The joy of lecturing — with a critique of the romantic tradition in education writing

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1 A sage on the stage speaks his mind

A cornerstone of the current mathematics education reform is the recommendation that teachers should cease being "the sage on the stage", but should instead assume the role of "a guide on the side".[1] Lecturing is discouraged; direct instruction is passé. Students should be working in small cooperative groups to discover the mathematics for themselves, and the instructor should be merely providing guidance on the side. Indeed, "students frequently working together in small cooperative groups" is second among what an eminent educator considered to be the five preeminent characteristics of the present reform effort ([6, p. 105]).

It appears to me that this rejection of the sage-on-the-stage method of

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[1] Although this "sage-on-stage and guide-on-side" dictum has been in existence since the late eighties, it appears difficult to find a precise reference to it in the literature. One place where it is mentioned unambiguously is footnote 15 on p. 17 of [5].

instruction is unjustified. There are situations where lectures are very effective, and in fact there are even circumstances which make this method of instruction mandatory. Furthermore, in recommendating the guide-on-the-side strategy, educators should have been more forthcoming about its limitations so that teachers can better decide for themselves whether or not to follow such a recommendation. The purpose of this appendix is to amplify on these remarks. Although there are many alternative methods of instruction other than lectures, I shall limit the present discussion to the guide-on-the-side format on account of its favored status in the current reform.

It may be assumed that a person who rises to the defense of lectures must be someone who has never taught any other way; it would not be unnatural to go even further and conclude that the only way he can teach is by giving lectures. Not so in this case. Although I have been lecturing in the classroom for all thirty-three years of my teaching life, outside of the classroom I rarely give lectures in the sense of systematically presenting a body of knowledge. When undergraduate students come to my office with questions, for example, I do not believe a short lecture by me giving complete answers would do any good in an overwhelming majority of the cases. Instead I try to engage them in a dialogue and employ the Socratic method to expose for their own benefit the gap in their understanding that led to their questions.[2] In other one-on-one situations, I also do not lecture. I have given reading courses to undergraduates, and in such cases, I make clear that learning can only be achieved by the student and that all I can do is to nudge him in the right direction and offer help when absolutely necessary. The student must do all the work and my contribution is essentially limited to asking key questions when we meet. It is the same when I find myself tutoring high school students on occasions. No lectures. The only thing I insist on is that no time limit be placed on any of the tutoring sessions: once we start on a topic, we stay on course until it is finished regardless of how long it takes. For a later purpose, let me describe one specific example of my tutoring experience.

I once had to teach someone the division algorithm for polynomials. I started by remarking that it was just a glorified version of the same algorithm for integers. I asked if he knew the latter (yes), and if he could prove it (no). So I suggested a proof by induction of the division algorithm for given positive

[2] Unhappily, most students are only interested in getting simple answers and getting out of my office as fast as they can. My attempt at fostering genuine education only results in bad student evaluations for my “unfriendliness”.

integers a and b in the form $a = qb + r$, $0 \leq r < b$. It took a while for him to decide, with some help from me, that one could attempt an induction on a, but in due course he succeeded in writing down a complete proof. Next came the polynomial version $f = qh + r$, $0 \leq \deg r < \deg h$. I asked him whether he could imitate the case of integers. It took some time for him to realize—again with some help from me—that $\deg f$ could be used for induction. However, he immediately saw that the usual induction step of "$P_{n-1} \Rightarrow P_n$" was of no use in this situation. At that point, it was time for me to step in to teach him about complete induction in the form of "$P_1, P_2, \cdots, P_{n-1} \Rightarrow P_n$". Then I let him figure out how to use it to prove the algorithm. Getting an appropriate q to start off the induction argument was not easy for him. While I saw the frustration, I left him alone because the frustration has to be part of the learning process. Finally he got it done. The whole session took something like two hours. I had no doubt that he really learned the algorithm through this tortuous process, and it is likely that for most students this is the only way to learn it. But could I teach in any fashion remotely resembling this in the usual junior level introductory algebra course? Absolutely not. In such a course, the polynomial algorithm merits a discussion of about 25 minutes. If I spent two hours to teach it, I would be fired for pedagogical turpitude,[3] and rightly so.

