4$ and $n = 3$ and give the proof in this case. The proof in general will be exactly the same, but with 4 and 3 replaced by m and n, respectively. Thus we will prove that $4 \div 3 = \frac{4}{3}$. We first look at $4 \div 3$: by definition we have to divide the segment $[0, 4]$ into 3 equal parts. We do so in a particular way: divide each of the sub-segments of length 1, $\{[0, 1],\ [1, 2],\ [2, 3],\ [3, 4]\}$, into 3 equal parts, so that the segment $[0, 4]$ is now divided into 12 ($= 3 \times 4$) equal parts:



If we take every fourth marker in this division, then we clearly obtain a division of $[0, 4]$ into 3 equal parts. But the markers in question are $\{0, \frac{4}{3}, \frac{8}{3}, \frac{12}{3}\}$, so that the first marker to the right of 0, $\frac{4}{3}$, is exactly $4 \div 3$, by definition.

I hope this proof demonstrates convincingly the advantages of having precise definitions *and* being able to make use of precise definitions. This proof also shows that it is not necessary to blindly follow somebody else's dictum that a fraction "may be viewed as a quotient, that is, as the result of division". If we can say what a fraction is, and what a "quotient" means, then we can *prove* that a fraction is a quotient.

The previously quoted passages about fractions are typical of the treatment of fractions in the literature on pre- or inservice professional development. What is true for the literature on fractions is of course true for any other mathematical topic. We cannot produce the kind of mathematics teachers we need so long as we engage in this kind of professional development.

My insistence on teaching prospective teachers precise definitions is not because I have the alleged tunnel-vision of a mathematician and only want school mathematics to serve future professional mathematicians. The reason is rather that if teachers do not make students aware of the inherent precision of mathematics, then mathematics would easily degenerate into the qualitiative morass that we sometimes witness in school classrooms. More importantly, logical explanations — the essence of mathematics *no matter how mathematics is defined* — cannot be given without precise definitions. The absence of logical explanations in school mathematics is a principal reason why teaching-by-rote and learning-by-rote so often prevail in mathematics classrooms.

For example, in introductory algebra, many students and teachers make a big fuss about the difference between the point-slope form and the slope-intercept form of the equation of a straight line, and about how to write the equation of a straight line when either two points or a point and its slope are given. This kind of fuss would be rather pointless if there is any understanding of *why* the graph of a linear equation is a straight line.[7] Unfortunately, such understanding is not always present in a typical algebra classroom because the explanation of why the graph of a linear equation is a straight line

[7] This is another illustration of the difference between advanced mathematics and school mathematics. In the context of the former, a straight line is, *by definition*, the graph of a linear equation. It would not be advisable to do the same in the eighth grade. Instead we take for granted that students have had some prior geometric experience with straight lines, and would therefore expect that (1) if two intersecting straight lines are given, then the right triangles obtained by dropping perpendiculars from points of the first line to the second are all similar to each other, and (2) if two angles are equal and have one side (a half-line!) in common, then the other sides either coincide or are symmetric with respect to the first side. Then on the basis of these acceptable geometric facts about straight lines, we can prove that the graph of a linear equation is a straight line.

is rarely given, either in textbooks or by the teacher. Plotting a few points and observing that they appear "straight" pass for logical explanations at the moment. Until our teachers realize the need to present basic proofs of this nature, such logical explanations will continue to be absent from school classrooms, basic facts about the straight line will continue to be taken on faith by students, and interconnections among these facts will continue to remain hidden. This is how learning-by-rote sets in.

If one examines carefully the explanation of why the graph of a linear equation is a straight line, one would see that — in addition to the use of similar triangles — a critical part of the reasoning depends on understanding the *exact* definition of the graph of a function ("the set of *all* points (x, y) so that $y = f(x)$"). Such an explanation cannot be given if the definition of the graph of a function is not taken seriously. Knowing the definition of the graph of a function then opens the door to an understanding of linear equations.

In case you are under the misapprehension that I am exaggerating the situation concerning the absence of precise definitions, consider the following concepts:

> remainder of a whole number division, decimal (finite or infinite), mixed number, ratio, rate, irrational number, real number, slope of a line, graph of an equation, graph of an inequality, rational expression, change of scale, dilation, area, volume.

These are basic concepts in the school mathematics curriculum. Now try to look through the available textbooks, both in schools and in pre-service professional development, to see how many of these are carefully and correctly defined.[8] Then try to find out how many teachers make the effort to give

[8] I used to believe that the failure of textbooks to provide correct definitions of mathematical concepts was due to their writers' inadequate knowledge of mathematics, but a parent, Cettina Cornish, brought to my attention a passage in a second grade textbook that seems to indicate that the failure may be more a matter of ignoring the basic requirements of human communication. The passage in question is an assignment given to parents: "Your child has been working to make up adding and subtracting stories. Have your child share with you some of her or his stories and the addition and subtraction sentences that tell about the stories." The parents are therewith invited to create these sentences and stories for their child to bring back to school. But in the absence of any examples of such "stories" or "sentences", or in fact any explanation thereof, many dumb-

correct definition of each of these *in spite of* the textbooks, as well as emphasize to their students the importance of these definitions. To drive home my point, consider the following two vignettes:

> (1) *Professional development for high school teachers.* The instructor asked his teachers whether they got anywhere with the investigation of why it only takes three points to determine a circle, but it takes 5 points to determine a conic. There were murmurs about parabolas and ellipses, but no definitive answer. A teacher mentioned that she had some idea of how to proceed but could not come up with a proof. Then a visitor asked if the teachers knew the definition of a *conic*. Apparently nobody did.
>
> (2) *Professional development for high school teachers.* The instructor showed how to use computer software to illustrate dilation by a factor of 2, 3, etc. The teachers were excited and they were given instructions on how to reproduce the same effect on the computers in front of them. Then the instructor talked about "dilation by a factor of $-\frac{1}{2}$". First he explained what the minus sign means: "goes the other way". So he showed the dilation by a factor of $\frac{1}{2}$, then by flipping it to the other side of the *center of dilation* (which he neither formally defined nor stressed), he showed graphically the effect of "dilation by $-\frac{1}{2}$". He described the computer commands that would produce the graphic effect of the minus sign. The teachers got busy with the the graphics and became even more excited. Someone then asked if the teachers near him knew the precise definition of "dilation". No one did, including the assistant to the instructor. One teacher got indignant about the question and said: "Do you want me to make my students memorize the definition? If I can make them understand the concept of dilation by using the computer, what do I care about definitions? What has definition got to do with conceptual understanding?"

Are these vignettes typical? I cannot say. Are they sufficiently common to be alarming? All the anecdotal and firsthand evidence points to an af-

founded parents are understandably upset by the assignment because they have no idea what is expected of them.

firmative answer. We clearly have a long way to go in impressing on our teachers the importance of both definitions and proofs, and the fact that one cannot have proofs without definitions.

3 Further comments on definitions

Precise definitions are important in mathematics. Without the precise definitions of concepts, logical explanations cannot be given and, without explanations, mathematics becomes nothing but the proverbial laundry list of theorems. It has been said once, but it deserves to be said again: precise definitions are not one more collection of facts for prospective teachers to memorize, but are instead the solid foundation on which they can launch their mathematical investigations.

