Must Content Dictate Pedagogy in Mathematics Education?*

H. Wu

May 23, 2005.

Department of Mathematics, #3840
University of California
Berkeley, CA 94720-3840
wu@math.berkeley.edu

Contents

*An expanded version of a talk given at California State University at Northridge on October 27, 2004. I owe a major reorganization of of this article to Jim Milgram and Ralph Raimi, and an important improvement to Jim Stigler. It is also my pleasure to thank Heidi Boley, Al Cuoco, Ed Dubinsky for corrections.

1 Issues Related to Content Knowledge

All of us share the common goal of improving mathematics education. Where we may differ is the method we employ to achieve it. The important role of content has slowly moved to the forefront of mathematics education in recent years due to a confluence of many factors, and the title of my lecture is meant to be provocative. Because mathematics is the underpinning of every aspect of mathematics education, including curriculum, pedagogy, and assessment, I should begin by limiting the scope of this talk: I will only discuss the teaching of mathematics in grades 5–12.[1] Please note that I am not at all implying that what I am going to say is not applicable to curriculum or assessment, or that it is false in K–4, but I am trying to avoid the pitfall of being mile wide and inch deep, to borrow the well-known phrase of Bill Schmidt. The exclusion of K–4 in my discussion of teaching is due to the fact that, in the early grades, the role of content knowledge tends to be not as dominant in affecting the quality of mathematics teaching. That said, let me remove the last vestige of suspense from my talk (if there was any to begin with) by answering the question of my title right away: *yes!* I want to drive home the point that sound pedagogical decisions can only be based on sound content knowledge.

I should hasten to add that I did not pose this question in a vacuum. Rather, it is an observation of mine that in the mathematics education mainstream of the past ten or fifteen years, there is an alarming trend which may be called the mathematics-avoidance syndrome. Proposals for educational improvement, be it in curriculum, pedagogy, or assessment, tend to skirt the importance of content. I do not want you to misunderstand me: everybody talks about *content* nowadays, but there is a difference between talking and doing. My observation is that, with a few exceptions (see for example [Ball] and [Ma]; you must know that in any sociological discussion, every statement would likely admit a small number of exceptions), people in mathematics education tend not to place the importance of content front and center.

Because some of you would be mightily annoyed by my last statement, I would like to ask you to withhold your anoyance until you hear the evidence I am going to present to you. Your annoyance may be due to a cultural or linguistic misunderstanding of my message. Perhaps it is not clear what I mean by placing "the importance of content front and center". This kind of misunderstanding has happened before. For example, back in 1997, I was on the Advisory Board of the California Mathematics Project (CMP).

[1] I could have said 4–12, but I'd rather play it safe.

After visiting four of the sites, I wrote a report and concluded that those sites (and most likely CMP as a whole) did not do much in the way of increasing teachers' content knowledge, especially elementary teachers' content knowledge. The accusation caused an uproar because it was thought that if a site spent time doing mathematics, *any* kind of mathematics, then it had to be contributing to teachers' content knowledge. I had assumed as obvious the fact that, in this context, "content knowledge" could only mean the mathematics teachers teach in the classroom, but I was wrong. Only when I looked back some years later did I begin to understand what the uproar was all about.

I believe most people had no idea back then that there is a nontrivial body of *mathematical* knowledge in arithmetic and beginning geometry that should be taught to elementary and middle school teachers in a serious and systematic way. Pre-service professional development for teachers is, as a rule, inadequate on two fronts. First, it does not directly address what teachers actually need in the clasroom, and second, it presents mathematics as a collection of facts but not as a coherent whole, where the coherence comes from logical reasoning. The in-service professional development in CMP (or, at least what I got to observe) tended to ignore both kinds of inadequacy and chose to do "fun" topics such as Eulerian circuit or chi square test. It did not make an effort to help teachers acquire an understanding of the mathematical coherence of what they teach in the classroom. I regret that in those days I did not see how to explain to the CMP community my perception of what had gone wrong in mathematics professional development. It was only much later when I started to do inservice professional development myself and made an effort to teach teachers the mathematics I believe they need (a small part of this can be seen in [Wu2] and [Wu3]) that I could demonstrate by example what I had in mind about "increasing teachers' content knowledge". I will come back to this point later in this talk. But I hope that in like manner, if you are willing to keep an open mind and listen to my parade of facts about the education establishment's lack of emphasis on teacher's content knowledge, you too may come to agree with me.