In an ideal world, I would like to teach all my classes the same way I teach my students in a one-on-one situation. But this is a dream largely unrealized except during the extra problem sessions I offer my students in upper divisional courses. With a sparse attendance and little time pressure, I can afford to let the students dictate the pace and the direction of the discourse half of the time. Otherwise, I find the obstacle of time-constraint almost impossible to overcome, and this constraint will be a recurring theme of this article.

2 The hows and whys of lecturing

No matter who says what, lecturing is *an* effective way of teaching in a university—and for that matter in grades 7-12—so long as our education system stays the way it is. Such a bald statement requires a careful description of its underlying assumptions, and I will proceed to do that. I assume

[3] In my contract with the University of California, there is a clause that says I could be fired for moral turpitude.

that:

(i) the instructor is mathematically and pedagogically competent,
(ii) only 12 years are devoted to public school educations and 4 years to college education,
(iii) after 12 years of school education students should be competent enough to function as useful citizens in society, and after 4 years of college students should be competent enough to start graduate work in their chosen disciplines, and
(iv) our education system continues to be one for the masses rather than for a select few, so that each teacher or professor must teach *many* students in each course.

Whatever I say below will apply equally well to teaching in grades 7–12, but for the purpose at hand I will specifically discuss only college teaching. The sage-on-stage style of lecturing has come under attack in the current mathematics education reform, but the attack seems to show no awareness of the basic constraints of (i)–(iv) above. For example, if the amount of material to be covered in a course can be greatly reduced (thereby violating (iii)) and students are expected to spend 8 years in college (thereby violating (ii)), then we can all safely abandon the lecture format and engage in a wholesale application of the guide-on-the-side philosophy in our teaching. To put this comment in context, let us continue with the above discussion of the polynomial division algorithm by considering it specifically as a topic in a junior level algebra course.

The purpose of a mathematics course is, naturally, to further students' knowledge of mathematics and logical reasoning skills, but there is also a practical aspect along the line of assumptions (ii) and (iii) above. Thus a junior level algebra course should enable its students to acquire a minimum mastery of the most basic techniques and ideas in algebra: the concepts of generality and abstraction, the concept of mathematical structure, and certainly the basic vocabulary of groups, rings and fields. The details may vary and the broad framework is susceptible to a certain amount of stretching (cf. [12]), but ultimately the course must serve to fulfill assumptions (ii)–(iii). Students coming out of such a course should be ready to embark on more advanced courses in mathematics and the sciences, deal with the basic technical issues in industry, or at least be able to look back on the high school materials of polynomials, triangle congruence, or fractions with renewed understanding. Given that such a course would typically meet for only 45 hours (a semester), class time must be used wisely. This is the reason why only half a lecture can be allotted to the explanation of the polynomial division algorithm.

Learning mathematics is a long and arduous process, and no matter how one defines “learning”, it is not possible to learn all the required material of any mathematics course in 45 hours of discussion. To make any kind of teaching possible, professors and students must enter into a contract. The contract can take many forms, but the following would certainly be valid: the professor gives an *outline* of what and how much students should learn, and students do the work on their own outside of the 45 hours of class meetings. Lecturing is one way to implement this contract. It is an efficient way for the professor to dictate the pace and convey his vision to the students, on the condition that students would do their share of groping and staggering towards the goal on their own. It should be clear that without this understanding, lectures would be of no value whatsoever to the students, especially to those who expect to come to class to be spoonfed all the tricks for getting an A in the course. In advocating “guide-on-the-side” over “sage-on-the-stage”, did educators weigh carefully the intrinsic merits of the lecturing format against the apathy of the students before putting the blame squarely on the former? Or is this simply a case of expediency over reason, because there are hidden forces at work which the educators did not bring to the table? Have they perhaps re-defined learning without telling us what they *really* have in mind? If so, then this is an illustration of what I call the *romantic tradition* in education writing: unpleasant details are left to the imagination because they might interfere with the attractiveness of the advocacy in question.