The precision of mathematical definitions is a double-edged sword. While it makes logical explanations possible, its often counter-intuitive nature is also the reason it makes many teachers and students shun definitions. Somewhere in professional development there should be a careful explanation of the fact that a mathematical definition is not meant to be the first thing that comes to one's mind when confronted with a concept, but is rather the end result of a (perhaps long) search for the balance between what can best serve our needs and what is intuitively clear. The failure to recognize this fact accounts for the fluctuation in many teachers' attitude towards definitions between two extremes, that of accepting by rote a mathematical definition-as-is, and that of circumventing it by verbose and ultimately mathematically meaningless descriptions. An example of the latter is the long description of fractions quoted in the preceding section. This is not to say that such a description has no value, only that *in the absence of a precise definition of fraction* it does not contribute to a reader's understanding. Equally harmful, but in a different way, is the other extreme of presentating a definition without any comments about background or motivation. This kind of teaching is what gives mathematics a bad name. As an example of the non-intuitive nature of a typical mathematical definition, consider that of a convex region. A region is by definition *convex* if "the line segment joining any two points of the region lies in the region", but the intuitive idea of a convex region is of course that it "bulges outward". A teacher should be able to explain that such a definition has the undeniable virtue of being easy to use and yet, exotic as it may seem, does manage to capture the essence of "bulging outward". Regions that arise naturally that "ought to be" convex can now be *shown* with ease to be convex, e.g., the intersection of any number of convex regions is convex. The exactness of this definition of convexity renders unnecessary a lot of futile hand-waving about "bulging outward". A successful program to prepare mathematics teachers would make a strenuous effort to explain the genesis of mathematical definitions and point out the crucial role definitions play in the unfolding of logical arguments. The available evidence is that such an effort is not there, yet.

It must not be assumed that when I say "precise definitions", I mean "one hundred percent correct mathematical definitions". It has already been pointed out in §1, for example, that a definition of "fraction" for upper elementary school emphatically does not mean an equivalence class of ordered pairs of integers, any more than a definition of "area" in high school means Lebesgue measure. So this is as good a place as any to make explicit the fact that everything in this article has to be understood in the context of school mathematics. Definitions should be precise, but they *may be only as mathematically complete and accurate as the context allows.* In school mathematics, there is always a tension between what is mathematically correct and what is teachable on a specific grade level. *A principal goal of pre- or inservice professional development is to help teachers resolve this tension as best we can.* In §5, I will briefly mention a usable definition of fractions for teachers in the fifth grade. Here, let me use the concept of area for illustration.[9]

In middle school, the most important point about the concept of area is the need to fix a *unit* of measurement — usually a square of side length 1 according to a pre-assigned choice of length — before the area of polygons can be defined as the total number of units that can be "packed" into the polygon, with "packed" given the usual meaning of cutting and pasting in the sense of congruence. Most of the discussion of area in middle school should be restricted to the special case of polygons because this is the class of geometric figures whose area can be computed *exactly* once the area formula of a rectangle (i.e., width times length) is accepted. Certainly, the area formula for rectangles whose sides have rational lengths should be proved

[9] Needless to say, anything said about the concept of area in the ensuing discussion applies equally well to volume.

completely.[10] The inevitable confrontation with the circular disk and its area then necessitates an intuitive "definition" of this area as the "limit" of the areas of approximating inscribed regular polygons. If the computation of the area of polygons has been adequately explained, such hand-waving should be sufficient for the purpose of giving a robust conception of what area means in general as well as a computation of the area of a disk as πr^2.[11] For high school teachers, essentially the same discussion would also serve, but the intuitive discussion of "curvilinear area" may be expanded to include geometric figures other than disks (ellipses and parabolic segments, for instance), and "limits" may be explained on a more sophisticated level, though by no means in full strength.

To return to the issue raised above concerning the need to resolve the tension between what is correct and what is teachable, we can see that the definition of area outlined above has many gaps when compared with the correct definition. Yet it would be inadvisable to cram more into a K–12 discussion of this concept because to do so would require a complete discussion of limit. We therefore end up teaching a notion of area which is technically incomplete but which is nevertheless correct. Few would argue that this incompleteness would cause great harm in a student's education. Thus part of the responsibility of pre-service professional development is to help prospective teachers find this middle ground between what is completely mathematically correct and what is teachable.

One can also approach the same issue from the perspective of a teacher's education. A high school teacher who has taken a good course in calculus — and all of them should have taken such a course[12] — would have a correct understanding of area[13] as a Riemann integral. Knowing that the Riemann integral can be used to define area and volume is necessary for teaching high school mathematics,[14] but it is far from sufficient for use in a high school classroom. For example, telling students in Euclidean geometry that "area can be defined by the Riemann integral" but nothing else

[10] But the general case of rectangles with irrational side lengths should be left to a college course.

[11] Incidentally, such an explanation is also accessible to elementary school teachers.

[12] I will not enter into the difficult subject concerning instruction by teachers without a single-subject credential.

[13] At least the area of almost every geometric figure that comes up in daily life.

[14] There is no other way, for example, to understand why the volumes of cones and pyramids always have a factor of $1/3$.

about area is bad education. The simplified definition of area in the preceding paragraphs would be more informative to these students than learning about the terminology of the "Riemann integral", so the simplified definition must be part of a teacher's education too. Unless we teach this definition explicitly in pre-service development courses, we cannot assume that every high school teacher knows it. We are therefore back to the position that pre-service professional development involves more than just teaching standard upper division mathematics.

If we ask why so many teachers at present do not put much stock in definitions, a look at their education experiences would provide the answer. Most school textbooks show such contempt for precise definitions as a first step in mathematical reasoning that a mathematician may rightfully question if these textbooks are about mathematics at all. (In case of doubt, look at the treatment of fractions.) One explanation is that with so little mathematical reasoning taking place in these textbooks, there is little incentive to give precise definitions. It would then be reasonable to surmise that not much emphasis is given to definitions and proofs in the classroom instruction either. Now look at what elementary school teachers get in their college education. The two or three mathematics courses they take do not by tradition talk about definitions or proofs with any serious intent, and are therefore not the ideal vehicles to counteract the mis-education of their first thirteen years. If anything, and the available college textbooks for these courses tend to confirm it, these courses reinforce their perception that precise definitions are not important and logical deductions are not valued. Many of the teachers are also exposed to recent pedagogical ideologies which consider precise definitions antithetical to the so-called "discovery" method of learning. Thus it comes to pass that when they become teachers themselves, they too become part of the vicious circle by conferring on their children the same attidude about definitions and proofs.

The situation with middle and high school teachers differs in the details but not in the outcome overall. Their college courses are, with rare exceptions, those required more or less of other mathematics majors, and while these courses are *in principle* helpful to their teaching, in practice they are not, or at least not at this stage of pre-service professional development. The subject matter is generally too far removed from that of K–12 and, unless a great effort is made, its relevance to K–12 would escape all but a few of them. To some extent we have seen this phenomenon in the above discussion of area. This aspect of pre-service professional development is discussed, as we said, in some detail in [Wu 1997]–[Wu 1999b], and is also one of the reasons why MET ([MET]) was written. The long and short of it is that although advanced courses in mathematics require meticulous attention to definitions and proofs, prospective teachers in such courses may not appreciate this fact if they have to struggle with the material or, worse, regard such material as irrelevant and tune it out.