Let us take a brief look at some of the major ideas regarding the teaching of school mathematics in the past fifteen years that lend credence to this syndrome. Around 1990, the California Department of Education (CDE) decided that the way to improve mathematics education was to change pedagogical techniques and make small group learning and the discovery method the centerpiece of mathematics instruction in every classroom. The fact that teachers in California (and elsewhere) were in dire need of better content knowledge was not mentioned. I must add that CDE's advocacy was nothing more than a mildly radical interpretation of a prevailing national trend. With hindsight, it is clear that an emphasis on pedagogy in an environment where teachers were ill-equipped to handle the content was a recipe for disaster, all the more so because the demand put on teachers' knowledge of mathematics by the discovery method is at times so enormous as to defeat even professional mathematicians. The consequences of this particular advocacy of CDE were, as they say, history.

As CDE was making its move, there were also other ideas on how to improve teaching in the mathematics classroom. One was to concentrate on children's cognitive capabilities. Notice that, once again, such a recommendation ignores the fact that interpreting children's mathematical thinking requires strong content knowledge. In the last few years, two new developments have emerged. One is a strong advocacy that lesson study be the principal activity of the professional development of teachers. The other is the emphasis given to teachers' acquisition of *pedagogical content knowledge* (a term introduced in [Shulman]), the kind of special mathematical-pedagogical knowledge teachers need to have in order to teach well. [2] As far as lesson study is concerned, a friend of mine mentioned some time ago how distressing it was to witness a group of teachers trying to refine a lesson plan on teaching fractions when none of them seemed to understand the mathematics underlying the subject of fractions. Without a firm foundation of content knowledge, lesson study is an exercise in futility.

The subject of pedagogical content knowledge was featured in a recent article in Education Week ([Viadero]) devoted to an interview with Deborah Ball on her research with Heather Hill and Brian Rowan ([H-R-B]) on this subject.[3] They studied third grade classes in many schools and, as such, this study falls outside the concern of my talk. Recall that I am concentrating on grades 5–12. But insofar as the article will be read as a general statement about teaching, it becomes important that we take its message seriously.

This article suggests that pedagogical content knowledge is the gold standard of teaching we have been waiting for. As the ultimate goal in good teaching, the idea that every teacher should possess pedagogical content knowledge to a high degree cannot be faulted. But in the context of the current state of professional development when we are very far from getting every math teacher to know the minimum content knowledge, leave alone the requisite pedagogical content knowledge, the basic message of this article

[2] It is well to note that the concept of "pedagogical content knowledge" is yet to be precisely defined; see the footnote on p. 12 of [H-R-B].

[3] During the NCTM Annual Meeting in April of 2005, Deborah Ball informed me that, in fact, the Hill-Rowan-Ball article [H-R-B] is about teachers' content knowledge, *not* pedagogical content knowledge.

will likely be misunderstood. Take, for instance, the following passage near the end of the article. It reads:

> *The results also suggest that efforts afoot now to require teachers to take particular math courses — or to open up the field to individuals who might have degrees in math but no education training — could be on the wrong track.*
>
> *"While those may be nice ideas," Ms. Ball said, "these results suggest that might not be the most promising way to think about improving math instruction."*

Because one thinks of teachers' content knowledge in terms of what they learn from the math courses they take, to most readers the the message is likely to be interpreted as asserting that, in order to improve the teaching of mathematics in general, teachers should bypass the attainment of mathematical knowledge (for who wants to be on the *wrong track?*) and make an all-out assault on pedagogical content knowledge instead.