Let us return once again to the polynomial division algorithm for a more detailed discussion. *Lecturing can take many forms.* In the 25 minutes or so allowed for the teaching of this topic in a junior level mathematics course, one way to approach it in the classroom is for the professor to indicate briefly the long process of possible trials and errors in arriving at the correct proof and to discuss the main points of the proof in precise terms. Most of the 25 minutes would therefore be spent on explaining the details of the induction on the degree. In order to understand such a lecture, students will have to retrace *on their own* the steps of the trials and errors omitted in class (see the discussion of the tutoring session in the preceding section). There are other ways to handle the lecture. For instance, if the textbook is reliable and readable, the professor may decide to let the students read the polished final account at home but use the class time to go through, as much as possible, the tedious learning process in the allotted 25 minutes. Or, the allotted 25 minutes of class time could be used to go through the initial segment of the trail and error process and, following which, students are told what they need to do in order to complete the investigation. For this kind of teaching to work, the students would have to be very mathematically mature. There can be other variations too. No matter. The fact remains that if we abandon lecturing and the underlying assumption of the sharing of labor between professor and students, and insist that the whole learning process (guided discovery, trials and errors, etc.) must take place within the 45 hours of class time, then the amount of material that can be covered in each course would be reduced by half if not more. Unless we stretch college education to 8 or 10 years, this is not a realistic option.

Last semester (Spring 1998) I taught a one-semester introductory analysis course, and I volunteered to give two additional problem-solving sessions each week. Attendance in these sessions was optional, and therefore poor.[4] Since in these sessions time pressure was not a serious concern, I could often indulge myself in my teaching method for private tutoring (see the preceding section). I did not insist on any kind of cooperative learning, but I let them decide for themselves if they wanted to discuss with their neighbors. Once I had about seven students, and I asked them to prove that the function $f(x) = \sqrt{x-5}$ is continuous at $x = 10$ by the use of ϵ and δ. Of course this requires a rationalization of the expression. The trick in question happened to have been discussed briefly several weeks before in the context of the limit of sequences, but it would appear that none of them had any recollection of it and, in any case, they could not make the connection.

I walked around the room, talking to each of them trying to coax at least one of them to come up with a reasonable plan of attack. After more than ten minutes of futility, it became clear that they had to be told. So I mentioned the word “rationalization”, and one student immediately caught on. A few more explicit hints later, all of them knew what to do. The actually doing, needless to say, took a while at that stage of their mathematical development. I looked at each student’s work and literally guided the hands of a few of them. After more than 15 minutes, they all got it done. Then I asked one of them to go to the board to give a complete exposition, and I followed with some general comments, partly to point out the pitfalls along

[4] I have volunteered these problem sessions often, and attendance has always been poor. Typically about 15-20%. I wish to let this fact be known.

the way and partly to bring closure. They probably learned something from the experience, but it must be pointed out that it took almost the whole 50 minutes of class meeting. At the risk of harping on the obvious, whatever might be the educational benefits of this way of teaching continuity, an introductory analysis course taught entirely—or just *frequently*—this way could hardly get off the ground.

There is another aspect to lecturing that deserves to be discussed. Lecturing allows the professor to share his insight with students beyond what is found in the textbooks. Again allow me to offer an example from my personal experience. Each time I teach calculus, I go through what I have come to call the "catechism of π". Most students believe they know what π is. To my question of "what is π?" usually comes the reply "circumference divided by diameter". So I ask "what is circumference?", the quick rejoinder of "2π times radius" is usually followed by nervous tittering. They know they have been had. Sometimes the catechism replaces "circumference" by "area", but the result is of course the same. Some years ago, I decided to address this issue directly by defining for them, right after the discussion of arclength, the number π as half the circumference of the unit circle:

$$\pi \equiv \frac{1}{2}\left(2\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}dx\right),$$

and then proceeded to *prove* for them that with this definition of π, the area of the unit disc is actually equal to π, *i.e.,*

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}dx = 2\int_{-1}^{1}\sqrt{1-x^2}dx.$$

This requires an integration by parts argument. Could I have guided my students on the side to the final conclusion? For 20% of them, maybe—that is just my guess— and only if I had two lectures at my disposal. But I had only half a lecture (25 minutes) to work with, because in the remaining 25 minutes, I also showed that the circumference of a circle of radius r is $2\pi r$, defined the *radian* measure of an angle correctly for the first time, and computed the arclength of a segment of a parabola together with an explanation of why Archimedes could find the area enclosed by the segment but not its arclength (he had no logarithm function, only polynomials). Such a pace is normal for a calculus course at Berkeley.