Any change in basic attitude or habit is painful, but this is what we hope to do for prospective teachers regarding definitions and proofs. Mathematicians can help if they are willing to show by examples how they use definitions in a positive way to do mathematics: how explanations rely on having precise definitions, and how precise definitions help make clear thinking possible. The task is not easy, and you may even ask if the payoff is worth the effort. But it is. There is some evidence that teachers do come around to appreciate the importance of definitions and explanations and, once they do, they vow never to go back to the "old habits" again ([Burmester-Wu]). The following anecdote may yet be the best argument for why we should try harder. Recently I had a soul-baring session with a colleague of mine whom I talked into doing some professional development work for the first time in his life. After an outpouring of his frustrations, he finally admitted that, even if he could convert only *one* teacher, he would have made an impact on the education of probably thousands of children.

4 Longitudinal coherence of the curriculum

It is a consequence of the way we teach university mathematics courses that details tend to be emphasized at the expense of interconnections among topics. We endeavor to make students see the individual trees clearly but, in the process, we shortchange them by not calling their attention to the forest. There is a valid justification for this approach: learning the technical details is so difficult for beginners that we must give them all the help they can get. On the other hand, we do pay a price. We produce many students who do not think globally — or to use a more common word these days, holistically — about mathematics. In the present context, teachers who come through such a program may know the individual pieces of the school curriculum, but they are less adept at seeing the interrelationships among topics of different grades.

The ability to see such relationship has an immense impact on teaching. I will use three among many possible examples to illustrate why we must give serious thoughts to firming up this aspect of our teachers' knowledge of mathematics. Consider the following logical progression in roughly grades 5 to 8:

$$\text{whole numbers} \longrightarrow \text{fractions} \longrightarrow \begin{cases} \text{finite decimals} \\ \text{ratio, rates, percent} \\ \text{algebra} \end{cases}$$

This progression recognizes that whole numbers and fractions are on equal footing and that the latter is nothing but a direct extension of the former. Such a recognition would compel a teacher to make an effort to give a definition of what a "number" is (e.g., a point on the number line), explain in what sense a whole number is a "number", and give a definition of a fraction as a special kind of "number". See, for instance, the discusions in §5 of [Wu 2001a] and §1 of [Wu 2001b]. The teacher would then be able to explain every single arithmetic operation of fractions on the basis of the corresponding operation of whole numbers (cf. §5 below). Such an approach gains the immeasurable advantage of making children see that fractions are very "natural" objects rather than some kind of mathematical mutants that obey no rule or regulation known to human kind — and need I add that at the moment most children truly believe the mutant theory. It should be obvious how this psychological edge can facilitate the learning of fractions. Once fractions are accepted by children in both the mathematical and pyschological senses, other common but abstruse concepts can be clearly explained. In particular, finite (terminating) decimals are no longer objects "very closely related to fractions", but are exactly those fractions whose denominators are powers of 10. I choose to be so precise here about decimals because children's failure to make sense of decimals is often attributed to every reason under the sun other than the fact that the definition of a decimal is not clearly stressed in schools. Other concepts such as rate, ratio, and percent — if we agree that school mathematics is essentially the mathematics of rational numbers — are now also seen to be special kinds of quotients of fractions (see §11 of [Wu 2001b]).

If real numbers are taken into account, then everything gets infinitely more complicated. School mathematics thus far has not dealt with real numbers in an — pardon the use of the word — honest manner. Suppose we can get over the hurdle of real numbers (see §11 of [Wu 2001b], especially the discussion of the Fundamental Assumption of School Mathematics), then formally at least, the meaning of rate, ratio, and percent will remain the same, i.e., the quotient of numbers (again see §11 of [Wu 2001b]). I would like to single out "ratio" for a brief discussion. This term was first introduced by Euclid in his *Elements* and, if he had the mathematical understanding of the real numbers as we do now, he would have said outright: "the '*ratio of A to B*' means the *quotient* $\frac{A}{B}$". But he didn't, and human beings didn't either for the next twenty one centuries or so. (See the discussion at the end of §11 in [Wu 2001b].) For this reason, the long tradition of the inability to make sense of ratio continues to encroach on school textbooks even after human beings came to a complete understanding of the real numbers round 1870. It may be 130 years too late, but now is the time for us to say unequivocally in all our school textbooks:

The *ratio of A to B* means the quotient $\frac{A}{B}$.

Unfortunately, most textbooks continue to take "ratio" as a known concept and fractions are defined as "ratios of two whole numbers". Or, prospective teachers are exhorted to "recognize that ratios are not directly measureable but they contain two units and that the order of the items in the ratio pair in a proportion is critical". There are good reasons why fraction-phobia is so commonplace.

An additional comment about one part of the preceding progression, namely,

$$\text{whole numbers} \longrightarrow \text{fractions} \longrightarrow \text{finite decimals}$$

would be appropriate here. The long division algorithm, a topic of intense controversy in elementary education, is of course part of the subject of whole numbers. A full appreciation of this algorithm cannot be acquired within whole numbers, however, because it is only when one tries to understand why fractions are finite or repeated decimals that the importance of the algorithm is revealed. Would that all elementary teachers had at least some understanding of the longitudinal coherence of this part of the school curriculum!

Fractions of course lead directly to the study of *rational expressions*.[15]

[15] More commonly called *rational functions* in mathematics.

Inasmuch as the teaching of fractions is confused, the teaching of rational expressions is also confused. It is not just that a small portion of school algebra depends on teaching fractions well, but that the whole of school algebra does. To children, the difference between whole numbers and fractions is that, while whole numbers can be understood in concrete terms, the understanding of fractions requires an element of abstraction. Consider the simplest of all number operations: addition. With whole numbers, the choice of a unit is usually so clear that, for example, no child would try to do the addition problem $3 + 5$ by combining 3 chairs with 5 apples. In his mind, he either fixes a "chair" as his unit and counts 3 chairs and then 5 chairs, or fixes an "apple" as a unit and counts 3 apples followed by 5 apples. Because for whole numbers the choice of unit is easily taken for granted, children coming to fractions run the danger of forgetting that such a choice must be consciously made at the beginning. For example, suppose they have to add $\frac{1}{3} + \frac{2}{5}$. They have to be explicitly aware of the fact that $\frac{1}{3}$ and $\frac{2}{5}$ refer to a third and two fifths of *the same unit*, respectively, before they can begin to think about the addition. If they fix the unit to be the area of a pie, then they have to carefully put two pieces of the *same* pie together: one piece is $\frac{1}{3}$ of the pie, and another is $\frac{2}{5}$ of the same pie. Then they need to think a little how to calculate the combined size of these two pieces of pie. This involves some abstraction. Children have been found to put $\frac{1}{3}$ of a small square and then $\frac{2}{5}$ of a bigger square together and try to estimate the combined size. Then of course they get stuck.

We can see from this simple example that some mental effort is needed for the understanding of the arithmetic operations with fractions. Nothing strenuous, but the need to give up one's complete reliance on the concrete is unmistakable.