I find this subliminal message of the preceding passage disturbing. There is by now a realization that the most important step in improving mathematics teaching is to bolster teachers' content knowledge by directly teaching them the mathematics they need in the classroom. (For an account of the history of this development, see the beginning of [Wu1]; a more detailed discussion of this issue can be found in Chapter 10 of [Kilpatrick] — especially pp. 375-6, — Chapters 1 and 2 of [MET], and [Wu1]). To get an idea of what kind of mathematics is considered to be the kind of "mathematics teachers need in their classroom", consider the anecdote given at the beginning of the Ed Week article [Viadero] of Ball's personal experience:

> *As a young teacher in an East Lansing, Mich., elementary school, Deborah Loewenberg Ball realized teaching mathematics required a special kind of knowledge. Unfortunately, she didn't have it.*
>
> *Take long division, for example.*
>
> *"There was no way of explaining to students what the procedure means, and what they're really doing," said Ms. Ball ...*

The tone of the article suggests that the explanation for the long division algorithm is part of a teacher's *pedagogical* content knowledge. The truth of the matter is that it is *straightforward mathematics*, namely, the repeated application of the common division-with-remainder. It is given in some of the recent mathematics textbooks written for elementary teachers such as §3.5 of [Wu 1], [Beckmann], [Jensen] and [P-B]. At this point, you may wish to recall my experience with CMP recounted earlier, especially regarding the discussion about increasing teachers' content knowledge.

The reality at present is that the professional development culture has not yet fully embraced the idea that there is an urgent need to focus on teaching straightforward *mathematics* to our teachers, partly because this culture does not yet recognize that there is a nontrivial body of mathematical knowledge on the level of K–12 that should be taught to teachers in a serious and systematic way. For this reason, the seductiveness of the Ed Week article about pedagogical content knowledge may mislead school district administrators and professional developers into abandoning the basic mission of teaching mathematics in favor of pursuing pedagogical content knowledge.

I do not believe a teacher can have pedagogical content knowledge without a firm command of "content knowledge" in the sense just described. The question then becomes one of how to help teachers acquire the requisite content knowledge. The experiences of myself and other mathematics colleagues who are engaged in professional development work all attest to the fact that learning the mathematics is very difficult for most teachers, and it does not make good sense to handicap their effort by asking them to also acquire pedagogical content knowledge at the same time. If we ask teachers to run before they can walk, they will fall flat on their faces. By asking them instead to first understand the mathematics they have been teaching for years, we may hope to put them on the path of professional growth so that they would acquire the needed pedagogical content knowledge gradually. One person's experience may not serve as a valid guide, but what I have observed among teachers I have taught bears out this learning trajectory (cf. [B-W]).

It should not be difficult to put my personal belief as outlined in the preceding paragraph to the test. Let me therefore put forth two conjectures. If we are to measure, in grades 5–12, the correlation not only between good pedagogical content knowledge and good teaching, but also between good *content knowledge* and good teaching, the two sets of data would be highly correlated. In addition, I also believe that, still with grades 5–12, those teachers with good content knowledge include those with good *pedgagogical* content knowledge. Why these conjectures are relevant is that, if they are true, then we would be able to offer the following procedure for improving school mathematics education:

> Good mathematics instruction requires good teachers, and good teachers are those with good pedagogical content knowledge who, in turn, are predominantly

those with good content knowledge. Improvement of school mathematics education therefore begins with teaching teachers the *mathematics* they need.

The virtue of this simple procedure is that by putting content knowledge in the limelight as the key to improving mathematics education, we can finally put the proper focus on this stumbling block in mathematics education. Coincidentally, the presence of the recent mathematics textbooks for teachers cited above should make this task a little easier.

2 Further Considerations

I have by now touched on all the main points of my lecture, and it remains for me to summarize them:

(1) The most difficult step in becoming a good teacher is to achieve a firm mastery of the mathematical content knowledge.

(2) Without such a mastery, good pedagogy is impossible.

(3) A firm mastery of the content opens up the world of pedagogy and offers many more effective pedagogical possibilities.

Before elaborating on these further, let me make sure you know what I am *not* saying, because this is as important as what I do say. I am not saying that knowing mathematics is all it takes to be a good teacher. Anyone who has gone through a four-year mathematics program in any university knows all too well that this is not true. What I do assert, on the other hand, is that if you can achieve a full grasp of the relevant mathematics, then you would be in an excellent position to become a good teacher. It is to be regretted that such a message has not been forcefully conveyed in mathematics education.