Now I do not wish to give the impression that each time my colleagues and I give a calculus lecture we always have this kind of interesting information to offer. How can this be true when so much spade work must be done in a basic course of this nature? Nevertheless, it is no stretch of the imagination to say that most of these lectures routinely carry elementary insights about mathematics which comes only with years of working at it. I see no reason why students cannot profit from such insight by making an effort to understand the lectures instead of adamantly resisting them by coming to class unprepared, or for that matter, leaving it without making an effort to understand it later. The teaching of calculus varies with each university, but to any conscientious lecturer, the preceding account must resonate to some extent. We hear that lectures are a relic of the past. Does this mean that it is improper for students to pick up essential information because they *have to* "discover" it by themselves? Or is it the case that, since students no longer want to put in the strenuous effort to learn, the universities must henceforth resolve to teach either the most simplistic aspects of mathematics or only the smallest possible amount consistent with the guiding-on-the-side philosophy?

For advanced (upper division) courses in mathematics, the professor's understanding and vision of the subject are even more important in providing proper guidance to students—recall that we are assuming a large amount of material is supposed to be covered in each course (assumptions (ii) and (iii) at the beginning of this section). This is especially true in view of how textbooks are written these days (cf. [12] again). Lecturing is not the only way to provide this guidance but, until there is data to prove otherwise, it is one way of doing it. However, since the lecturing format is most heavily criticized in the context of the teaching of calculus, let me continue with the example of π and discuss calculus lectures. It has been said that the typical calculus lecture is *worthless, because it has virtually no conceptual development in it, is boring, and focusses mostly on techniques.* If this a judgment on the average calculus instructor's pedagogical or mathematical deficiency, then it has no bearing on our present discussion of lectures *per se.* On the other hand, learning about a correct definition of π and understanding for the first time how π enters into the circumference or area formula certainly gives a good account of the conceptual development in mathematics, makes interesting topics for students, and convincingly demonstrates how technique is inseparable from conceptual understanding in mathematics. Such being the case, it is clear that we have not even begun to exhaust the potential of lecturing. The sage on the stage still has work to do.

3 Time-compressed instruction

In the preceding sections, I have repeatedly emphasized the *time-compressed* nature of classroom mathematics instruction. In order to learn what is taught in class, students must be willing to spend two to three times the amount of time by themselves. For example, a survey conducted by the PDP (Professional Development Program) unit on the Berkeley campus in the 1980's shows that those students who got A's and B's in freshmen calculus spent an average of 10 to 14 hours on the course material per week outside of regular class meetings. To put these figures in context, since all the calculus courses are only 4 units each, conventional wisdom would have students spend only 8 hours per week instead of 10 to 14 hours. As another example, a recent article ([10]) makes the following comparison between the work habits of Japanese and American school students:

> Another after-school activity that occupies the time of adolescents is homework. Great emphasis is placed on homework as the basis of the excellent performance of Japanese adolescents in mathematics and science. In a typical survey, therefore, one might ask high school students how many hours they spend doing homework each day. The answer often given by Japanese students is unexpected: none. Only by pursuing the topic further does the actual state of affairs become clear. Additional discussion and questioning reveals that homework is often not assigned, but high school students are expected to spend several hours a night reviewing the day's lessons and anticipating the lessons for the following day.
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> It is increasingly common in both middle schools and high schools in the U.S. that homework is done in school and simply represents work that teachers expect to be done before the next class meeting. The apparent lack of homework assignments was lamented by American parents and teachers. Parents questioned how their children could complete their homework during school hours, a practice very different from what they remember of their own school days. Teachers were concerned about the tendency of students to equate homework with studying; if there was no homework assignment, there was no studying.

Would it be fair to conclude from this that the bashing of lectures in the U.S. is a direct consequence of the infusion into our campuses students who are rarely asked to work outside of class all through K–12? In an inspiring address commemorating the centennial of the *American Mathematical Monthly* [11], Herbert Wilf bluntly stated: "In recent years, we have witnessed serious decline in the demands that we make on our students for intensive and solid intellectual achievement in mathematics. When we feed them more baby food every year, we thereby become accomplices to their intellectual softening." Wilf did not concern himself with the abolition of lectures, but he might as well have.