Thus the first step towards abstraction takes place naturally in the study of fractions ([Wu 2001c]). A teacher fully aware of the demands of algebra would know that, rather than shielding her students from the intrusion of abstraction in their study of fractions, she should welcome it but introduce it slowly. She knows that in her teaching of fractions, symbolic computations and arguments for the general case — rather than just specific numbers one at a time — will have to gradually take center stage. Doing fractional computations *only* by the drawing of pictures or some concrete counting methods will no longer be sufficient (see [Wu 2001b], §8, for a fuller discussion). For example, one should not teach the division of fractions *only* as repeated subtraction by using simple fractions such as $\frac{2\frac{1}{8}}{\frac{1}{4}}$ as prototypical examples and by drawing pictures to see the number of $\frac{1}{4}$'s there are in $2\frac{1}{8}$. Let me emphasize that the drawing of pictures and similar methods are important for developing intuition about fractions and the division thereof. Nothing wrong with using these as an aid. But sooner or later, one must confront "invert-and-multiply" because this is the way to do division that works under any circumstance (regardless of how big the fractions may be) and which subsumes all the usual interpretations of division. Here is a brief demonstration. If we believe in the fact that fractions are a natural extension of the whole numbers, then before defining the division of fractions, we should try to understand the division of whole numbers. If A, B, and C are whole numbers and A is a multiple of B, then

$$\frac{A}{B} = C \quad \text{means} \quad A = C \times B.$$

In a reasonable presentation of school arithmetic, such an interpretation of the division of whole numbers should be emphasized all the way through grade 5 (cf. §4 of [Wu 2001a]). This then suggests that we *define* the division of fractions in the same manner: if A, B, and C are fractions, then

$$\frac{A}{B} = C \quad \text{is defined to means} \quad A = C \times B.$$

A simple argument then shows that there is one and only one fraction C that satisfies $A = C \times B$ when $A = a/b$ and $B = c/d$ are given, namely, $C = ad/bc$. It follows that we now have the usual invert-and-multiply rule as a *theorem*, i.e.,

$$\frac{a/b}{c/d} = \frac{ad}{bc}.$$

See for example equation (29) in §8 of [Wu 2001b].

Now we reap dividends from such a seemingly abstract discussion of division by applying the invert-and-multiply theorem to the problem above. We will show how to solve it in a far simpler way:

$$\begin{aligned} \frac{2\frac{1}{8}}{\frac{1}{4}} \quad &= \quad \frac{\frac{17}{8}}{\frac{1}{4}} \quad = \quad \frac{17}{8} \times 4 \\ &= \quad \frac{17}{2} \quad = \quad 8\frac{1}{2} \end{aligned}$$

But according to the definition of division, what we have proved using invert-and-multiply, namely, that

$$\frac{2\frac{1}{8}}{\frac{1}{4}} = 8\frac{1}{2},$$

means that

$$2\frac{1}{8} = 8\frac{1}{2} \times \frac{1}{4},$$

so that

$$2\frac{1}{8} = (8 + \frac{1}{2}) \times \frac{1}{4} = \left(8 \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right).$$

This says explicitly that $2\frac{1}{8}$ contains 8 of the $\frac{1}{4}$'s plus a remainder which is only $\frac{1}{2}$ of $\frac{1}{4}$.

Needless to say, the reasoning is general: what one obtains by invert-and-multiply always gives the number of multiples of the denomintor in the numerator, regardless of the size of the numerator or denominator. To illustrate, suppose you want to know how many $1\frac{5}{7}$'s there are in $48\frac{2}{3}$. You may not wish to solve this problem by drawing pictures! More simply, you compute instead:

$$\frac{48\frac{2}{3}}{1\frac{5}{7}} = \frac{146}{3} \times \frac{7}{12} = \frac{1022}{36} = \frac{511}{18} = 28\frac{7}{18}.$$

Then you know that there are exactly $28\frac{7}{18}$ of the $1\frac{5}{7}$'s in $48\frac{2}{3}$. And the reason? By the definition of division, the meaning of the quotient $28\frac{7}{18}$ is that

$$48\frac{2}{3} = 28\frac{7}{18} \times 1\frac{5}{7}.$$

So:

$$\begin{aligned} 48\frac{2}{3} &= (28 + \frac{7}{18}) \times 1\frac{5}{7} \\ &= \left(28 \times 1\frac{5}{7}\right) + \left(\frac{7}{18} \times 1\frac{5}{7}\right), \end{aligned}$$

which says that there are exactly 28 of $1\frac{5}{7}$ in $48\frac{2}{3}$, plus a remainder which is only $\frac{7}{18}$ of $1\frac{5}{7}$.

In general, if one understands fully what invert-and-multiply is about, then one can *correctly deduce* any and all interpretations about the division of fractions.

The saga of the infamous "invert-and-multiply" rule calls to mind a well-known story in Zen Buddhism.[16] A young monk was undergoing training to attain enlightenment. At the beginning, he saw a mountain and said: "This is a mountain". After some years of meditation, he looked at the same mountain again and said: "This is only an optical illusion and not a mountain." Then years passed and he finally found enlightenment. When he looked at the mountain again, he said simply: "Yes, this is a mountain."

It may be the same with invert-and-multiply: when all is said and done, it will be realized that invert-and-multiply *is* good mathematics, provided you understand division.

In any case, I hope I have made my point that a teacher who teaches algebra without making a conscientious effort to gradually increase the abstract reasoning and symbolic computation is not likely to give students the proper preparation for algebra that they deserve.

Let us take up another logical progression:

$$\text{similar triangles} \longrightarrow \text{trigonometry} \longrightarrow \text{calculus}$$

If I want to be picturesque about it, I would say that trigonometry is nothing but a shorthand bookkeeping system for various collections of similar right triangles. One's understanding of trigonometry would be very defective indeed if one fails to see that basic facts about similar triangles have to precede the definition of trigonometric functions. This fact is obvious, yet many teachers may not be fully aware of it, for two reasons. School textbooks as a rule do not clearly underline the fact that the the correctness of the definition of trigonometric functions depends on the properties of similar triangles. This harks back to the recurrent malaise of not emphasizing definitions. At the same time, college textbooks on trigonometry also tread lightly on this issue because they take for granted that such a logical dependence is known from high school. School teachers are therefore caught in a no man's land, and this is another obvious reason why we must rethink preservice professional development. The problem with school textbooks is actually more severe than I have indicated. There is one series which treats the tangent function before fully explaining similarity. Another defines trigonometric functions in such a convoluted manner that most likely the authors themselves are not aware of

[16] Contrary to common misconception, Zen Buddhism was strictly a Chinese creation, not Japanese.

the importance of similarity. And so on. You can picture in your mind the nightmarish scenario of teachers trying to do a Japanese style lesson study on the basis of such material.

Why is the connection of trigonometry to calculus important? One reason is that the use of trigonometry in calculus sheds light on trigonometry itself. Too often trigonometry is taught in school as a collection of definitions and identities which are disjointed, are unmotivated except by some simple-minded applications, and are all supposed to be equally important. Learning such a collection of facts is difficult, as anyone who has tried learning a language by going through a dictionary knows. A teacher who understands why the derivative of $\sin x$ is $\cos x$ and who has integrated a few trigonometric polynomials knows however that the Pythagorean theorem $\sin^2 x + \cos^2 x = 1$ and the sine and cosine addition theorems clearly stand out among all identities. The derivative formulas for sine and cosine also explain why the radian measure of angles is introduced in trigonometry,[17] namely, to insure that $\frac{d}{dx}\sin x = \cos x$. If degree is used instead, then one would get $\frac{d}{dx}\sin x = \frac{\pi}{180}\cos x$, and carrying around the factor $\frac{\pi}{180}$ would be too painful to contemplate. Furthermore, the derivative formulas for $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ explain why it is worthwhile for students to put up with the unpleasantness of learning the inverse functions and their graphs: we need to integrate algebraic functions and, when we do that, these inverse functions naturally appear. So all of a sudden, the disparate little pieces begin to take shape and make sense.