The lack of appreciation of the centrality of content knowledge in mathematics education is of course not limited to the recent ideas concerning teaching that I recounted above. Among the many other examples I can cite, let me direct your attention to three of them. The first is the deterioration of the quality of the mathematics in high school geometry classrooms. It is certainly no news, and it has been even documented in a paper written by an educator in 1988 ([Schoenfeld]), that proofs in high school geometry have often been replaced by rote memorization of what-the-teacher-says. In other words, many teachers no longer know what proofs are about and consider a geometric proof to be nothing more than a regurgitation of procedures dictated by the teacher. In the context of *mathematics* education, such a crisis in teacher's content knowledge demands to be urgently addressed. Yet, for the past fifteen years, this mathematical problem was left completely unattended while the mathematics education research of many other topics have fluorished as never before. There is a more thorough discussion of the issues surrounding the teaching of geometry on pp. 309-310 of [Wu4].

A second example is the announcement by the U.S. Department of Education in October of 1999 that ten mathematics programs were to be regarded as *Exemplary* or *Promising*. If a program can be considered among the ten best the nation has to offer, it may be taken for granted that the *mathematics* of each of these ten programs meets the minimum standard of being coherent and free of significant errors. Yet, to take the most obvious example, the mathematics of *Mathland*, one the five *Promising* programs, can be objectively demonstrated to be shallow, incomplete, incoherent, and not infrequently just plain wrong. So how did this travesty come about? In a recent authoritative publication from the National Research Council *On Evaluating Curricular Effectiveness* ([Confrey-Stohl]), the inattention to mathematical content in the review process of the U.S. Department of Education is revealed. The Department appointed an Expert Panel to set up a procedure for examining the evidence of success of the submitted programs. According to Richard Askey of the University of Wisconsin, in the 48 reviews of the initial 12 exemplary or promising programs, "no mention of any mathematical errors was made" (see p. 79 of [Confrey-Stohl]).

Now you must understand that any of the existing curricula, old or new, is so riddled with errors that it would take a superhuman mental effort to blot them out. How then did the dozen or so Expert Panel members and almost 95 Quality Control Panel members manage *not* to notice any of these glaring errors? One reason may be because there was only one mathematician on the Expert Panel, and only two on the Quality Control Panel. Without panel members who were able to spot the errors, the panel was led to call programs "exemplary" even though they contained serious mathematical errors.

A third example is the way blatant mathematical flaws in mathematics lessons are handled in some recent well-known case books, e.g., [Barnett], [Merseth], and [Stein]. These books would first present a lesson taken from a real classroom, and the facilitator (i.e., commentator on the lesson) would suggest directions that discussions of such a lesson might take in a professional development environment. In principle, this is certainly a powerful way to make teachers aware of the elements of good teaching. In an overwhelming majority of the cases in these case books, however, the mathematical flaws of a lesson are either completely ignored or glossed over in the facilitator's comments while attention is lavished on non-mathematical issues,[4] such as pedagogical skills, the teacher-student interface, students' thinking, etc. If we believe that the purpose of mathematics education is to teach students correct mathematics rather than to showcase excellent pedagogical strategies without regard to content, then the possibility is very real that these three volumes would corrode the concept of mathematics teaching and lead a generation of teachers astray. Of particular concern is the fact that these case books are very well received in the educational mainstream. Does content really play such a small role in mathematics education? For lack of space, I will refer to [Wu6] for a more thorough discussion of these case books.

You may recall that the title of my talk is "Must content dictate pedagogy?" My original intention was to hand out three cases at the beginning of my talk, one from each of the three case books mentioned above, and then discuss them with you after you have looked them over. I wanted to show you, in a concrete way, how a well-intentioned pedagogical decision in the classroom can be betrayed by faulty content knowledge, and how a deeper understanding of the underlying mathematics could lead to change in the pedagogical approach and render a lesson more clear and more understandable. But I subsequently decided that I should not penalize you for attending my talk by giving you such hard work. Instead, I will try to illustrate why content dictates pedagogy by showing you three statements from the education literature about recommended pedagogical strategies on teaching:

the comparison of (finite) decimals,

multiplication of fractions, and

division of fractions.