There are at least two special features about mathematics that dictate the time-compressed nature of mathematics instruction in the classroom: it is *cumulative* and it is *precise*. By cumulative, I mean that at any given point of a mathematical exposition, it is virtually impossible to understand what is taking place without first acquiring a thorough understanding of all that has gone on *before* that point. The failure to confront this rather brutal fact—in a mathematics class, once behind, forever behind—may be the single factor most responsible for the undoing of our mathematics students. The precision of mathematics stems from its abstract nature. Whereas even in a rigorous discipline such as physics, a photograph or a measurement by a laboratory equipment can render verbal explanations superfluous, the basic concepts of mathematics reside only in the realm of ideas and therefore must be meticulously described. Students must learn to concentrate fully on *every* word that is used in the description, or they run the risk of missing the point entirely. We all remember how as students we had to struggle with the seemingly innocuous quantifiers "for every" and "there exist" in the definitions of limit and continuity. And that is only for starters. The precision in mathematics *is* unforgiving indeed.

These two features make it difficult for students to learn from mathematics lectures if they are unwilling to also invest time and energy before or after the lecture for this purpose. In a videotape ([1]) made available by TIMSS (Third International Mathematics and Science Study) in 1997, two Japanese teachers were shown to give lessons—unequivocally based on direct instruction—with the rapt attention and active participation of their students.[5] Now that we have the preceding account ([10]) of the kind of preparation Japanese students routinely make before coming to class, we are

[5] In particular, no cooperative learning there.

finally in a position to understand why the teaching of mathematics in Japan achieves such good results and why their students always score so well in interntational tests. The last time I checked, the slogan of "guide-on-the-side but not sage-on-the-stage" has not been aggressively promoted in Japan.

4 The importance of being honest

The folk wisdom that there is no free lunch in this world seems for some reason to be missing in current education writing, and this fact may be the genesis of the *romantic tradition* mentioned in §2. Wonderful new prescriptions for ailments of long standing in education are given on a regular basis with nary a hint of the likely detrimental side effects. On the K–12 level, for instance, "real-world" relevance of mathematics has been trumpeted as the salvation of the school mathematics curriculum *without* the caveat that unless this is done in moderation, the abstract nature of mathematics as well as its internal coherence would be jeopardized. Sure enough, the worst fears were realized in most (if not all) of the recent school mathematics texts, which emphasize "real-world" relevance (cf. p. 1535 of [13] and the references given therein).

The advocacy of the abolition of lectures as we know them is another case of promoting an idea without any explicit warnings of the possible losses and gains. For example, a more balanced approach to the subject of lecturing might begin by listing its strengths, its weaknesses, and the range within which it would be effective. A summary of the preceding discussion would include the following among the strengths of lecturing:

> (a) It allows the instructor to set the pace of the course. This is an important consideration if the basic parameters of school and college education as we know them are to remain intact. See assumptions (ii) and (iii) of §2.
> (b) It allows the instructor to share his insight into the subject with students. If we still believe that education is the process of passing the torch from generation to generation, this too is an important consideration.

A side benefit of the lecturing format, one that has not been discussed thus far, is that it forces students to stretch their concentration spans. In these days of MTV when everything is interactive and instantaneous, such a beneficial effect should not be dismissed lightly. In fact, one can speculate on the possible correlation between the onset of the computer-age and the dissatisfaction with lectures. As to the weaknesses of lecturing, the most serious is that, unless students are willing to do their share of the work outside the class and meet the instructor halfway, lectures are a waste of time. It is possible that this aspect of the lecturing format has never been made explicit to a large percentage of our college students, and the dismal student performance in mathematics courses is the result. The guide-on-the-side advocacy then becomes a facile, one-dimensional response to a multifaceted challenge. One reason why lecturing has been the accepted mode of instruction in most universities for so long is probably the assumption that students are there to work. Are we at the dawn of a new era when even such standard assumptions must be re-examined? Perhaps universities can no longer survive as institutions of higher learning but must transform themselves into "caring, nurturing", glorified high schools.