The preceding considerations suggest a way to prepare teachers in trigonometry so that they could achieve a depth of understanding for better instruction. We need to clarify for teachers that the central theorems are the Pythagorean theorem $\sin^2 x + \cos^2 x = 1$, and the sine addition theorem. (The cosine addition theorem is a consequence of the latter.) One way to underline the fundamental nature of the addition theorems would be to prove that, among differentiable function, $\sin x$ and $\cos x$ are the *only* ones satisfying the sine and cosine addition theorems and the initial condition of $\sin 0 = 0$ and $\cos 0 = 1$. A teacher who has gained such a perspective would be less likely to fixate on teaching the proof of identities for its own sake,

[17] It has always been a mystery to me why students accept the weird definition of radian measure without protest. Even if the arclength of the unit circle is brought in for motivation, it still begs the question of what is so wonderful about arclength. If we believe in our decimal system, should we not change the measure of a $360°$ angle to 10 rather than 2π?

but may instead choose to spend the time on giving more than one proof of the sine addition theorem and exploring its various implications, such as De Moivre's theorem. The latter is usually presented without much of an explanation. We can help promote this new thinking by discussing, in a teacher preparation program, how the addition theorems made the compilation of the trigonometric tables possible in the days before the advent of the computer, and how the tables were the bread and butter of every scientist in those days. We can discuss how some simple consequences of the addition theorems inspired Napier to discover the logarithm function and compile the logarithmic tables, another landmark in mathematics. We can also mention the cubic equation derived from the addition theorems which shows that a 60° angle cannot be trisected by ruler and compass. Other options include a discussion of the relationship between the addition theorems and rotations of the plane (how the fact that rotation by angle θ followed by rotation by angle ϕ is the same as rotation by angle $\theta+\phi$ leads to the addition theorems), and a qualitative description of Fourier series to disabuse teachers of the possible misconception that trigonometry is a purely geometric subject.

Finally, let us look at the logical progression

$$\text{proportional reasoning} \longrightarrow \text{linear functions without constant term}$$

I will introduce a piece of *ad hoc* terminology here to make life simpler. A function $f(x) = mx + b$ with constants m and b is of course a linear function. We are now interested in those f's with the constant term b equal to 0, so that $f(x) = mx$. I propose to call such a function a *special linear function*, so that we can rewrite the preceding progression as

$$\text{proportional reasoning} \longrightarrow \text{special linear functions}$$

The fact that problems about proportional reasoning are those modeled by special linear functions is of course easy to see.[18] What is surprising is the fact that, in spite of all the soul-searching that has gone into the teaching of proportional reasoning, the recognition that teachers should use their knowledge of special linear functions to inform their teaching of proportional reasoning does not seem to be widespread. For students, proportional reasoning is the laboratory in which they get to witness the raw data of linearity

[18] Nevertheless, it is well to point out that it is mathematically incorrect to say linear functions *illustrate* proportional relationship.

without being encumbered by the technical concept of a function. Teachers, on the other hand, must use what they know about linear functions to steer students in the optimal direction of learning. It follows that in preparing middle school teachers, we would want them to understand that

> proportional relationships are examples
> of special linear functions.

This means that, in teaching proportional reasoning, teachers should always keep special linear functions in the back of their minds. They have to isolate the key linear relationship, impress on students why it is important for the solution, and try to convey to students the essence of linear functions — without ever mentioning "function"— by plotting graphs, by compiling tables of values, or by whatever means that happens to be at their disposal.

I will use an example to illustrate why it is critically important for middle school teachers to realize that proportional reasoning is nothing but an elementary way to approach special linear functions without the symbolic notation and without the formal concepts of function and linearity. Consider the following typical problem in proportional reasoning:

> If a person weighing 98 pounds on earth weighs only 86 pounds on Venus, how much would a person weigh on Venus if he weighs 120 pounds on earth?

We first analyze this as a problem in algebra. If the weight of a person on earth is x pounds, let his weight on Venus be $v(x)$ pounds. From physics, we know that $v(x)$ is a special linear function: $v(x) = mx$, where m is some constant. This is the key step because the rest of the solution is simple. It is given that $v(98) = 86$, so $98m = 86$, from which $m = \frac{86}{98}$. Thus $v(120) = 120m = 120 \times \frac{86}{98} = 105.3$ pounds, approximately. The 120-pound earthling weighs about 105.3 pounds on Venus.

To solve this problem in the context of proportional reasoning, a traditional method is to say that if the 120-pound earthling weighs x pounds on Venus, then the ratio of 98:86 should be the same as the ratio 120:x, and therefore we set up the proportion $\frac{98}{86} = \frac{120}{x}$ and solve. The central issue is why these enigmatic "ratios" should be equal, but the explanation of this issue is usually not forthcoming in a typical classroom. Another way is to say that if a 98-pound earthling weighs 86 pounds on Venus, then a

1-pound earthling weighs $\frac{86}{98}$ pounds on Venus, and therefore a 120-pound earthling would weigh $120 \times \frac{86}{98}$ pounds on Venus. This reasoning has a lot of intuitive appeal except for the chasm separating the given data and the conclusion that a 1-pound earthling must weigh $\frac{86}{98}$ pounds on Venus. *Because we already understand the linear relationship between an object's weights on earth and Venus*, we naturally consider such a conclusion obvious. The danger of trying to convince students to make this intellectual leap *assuming a linearity relationship that they do not even understand* is that they would henceforth automatically assume everything is linear. In fact, this implicit urge to "linearize the world" is a virus that infects not only students but also textbooks and standardized tests. One manifestation is the common appearance of multiple choice questions of the type "if a sequence is 2, 5, 8, 11, . . . , what is the 27th term of this sequence?"[19]

Thus the challenge inherent in the teaching of this and related problems lies in explaining the information encoded in the special linear function $v(x) = mx$ *without* using the language and notation of functions. One way is to proceed as follows. Explain to students that the weight x of an object on earth and its weight v on Venus satisfy the relationship

$$\frac{v}{x} = \text{ a fixed number no matter what } x \text{ may be.} \tag{1}$$

This is the crucial fact, and this fact comes from physics. No one claims that this mysterious fact would be immediately embraced by young kids in middle school, and precisely because this fact is mysterious, *it is incumbent on the teacher to explain this fact fully and to make sure that students recognize its central role*. One may start with $x = 1$, $x = 2$, $x = 3$, etc. and ask what the "fixed number" in (1) might be in order that when $x = 98$, v is exactly 86. One may also plot graphs, and so on. Once this is done, the rest is easy: We are given that

$$\frac{86}{98} = \text{ this fixed number,}$$

and if the weight on Venus of the 120-pound earthling is v_0, we also know

$$\frac{v_0}{120} = \text{ this same fixed number.}$$

We thus obtain:

$$\frac{86}{98} = \frac{v_0}{120},$$

[19] The answer is not $2 + (3 \times 26)$, unless you have been told that the sequence proceeds linearly.

from which we get $v_0 = 105.3$ pounds, approximately, as before.

Notice that we have just solved the problem seemingly by the routine method of setting up a proportion and finding the missing-value. The difference lies in isolating the key concept embodied in (1), from which the proportion follows naturally. When this method is done correctly, it is efficient and the reasoning is easy to understand, but it is so only if the linearity relationship in equation (1) is clearly understood in the first place.

It is customary in the education literature to bemoan students' lack of understanding of proportional reasoning and their propensity to set up mindless proportions and solve the wrong missing-value problems. Before blaming the students, however, we should first ask if teachers and textbooks explain proportional reasoning correctly. The concept of "setting up a proportion" is a very elusive one, and unless great effort is devoted to — as in the Venus problem — explaining why equation (1) (or its counterpart in another situation) is true, students won't get it. At present, such an effort is not always there in most classrooms.