Then I will show you how a better grasp of the content in each case naturally leads to a different pedagogical strategy and, of course, a mathematics lesson that is easier to understand.

Notice that I have chosen to limit myself to mathematics of grades 5–7 for this purpose. Were I free to choose among materials in grades 5–12, the horizon would be limitless. Examples from school algebra, for instance, would fill a separate article (see section two in [Wu5]).

[4] Which can of course be important too.

3 The Comparison of (Finite) Decimals

The first example we wish to look at is the comparison of decimals. *Let it be understood that, throughout this discussion, a decimal is automatically assumed to be a* finite *decimal.* Students' ignorance of decimals, to the point of not knowing that 0.09 is smaller than 0.2, has inspired a whole industry on how to increase students' conceptual understanding of this topic. The following is a commentary[5] on this very issue of teaching decimals *in middle school* with conceptual understanding:

> *The decimal point indicates that we are beginning to break our unit — one — into tenths, hundredths, thousandths, and so on. But the number* one, *not the decimal point, is the focal point of this system. So really,* 0.342 *is 342* thousandths of one. *Put another way,* 0.3 *is three-tenths of 1, while 3 is three ones, and 30 is three tens, or 30 ones. But by the same token that* 0.3 *is three-tenths of one, 3 is three-tenths of 10, 30 is three-tenths of 100, and so on up the line. Or starting further down,* 0.003 *is three-tenths of* 0.01, *while* 0.03 *is three-tenths of* 0.1. *Moving in the opposite direction, 3000 is 30 hundreds, 300 is 30 tens. 30 is 30 ones, 3 is 30 tenths,* 0.3 *is 30 hundredths,* 0.03 *is 30 thousandths, and so on.*
>
> *All of this might sound more confusing than it really is. To compare* 0.45 *and* 0.6, *students are often told to "add a zero so the numbers are the same size." (Try figuring out what this might mean to a student who does not really understand decimal numbers in the first place!) This strategy works, but since it requires no knowledge of the size of the decimal numbers, it does not develop understanding of number size. Instead of annexing zeros, couldn't we expect students to recognize that six-tenths is more than 45 hundredths because 45 hundredths has only 4 tenths and what's left is less than another tenth?*

Let me paraphrase the main points of this commentary:

(A) To insure that students understand decimal numbers, we should teach

[5] I am trying to criticize a prevailing trend in mathematics education. For this purpose, I have to select from the education literature certain passages that illustrate this trend to form the basis of my criticism. When a passage comes from individuals, I will omit any explicit citation because my message would not be enhanced by the added knowledge of authorship. On the other hand, when a passage comes from a central document such as [PSSM], then an explicit citation is very relevant because [PSSM] is nothing if not synonymous with a major trend.

them the meaning of place value by using the language of tenths, hundredths, thousandths, etc.

(B) The usual way of comparing, for example, 0.45 and 0.6 through the routine of "add a zero so numbers are the same size" does not require a knowledge of the size of decimal numbers nor develop an understanding of number size and should therefore be avoided.

(C) The comparison of $0.45 < 0.6$ can be understood through the use of verbal descriptions such as: "six-tenths is more than 45 hundredths because 45 hundredths has only 4 tenths and what's left is less than another tenth".

I have a different take on this situation. I think most of the blame on students' non-learning of decimals can be placed squarely on the following two facts: (i) decimals are mostly taught as a topic independent of the subject of fractions, and (ii) no clear and precise definition of a decimal is ever given. For example, the preceding commentary implies that to understand decimals, it sufffices to concentrate on tenths, hundredths, thousandths, etc,. of the unit 1, but nowhere does it say what a decimal really *is*. In the learner's mind, a decimal becomes something elusive and ineffable: it is something one can talk about indirectly, but not something one can say outright what it is. This violates the basic principle of *literalness* in mathematics, WYSIWYG, i.e., what you see is what you get. If we cannot say explictly what a decimal is, then it is not a concept we can expect students to understand. Put differently, if we expect students to be able to fluently compute with decimals, then we have an obligation to tell them what a decimal is.