We now come to the guide-on-the-side method of instruction which, as mentioned, means the guided discovery method *in the context of cooperative learning*. What are its implicit assumptions and what are its strengths and weaknesses? By transferring what used to be activities outside of class into the classroom, the discovery-via-cooperative-learning pedagogy tacitly assumes either that students can no longer be trusted to do their share of the work or that they are incapable of doing it. The great advantage of this method of instruction lies in its seeming ability to make mathematics accessible to a much wider audience than is possible in the lecturing format.[6] The slower students who do not wish to put much energy into a mathematics class would certainly find participating in cooperative learning more congenial than listening to lectures. On the debit side, guided discovery and cooperative learning slow down the pace of a course, at least by half. One may surmise that the authors of some textbooks which advocate this particular pedagogy are well aware of the attendant loss of class time, and therefore deliberately set out to cut down on the more mathematically substantive topics. Thus we find calculus texts which do not even present the proof of something as basic as the Fundamental Theorem of Calculus (cf. [7] and [9]). Another drawback of this particular pedagogy has also been discussed: a guide-on-the-side has fewer occasions to share his vision or insights with

[6] But by no means to ALL students. I will not reproduce here the oft-repeated anecdotes about how some members of a study group sit and do nothing while one or two members take charge and do all the discoveries for them.

the students than a lecturer. Those who would otherwise profit from the knowledge and experience of their instructors end up being short-changed by this pedagogy. If we look past the heroic efforts of a few extremely talented instructors, it would be fair to say that students in currently advocated programs of guiding-on-the-side typically learn the details in a small area but not acquire much of a perspective overall.

In this context, an additional comment about the possible omission of topics in a guide-on-the-side classroom may not be out of place. It is a fact that American high school graduates are among the least mathematically knowledgeable compared with their counterparts in nations that did well in TIMSS (cf. [2]–[4]). We can also verify directly from our own collective experience that American students are generically the least prepared among our graduate students. Would it not be fair to say that our undergraduate as well as K-12 mathematics curricula are already down to the bone and have no more fat to be trimmed?

The preceding discussion of lecturing and its common alternative is by no means exhaustive, but even this much critical analysis would have been beneficial to the current debate on the mathematics education reform. For instance, where in this advocacy for guided discovery in the classroom do we find an explicit reference to the underlying assumption about the students' unwillingness or inability to work on their own? (Consult [11] again.) Or is it the case that this assumption is a misapprehension? These issues should have been openly debated long ago so that teachers who opt for one or the other pedagogy would have the benefit of knowing what they are getting into. It would be wrong to say that this advocacy has produced nothing of value thus far. Quite the contrary. Because of this advocacy, some of us who had to struggle to become mathematicians—and have always assumed that all students must know the need to do the trials and errors on their own—have been awakened to the fact that we must *tell* the students of this need or even *demonstrate* to them this need by use of examples. But given the human tendency to oversimplify, the danger of a passionate advocacy in a subject such as pedagogy—which is far from a hard science as of 1998—is that blind acceptance and a reckless pursuit would inevitably follow. The classic dictum that if a little bit is good then a lot must be better unfortunately applies only too well in this situation.

Lest this article sound like a defence of the status quo of the lecture format, let it be said—however briefly—that perhaps the quality of some lectures does raise legitimate concerns. There are lecturers who fail to observe the basic etiquette of lecturing (cf. §§1.6 and 2.13 of [8]), and there are also those who still cling to textbook-writing-on-the-boards as a legitimate form of lecturing in spite of the present super-abundance of adequate textbooks on almost every standard topic. For lecturing to survive, the practitioners of this art must continue to be vigilant (see assumption (i) of §2). Nevertheless, the overriding fact remains that the current discussion of pedagogy fails to meet the most basic requirements of scholarship: any advocacy should state clearly its goal, its benefits, and its disadvantages. In this light, the advocacy of the guide-on-the-side pedagogy has been presented more like an info-mercial than a scholarly recommendation. It is all good and nothing bad could possibly come of it.

In the field of medicine, the FDA has made the listing of the precise range of applicability and the side effects of each drug mandatory. Would it be too much to ask that the same consideration of fairness be also extended to teachers so that all education writings are always accompanied by an analysis of the limitations of a particular proposal, including its drawbacks and the conditions under which it would not be applicable?

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