I want to make a point in this connection, which is a side issue in the present context but which is important in mathematics education in general. It is likely that many textbooks as well as teachers would choose not to explain equation (1) but rather leave it to students to figure it out on their own, on the ground that this is the kind of proportional reasoning that students ought to possess. The truth is that this equation cannot be deduced by mathematical reasoning alone: a fundamental fact about how gravity behaves is involved. Unless we expect every 6th grader to be well-informed in physics, or unless a teacher has taken the pain to explain the relevant physics ahead of time, the above Venus problem is *mathematically unacceptable* because it asks students to draw a conclusion without providing the necessary hypothesis (about gravity). We recall at this point one of the statements of §2 concerning the teaching of mathematics: "Whatever information is given to students, it should be one hundred percent sufficient to provide the basis for logical deductions in everything else that follows". The failure to observe this basic principle in the posing of problems in school mathematics may be one of the reasons why students do not grasp the concept of proportional reasoning.

The fact that this kind of oversight is commonplace can be seen by inspecting another typical problem in proportional reasoning:

> If I was charged \$1.30 in sales tax when I spent \$20, what would be the sales tax on a purchase of \$50?

In this case, the concept of *sales tax* must be clarified before the the problem can be solved. This concept is a challenging one for 6th graders because if $\$x$ is the sales price of an item and the sales tax rate is $m\%$, the fact that the sales tax $T(x)$ on this item is a special linear function $T(x) = (m\%)x$ is by no means obvious. (Here as elsewhere, $m\%$ means the fraction $\frac{m}{100}$.) It is a succinct mathematical formulation of an idea that is talked about intuitively but rarely with precision. For example, one can explain a 6% sales tax to a twelve-year-old as 6 cents for every dollar, and therefore 12 cents for every two dollars, 18 cents for every three dollars, etc. But what about the sales tax for 87 cents? \$6.11? \$285.49? It is not reasonable to expect a twelve-year-old to know the answers to these questions. In fact, it is not reasonable to expect an average adult to know the answers to these questions; the Internal Revenue Service recognizes this fact and the income tax form has an elaborate explanation of what a constant tax rate means. Therefore, if the precise meaning of "a sales tax of $m\%$" is not fully explained to students before asking them to do problems of this nature, we only have ourselves to blame if they cannot do it correctly. In any case, we have the obligation to explain to students that "a sales tax rate of $m\%$" means precisely that if T is the sales tax on an item of $\$x$, then

$$\frac{T}{x} = m\% \qquad \text{regardless of what } x \text{ is.} \tag{2}$$

Once this understanding is in place, the solution is simple. We are given that

$$\frac{1.3}{20} = \text{ a fixed number,}$$

and if the sales tax on \$50 is T, then

$$\frac{T}{50} = \text{ the same fixed number.}$$

Therefore,

$$\frac{1.3}{20} = \frac{T}{50}.$$

This missing-value is easily seen to be \$3.25. So the key point is that students must understand equation (2), and we should prepare middle school teachers so that they understand why they must explain to students this fundamental linear relationship.

At the risk of harping on the obvious, let me say that I do not object to the use of problems such as the two examples above in teaching middle school students about proportional reasoning. What needs to be stressed is that the extra-mathematical information crucial to their solutions must be explained very well and the implicit linearity of the function in question made explcit. To illustrate the importance of the last point, it suffices to note that if "sales tax" in the second problem is changed to "income tax", then the problem would no longer be one about proportional reasoning because — as every law-abiding citizen knows only too well — income tax does not go up linearly with income! Thus far, there is no indication that any effort is being made to inform prospective teachers (or school students) of these two key points.

It may be mentioned in passing that not every country regards proportional reasoning as the cornerstone of middle school mathematics education. I was made aware of this fact when I went to Hong Kong in 1999 to do professional development for middle school teachers, and I have subsequently been informed by Bill Schmidt that the same holds true in other countries ([Schmidt]).

5 Fractions

I will now make a case for the need of pre-service professional development to give careful and systematic treatment of fractions and *school* geometry. Although on a superficial level, only elementary and middle school teachers need instruction in fractions and only middle school and high school teachers need instruction in school geometry, in fact all teachers need instruction in both.

Let us begin with fractions. Like nothing else, the subject of fractions illustrates the quintessential difference between professional development and the teaching of standard mathematics courses in a university. From the perspective of university mathematics teaching, the subject of fractions is quite trivial. Two to three lectures in an abstract algebra class would routinely complete the discussion of how to extend an integral domain to its quotient field. This then includes a rigorous construction of the rational numbers from integers together with the justification of all the arithmetic operations. But can a teacher make direct use of this knowledge in any part of the school curriculum? Not likely. Fractions are taught between grades 2 to 7, but the kind of mathematical instruction on the arithmetic of fractions under discussion here takes place during grades 5 to 7 (or 6 to 8). Under the circumstance, a teacher cannot afford to indulge in abstractions about ordered pairs or equivalence classes, but rather must teach fractions by building on children's intuitive knowledge of whole numbers and their intuitive conception of fractions as part-of-a-whole. The teaching of fractions therefore requires a completely different starting point from the considerations of abstract algebra. A teacher coming out of an abstract algebra course is hardly equipped to tackle a subject which calls for a completely different line of logical development from what she is accustomed to in rings and fields. To put it bluntly, standard mathematics education on the university level does not fill the void created by the special needs of teaching fractions in schools.

Such an unhappy situation would not have existed if current school textbooks or professional development materials addressed the mathematical issues of fractions adequately. It would be fair to say that most of them do not. Teachers caught in this predicament either teach fractions according to the (bad) script, or try to fight it through by creating their own solutions. The latter method is in general not feasible because the amount of work involved is far beyond the call of duty. The university mathematics departments must live up to its obligations by teaching these teachers what they truly need. They must teach fractions in a way that teachers can use in their classrooms.

This is not the place to go through all the mathematical problems of a logical development of fractions for use in schools,[20] but the fundamental problem of defining a fraction is very germane to the main concerns of this article. We have to put whole numbers and fractions on the same footing as "numbers", and for grades 5 to 7, this difficult concept can be defined only if we are willing to stop short of being 100 percent correct. The most common definition that is used in this context is that a *number* is a point on the *number line*.[21] Of course this definition is circular in a strict sense[22]

[20] But see the Appendix of [Wu 2001b].

[21] More commonly called the *real line* in mathematics.

[22] As a purely technical matter, Ralph Raimi pointed out to me that it would not be circular if we build all of school mathematics on axiomatic Euclidean geometry.

but — compared with all other options — this is the preferred definition pedagogically because the number line is psychologically easy for children to accept. Moreover, on this basis a consistent and fairly concrete mathematical development of fractions can be built (cf. [Jensen] or [Wu 2001b]; similar presentations can be found in [Beckmann] and [Parker-Baldridge]). So until a better version can be found, this definition will have to do.

It seems to me that such a development of fractions should fulfill at least three key objectives. The first one is that it defines all the concepts precisely. This includes not only the obvious ones such as ratio, rate, and percent, but also the less obvious ones such as the sum, product and division of fractions. For example, teachers should be aware of the fact that if A and B are fractions but not whole numbers, the meaning of the symbol $A \times B$ has to be specified before the usual operations with multiplication can be discussed. Same for the division $\frac{A}{B}$.