Let us therefore begin with a definition of **decimal**: it is a fraction[6] whose denominator is a power of 10. (Recall that we are in middle school so will freely use the language appropriate to this level. But of course "a power of 10" is a very simple concept to define in any case.) Historically, this was exactly how decimals arose; they used to be called **decimal fractions** (as distinct from **common fractions**, which are the *fractions* of today). Here are some examples of decimals:

$$\frac{271638}{10^4}, \quad \frac{6}{10}, \quad \frac{45}{10^2}, \quad \frac{730}{10^5}$$

[6]This means it is a number of the form $\frac{m}{n}$, where m, n are whole numbers and $n \neq 0$. More precisely, each $\frac{m}{n}$ (with fixed m, n) is the point on the number line obtained by dividing each of the segments $[0,1]$, $[1,2]$, $[2,3]$,... into n parts of equal length, and $\frac{m}{n}$ is the m-th subdivision on the number line to the right of 0.

Because one gets tired of writing the denominators (which are all so similar), the following convention of abbreviations for these decimals is commonly accepted in the English speaking countries:

$$27.1838, \quad 0.6, \quad 0.45, \quad 0.00730$$

The convention is easy to describe: omit the denominator, but keep track of the power of 10 (say 5) in the denominator by counting 5 times from the last digit on the right of the numerator and place a dot in front of that digit. That dot is referred to as the **decimal point**, of course. The 0 in front of the decimal point in case there is no nonzero digit to its left is added purely for the purpose of clarity, e.g., 0.6 in place of .6. In case the number of digits in the (whole number) numerator is smaller than the power of 10 in the denominator, just add the appropriate number of 0's to the left of the first digit of the numerator to keep track of this power. For example, for $\frac{730}{10^5}$, the power is 5 and there are only three digits in 730, so we add two zeros in front of 730, getting 00730, and then place the decimal point in front of the 0 on the left, thus: 0.00730

Incidentally, we see immediately that 0.00730 is the same as 0.0073, because by definition,

$$0.00730 = \frac{730}{10^5}$$

while

$$0.0073 = \frac{73}{10^4}.$$

Since $\frac{730}{10^5} = \frac{73}{10^4}$ because of equivalent fractions, we have $0.00730 = 0.0073$. In a similar fashion, we see why *adding zero's to the right of the decimal point does not change a decimal*. For example, $0.6 = 0.60 = 0.6000000$.

Let us see in what way a precise definition of a decimal can facilitate the the comparison of decimals. Consider the original example in the quoted commentary, 0.45 and 0.6. *By definition*, what we are comparing are the two fractions

$$\frac{45}{100} \quad \text{and} \quad \frac{6}{10}.$$

We know the fundamental fact concerning the comparison of fractions: *rewrite them as two fractions with the same denominator* (made possible by equivalent fractions). In this case, both denominators are powers of 10, so it is trivial to arrive at a common denominator, which is the denominator with the larger power of 10 (i.e., 100 in this case, which is of course 10^2). This we have a comparison of

$$\frac{45}{100} \quad \text{and} \quad \frac{60}{100}.$$

But clearly, $45 < 60$, so

$$\frac{45}{100} < \frac{60}{100}.$$

In terms of the decimal-point notation, we therefore have $0.45 < 0.6$.

Once the basic principle of comparison is understood, it is a no-brainer to compare *any* two decimals. For example, 0.0120481 and 0.0097. These are the fractions

$$\frac{120481}{10^7} \quad \text{and} \quad \frac{97}{10^4},$$

which we proceed to rewrite as

$$\frac{120481}{10^7} \quad \text{and} \quad \frac{97000}{10^7}.$$

The comparison of size is immediate: 120481 is bigger than 97000, so $0.0120481 > 0.0097$.