A second objective is that explanations (proofs) are given to every statement about fractions. Although it has already been discussed quite extensively in §§2–3 that both definitions and explanations are important in mathematics and that explanations cannot be given without precise definitions, the subject of fractions is particularly notorious for the absence of both. At this point, we recall the two additional reasons why teachers must be able to explain everything about fractions. One is that what students learn about fractions in grades 5–7 would have to serve them for the remainder of school mathematics. There is no review from a higher standpoint in high school — the same way calculus is revisited in a course on analysis. Therefore students' knowledge of fractions must be robust, and this cannot be so unless teachers' knowledge is likewise robust. Incidentally, it is for the same reason that high school teachers should also receive instruction on fractions in a teacher preparation program. Students' difficulties with fractions do not go away in high school or college, so it would behoove a high school teacher to learn how to answer questions about fractions correctly.

Another reason for being especially careful with teachers' ability to give explanations is that grades 5–7 make a gradual transition to formal mathematics. Beginning with algebra, logical explanations should begin to dominate, at least in principle. Teachers of grades 5–7 must therefore convey the latter message by explaining every step in their teaching. But the subject of fractions is full of pitfalls in this regard because it is laden with many "interpretations" of each concept or operation. For example, one "interpretation" of a fraction $\frac{a}{b}$ is that it is a division $a \div b$, another one is that division of fractions is repeated subtraction, etc. If a teacher does not recognize that every one of these "interpretations" is in fact a *mathematical statement waiting to be proved*, then he or she would unwittingly contribute to students' mis-education.

I have heard often that students' failure to learn fractions is due to their lack of a "conceptual foundation of fractions". It has been difficult for me to pin down what this conceptual foundation is. Let me suggest in the meantime an alternate assessment: students cannot learn fractions when they are not provided with precise definitions and explanations. As a first step towards a remedy, we should make sure that our teachers can provide both.

A final objective is to make teachers aware of the *mathematical* similarity between whole numbers and fractions.[23] It is well-known that most people's mathphobia begins with the arithmetics of fractions. The reason — beyond the absence of clear explanations of concepts and procedures — may be that fractions are presented as something distinct from whole numbers (cf. the discussion in §4). Because whole numbers are the major source of children's mathematical intuition in grades 5–7, cutting them off from their source leaves them rootless. For this reason, we want our teachers to restore children's confidence in their source by pointing out (and proving it with deeds!) that fractions behave in many ways exactly the same as whole numbers (see [Wu 2001b]). For example, once the definitions of the four arithmetic operations for whole numbers have been properly reformulated — so that, for example, addition is the concatenation of line segments and multiplication is the area of a rectangle — the definitions of these same operations for fractions can be given that are *formally* identical to these reformulated definitions for whole numbers (cf. [Wu 2001a], §4).

Conventional wisdom in mathematics education holds that we need to improve the pedagogical aspect of the instruction on fractions, and copious research has been done in that direction. This may well be true, but if the *mathematical* aspect of the instruction on fractions continues to be generally as poor as it is now, it is not clear that pedagogy matters. Let us get mathematically better-informed teachers first.

[23] In other words, stress the fact that the integers are a subring of the ring of rational numbers.

6 Geometry and proofs

To say that the teaching of high school geometry is in crisis would be an understatement. This crisis comprises at least the following three components:

(i). Teachers' inability to construct geometric proofs.

(ii). Teachers' lack of understanding of axiomatic systems.

(iii). General confusion over the content of high school geometry.

Given the turgid, inflexible, and boring presentations of most of the current texts on Euclidean geometry, it is something of a surprise that some students do learn something about the subject. Obviously most don't. It must also be said that there are at present high school textbooks that pretend to be about Euclidean geometry but do not contain any proofs. It goes without saying that students do not learn how to construct proofs from the latter either.

Before going into any analysis of the problem presented by (i), it must be asked why, after so much space has been devoted to definitions and proofs, we must discuss geometric proofs yet again. The reason is that geometric proofs present a different challenge. Unlike proofs in algebra or about numbers, geometric proofs require that students translate a visual image into an analytic framework. Given the complexity of the visual input, the translation process is not simple. So it involves more than straightforward logical reasoning. One must be at ease in organizing one's thoughts in the visual domain before one can bring the analytic faculty to bear on a given geometric situation. It is therefore very difficult, not to say impossible, to arrive at a geometric proof of any theorem without rather extensive experience with experimental geometry. In other words, no intuition, no proof. Furthermore, almost all the working mathematicians I know are of the opinion that, at least in the initial stage, geometric intuition cannot be obtained indirectly or through computer software. It has to be acquired through one's fingertips: construction of models with one's hands, drawing with ruler and compass, etc. My contention is therefore that, somehow, the tactile aspect of the learning of geometry cannot be bypassed. From this point of view, the inability to construct geometric proofs in our schools would seem to be directly linked to the failure of our elementary and middle schools to provide students with what might be called a "geometric experience". It is for this reason that all teachers, and not just high school teachers, have to be aware of the problems with high school geometry: the trouble starts way back in elementary school.

An additional difficulty with geometric proofs is that, compared with other theorems about numbers or school algebra, they do not readily yield to a standard collection of techniques or algorithms. A student can get *really* stuck in a geometric problem, in the sense of not being able to get started at all. This aspect of geometry does not add to students' feeling of security. Finally, if students try to learn from written proofs about how to construct proofs of their own, they often do not realize — and their teachers may not tell them — that the order in which the steps of a written proof is presented is often the reverse of how a proof was first arrived at.

Prospective geometry teachers therefore have their job cut out for them: how to help students overcome these multiple obstacles. Because they were not so long ago high school students themselves and they had gone through exactly the same experience,[24] our teacher perparation program must retroactively provide the opportunity for these prospective teachers to acquire the necessary geometric experience. Lots of hands-on activities. The program must also give them the opportunity to analyze many geometric proofs by deconstructing them and putting them back together in the order they might have been conceived in the first place. They need many examples because, in the same way that one doesn't learn to speak a language after listening to three sentences, one does not learn how to construct proofs after reading three or four trivial ones. Indeed, the way geometry is usually taught, students are asked to write proofs with almost no model to learn from. We must change that in preservice professional development.

Just as a written proof is an organizational afterthought of an intellectual conquest, an axiomatic system for a subject such as Euclidean geometry is also an organizational afterthought. Axioms are the means of making possible an orderly and efficient exposition of a body of knowledge as well as exposing its logical structure. The purpose of an axiomatic development is to pare the number of starting points of a subject down to a minimum, and to demonstrate that this minimum is actually sufficient. Once the axiomatization is done, there is no reason for others to retrace it step-by-step, least of

[24] This discussion implicitly assumes that every prospective high school teacher has had a high school course in Euclidean geometry. Unhappily, this assumption may be invalid these days.

all young kids in high school (unless they want to become professional mathematicians one day). For the purpose of education, the greatest benefit that can be reaped from the study of an axiomatic development may be to learn how to make extensive deductions from a small collection of accepted facts (the axioms). Clearly it is possible to learn such deductions without having to start from ground zero, i.e., the axioms themselves. One can assume a collection of theorems, for example, and go on from there. This is in fact a very valid way to approach geometric proofs and has even acquired a name: *local axiomatics*. This approach will be discussed again presently.

Most beginners do not learn well when they are forced to start from the basic axioms of an axiomatic system. This is because below a certain level in the process to pare things down to a minimum, many intuitive ideas may have been hidden or thrown away. Unfortunately, school textbooks on Euclidean geometry do not make this clear. Worse, they usually try to develop the whole subject *strictly* from the level of the axioms themselves. This is a pedagogical disaster because the initial deductions are generically boring and the initial theorems tend to be insufferably trivial. As a consequence of the mathematical vapidness, geometry classes often degenerate into a game of "following the teacher's rules" (cf. [Schoenfeld]). This sad state of affairs may be what inspired some students to proclaim that "To convince someone you use reason, but to show something obvious you use proofs."

Preservice professional development in geometry will have to counteract this misconception of axiomatic systems. An easy first recommendation is try not be too stuffy about the formalism of proofs. Insisting on enunciating "AB=AB" together with the reason "Reflexive property of congruence" as a separate step in a two-column proof, for example, should be avoided. Beyond this simple observation, there are at least three different models worthy of consideration:

> (a) The series of Japanese textbooks [Japanese] for grades 7–9 contains a treatment of Euclidean geometry. In grade 7, it is entirely informal. No proofs, only (computational) "problems" and discussions of basic teminology. The words "theorem" and "proof" appear only halfway through the still somewhat informal discussion of geometry in grade 8. There, it is clearly stated that, from then on, a few facts will be used, and they include: equality of vertical angles, lines are parallel if and only if the alternate

interior angles are equal, congruence criterion SSS, congruence criterion SAS, and congruence criterion ASA. In effect it uses *local axiomatics* by assuming all these facts to be axioms. With such a good starting point, the theorems and proofs that follow are much more interesting. It continues in grade 9, but the tone remains relaxed. No hairsplitting and no ritualistic mumbo jumbo. The emphasis throughout is on geometric substance.

(b) The book by Lang and Murrow [Lang-Murrow] states an axiom on page 1, but contrary to this ominous opening, no proof or theorem appears until p. 30. The first thirty pages are devoted to hands-on experiments and geometric constructions without proof. Most importantly, the tone is as relaxed as the Japanese texts. Although not explcitly stated as such, local axiomatics is essentially employed. The axioms of Lang-Murrow are not the standard ones of Eucldiean geometry, though they are spiritually similar. Coordinates are set up from the beginning, which affords the use of algebra at the outset. Only half of this 400-page book is about traditional Euclidean geometry (including the discussion of area). The last half is devoted to dilation and similarity, volume, vector algebra and perpendicularity of lines, tranformations of the plane, isometries of the plane, and congruence.

(c) Appendix D of the California Mathematics Framework ([CA Framework], especially items 5 and 6 on p. 280) gives a demonstration of how, if you are stuck with an axiomatic development from the beginning, you can still minimize the pitfalls by a judicious application of local axiomatics and an emphasis on geometric substance rather than formal logical details. It is noteworthy for demonstrating the possibility of getting at interesting geometric facts in a short time. The exposition here is a bit more formal than the preceding two, but it is still more informal than most textbooks.

My suggestion is that a geometry course in preservice professional development should spend at least half a semester going over a good part of Euclidean geometry along the line of any of the above three approaches. This is one way to help teachers overcome their fear of geometric proofs. Sometimes the only way to get over a bad experience is to relive it under controlled conditions. The course can be made more attractive if a lecture or two are devoted to proofs (by local axiomatics if necessary) of some gems of

Euclidean geometry such as the nine-point circle or the Simson line (for the latter, see [Wu 1996a], Appendix A). The whole discussion should be capped by a brief examination of a complete set of axioms of the subject, such as the S.M.S.G. axioms (cf. Appendix D of [CA Framework]). Teachers will need this basic knowledge in the classroom. This examination of the axioms then provides a context to highlight the important role played by the the parallel axiom in the development of mathematics up to around 1830.

In 2001, it is not likely that a whole year of a school geometry course would, or should be devoted to classical Euclidean plane geometry. Although what constitutes the standard geometry curriculum in schools is still in a state of flux, it appears certain that what is contained in Lang-Murrow ([Lang-Murrow]) as described above is the rock bottom minimum of what every geometry teacher should know. Thus the remainder of the semester should be devoted to transformations in the plane and 3-space, the relation of isometry to congruence, the relation of dilation to similarity, and the effect of dilation on area and volume.

A semester course on geometry that gives prospective teachers the needed empirical geometric experience, revisits the axiomatic development of Euclidean geometry, and treats the basic geometric transformations is a very hectic one. In this case, offering a companion seminar in support of the course may be a necessity.

7 Pedagogical considerations

This section gives a very brief discussion of two pedagogical issues implicitly raised by this article. The first is how one should approach the teaching of pre-service teachers. For elementary teachers, there is at present a feeling that they have been so damaged by their K–12 experience — defective curriculum, defective textbooks, and defective teaching — that we owe it to them to treat them with kid gloves. Not having any experience with primary teachers, I will restrict my comments to teachers of grades 4 to 6. Those that I have encountered are generally eager to learn and are willing to work hard (cf. [Burmester-Wu]). The kid-glove treatment would seem to be hardly necessary. I found that if we show teachers by example and not just by words that mathematics can be taught according to reason, and that the teacher's whim or authority need not intrude, then they invariably respond positively. There is another school of thought arguing that for elementary teachers, one should teach them not only the mathematics of their classrooms, but *at the same time* also how children think about the mathematics. Again, I can only speak from my own experience. The teachers I observed usually had so much difficulty just coming to terms with the mathematics itself that any additional burden about children's thinking would have crushed them. We must remember that the mathematics of elementary school is not trivial. It is also for this same reason that I would argue against another common proposal about preservice professional development, which is that one should teach elementary teachers *only* the mathematics they teach. Should we do that, then the teachers would always be teaching from the outer edge of their knowledge and would hardly have the flexibility to maintain a dialogue with their children and be able to correct the latter's mistakes on the spot. In the long run, their teaching would rigidify and become formalistic. We certainly want to avoid that.

Teaching prospective teachers makes heavy demands on the instructor's pedagogical competence in addition to mathematical competence. This is because the teaching style of prospective teachers is more likely to be influenced by what they observe in their instructors' teaching than by what they are told. Unfortunately, the number of university professors who are both mathematically and pedagogically competent and are interested in professional development is not large. I was happy to learn in the National Summit of November 2001 that there would be workshops in the future designed exactly to address the problem of attracting more mathematicians to do professional development. Accomplishing the latter goal is every bit as critical as getting enough good school mathematics teachers.

A second pedagogical issue is one that perhaps has been inadvertently created in this article, namely, whether I believe content knowledge is all it takes to be a good teacher. The answer is no (cf. [Burmester-Wu]). What I do believe is that a solid knowledge of mathematics is the *sine qua non* of competent mathematics teaching. This said, it is well to note that pedagogical concerns were implicit in all of §§1-6. We have seen, for example, how a knowledge of special linear functions suggests a different approach to the teaching of proportional reasoning, how an understanding of division would help with the teaching of the invert-and-multiply rule, or how a knowledge of calculus can change the emphasis in a trigonometry lesson. If one agrees that these are fairly typical situations in a school classroom, then it is easy to also agree that it is in general difficult to discuss pedagogy *per se* without reference to mathematical content. In addition, I believe that pedagogical considerations make sense only after teachers are at ease with the content. Thus a method course not only needs to assume a sound knowledge of mathematics, but should, as a consequence, address specific issues arising from teaching nontrivial mathematical topics. In my own work with teachers, pedagogical discussions took place only after teachers had achieved the mastery of the relevant content ([Burmester-Wu]).

I will close this discussion of the preparation of mathematics teachers by stating a basic conviction of mine:

> In mathematics, content guides pedagogy.

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