To summarize: the comparison of decimals — in accordance with the proposed strategy — is reduced to the comparison of *whole numbers*. This strategy makes explicit the relationship between decimals and whole numbers. I may add that this same relationship elucidates the concept of the place value of decimals, e.g.,

$$\begin{aligned} 4.215 &= \frac{4215}{10^3} = \frac{(4 \times 10^3) + (2 \times 10^2) + (1 \times 10) + 5}{10^3} \\ &= 4 + \frac{2}{10} + \frac{1}{10^2} + \frac{5}{10^3} \end{aligned}$$

In other words, the place value of decimals is nothing more than a rewrite of the place value of the numerator when the decimal is expressed as a fraction according to its definition. The pronounced advantage of this approach to the concept of place value of decimals is that students only need to learn about the place value of whole numbers, and nothing more. The place value of decimals becomes a consequence. At a time when mathematics education tries to preach the virtue of interconnectedness, this approach to decimals should be given serious consideration for this reason alone.

Recall that the recommended pedagogical strategy of the commentary for teaching $045 < 0.6$ (i.e., item (C) above) is to explain that "six-tenths is more than 45 hundredths because 45 hundredths has only 4 tenths and what's left is less than another tenth". I am not at all convinced that most students find such verbal explanations easy to follow. (I certainly didn't.) Moreover, this strategy won't work well with the comparison of $0.0120481 > 0.0097$, and would be even more awkward with comparisons of (say) 0.000068485749123 and 0.0000685. This is hardly the only drawback of the verbal approach to teaching middle school mathematics when you consider how one would try to explain what the decimal

$$0.12345678901223334444$$

is. The recommendation in item (A) of strictly using words such as hundredths and thousandths for the attainment of understanding would clearly be inappropriate, but according to our definition above, this decimal is nothing more than the fraction

$$\frac{12345678901223334444}{10^{20}}.$$

One should not dismiss offhand, in my opinion, the psychological advantage — in the context of learning — of knowing that any decimal, no matter how unwieldy, is something concrete that one can explicitly write down. It would be instructive as well as interesting to have some research done to confirm the importance of such an advantage.

It may be argued that in school mathematics, one never comes across decimals such as 0.12345678901223334444. But this seemingly senseless example actually raises a serious issue: should we pass off *partial* information as *the complete* information? In the case at hand, the verbal approach to decimals is obviously efficient for handling simple decimals such as 0.6 and 0.45, but as we have just seen, it does not enable students to handle decimals *in general*. Therefore the teaching of decimals in terms of verbal descriptions only provides *part* of the knowledge about decimals that students need. In middle school, students must begin to acquire the concept of generality because algebra is looming in the immediate horizon. If they confuse knowing a few special cases (of a concept or skill) with knowing the general case, their chances of learning algebra would be minimal. It is a heartening recent development that mathematics educators are beginning to take note of the need to distinguish special cases from the general case (cf. [Car-Rom]).

Thus when we canonize the verbal approach to decimals as one that brings conceptual understanding to students, and do so with no further qualifications, we are guilty of misleading our students. To avoid any misunderstanding of this message, let me illustrate with the same example of $0.45 < 0.6$. Suppose the symbolic argument as given above is already in place, then it would be entirely appropriate for a teacher to emphasize that the symbolic computation

$$0.45 = \frac{45}{100} < \frac{60}{100} = \frac{6}{10} = 0.6$$

includes the statement that 0.45 is 45 copies of a hundredth, whereas 0.6 is 60 copies of a hundredth, and this is the reason for the validity of the inequality $0.45 < 0.6$. So in this simple case, the verbal information succeeds in reinforcing the symbolic information. No doubt, verbal interpretation has its place in middle school mathematics education, but it is not a *replacement* of symbolic arguments.

To emphasize the fact that the preceding pedagogical recommendation is for grade 5 and up, I would recommend, by contrast, that the teaching of decimals (up to two decimal places) to third graders be accomplished by the use of money without any formal definition of a decimal. Thus 0.45 is simply 45 cents and 0.6 is six-tenths of a dollar, and third graders are welcome to simply count six-tenths of a dollar and 45 cents to see that 0.6 is more. By the time these third graders get to middle school, however, they need to reconcile naive discussions with more formal ones, and the 45 cents of yesteryear should be clearly identified with the decimal fraction $\frac{45}{100}$.

The argument against replacing the precise symbolic approach with the use of verbal descriptions should be pushed further. A critical step in this symbolic argument is the equality that $\frac{60}{100} = \frac{6}{10}$, which in decimal notation is $0.60 = 0.6$. The commentary disparages the use of the last equality for the reason that "since it requires no knowledge of the size of the decimal numbers, it does not develop understanding of number size" (see item (B) above). Such a statement is incorrect because — as we have explained — one rewrites 0.6 as 0.60 (i.e., $\frac{6}{10}$ as $\frac{60}{100}$) in this case because, to compare two fractions, one should rewrite them as fractions with the same denominator. Such an understanding about fractions is truly basic. Therefore rewriting 0.6 as 0.60 in this particular context, far from being a mindless rote procedure, is based on a conceptual understanding of the situation at hand. Students ought to learn how to use this procedure and, of course, the reason behind this procedure.

Mathematics relies on the symbolic language, and we have the responsibility as mathematics educators to make our middle school students fluent in the use of symbols. The undesirability of using only the words *tenths, hundredths*, and *thousandths* to teach decimals in middle school, in place of the symbolic definition of a decimal, can in fact be understood from the historical perspective. The earliest form of algebra, it may be argued, is the *Algebra* of al-Khwarizmi (circa 780–850 A.D.). It is perhaps not commonly known that al-Khwarizmi presented algebra entirely verbally, without the use of any letter-symbols, because the symbolic language was non-existent in his time as well as much later. Here is a typical passage:

> *.. a square and 10 roots are equal to 39 units. The question therefore in this type of equation is about as follows: what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type*

of equation is to take one-half of the roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3. The number three therefore represents one root of this square, which itself, of course is 9. Nine therefore gives the square. ([Rosen])

If you find this a little painful to read, then know that European mathematicians of the twelfth century also shared your pain.[7] Progress towards the use of symbolic notation was slow, however, and it wasn't until the one hundred years preceding the publication of Descartes' *Discours* in 1637 that the modern symbolic notation was essentially consolidated. In this notation, what al-Khwarizmi wrote can be rendered this way (his "root" is what we call the unknown):

> If a number x satisfies $x^2+10x = 39$, what is x^2? We have $x^2+10x+(\frac{10}{2})^2 = 39 + (\frac{10}{2})^2$, so that $x^2 + 10x + 25 = 64$. Thus $(x + 5)^2 = 8^2$. Taking square roots gives $(x + 5) = 8$, so $x = 8 - 5 = 3$ and $x^2 = 9$.

This is easier to read for most of us, and it is undoubtedly more clear. More importantly, anyone who cannot make sense of the symbolic version has not yet mastered middle school mathematics.

I hope I have made my point: the use of symbols is the gateway to mathematics, and the use of purely verbal descriptions, while it has its place in a pedagogical context, cannot replace the symbolic language. The realization that, in middle school, we have to develop students' fluency in the use of symbols is also part of teachers' content knowedge. This realization, together with the awareness of the need for precise definitions and the WYSIWYG principle, lead to a different way of teaching decimals which is at once more precise, more clear, and perfectly general.

4 Multiplication of Fractions

In this and the next section, we look at the multiplication and division of fractions, two topics which have invited the intense scrutiny of many mathematics educators because these topics have a reputation of being difficult to teach.

[7] al-Khwarizmi's algebra was translated from Arabic into Latin around 1150.

We have seen in the last section the importance of precise definitions. Unhappily, a precise definition of the product of two fractions $\frac{k}{\ell}$ and $\frac{m}{n}$ (k, ℓ, m, n are whole numbers and $\ell \neq 0$, $n \neq 0$) is generally missing in textbooks and professional development materials. This is again a violation of the basic WYSIWYG principle of mathematics (see preceding section). If students and teachers do not know what kind of an object the product of two fractions is, how can they work with the concept beyond making computations by rote? Nevertheless, educators forge ahead under this handicap. The education literature has a surplus of suggestions on how to deal with the pedagogical problem of helping students and teachers achieve a conceptual understanding of multiplying fractions in the absence of a precise definition. One approach to do this *in middle school* is the following:

> *We know that teachers and most other adults in our country have a limited understanding of the meaning of multiplication and division of fractions. . . . Teachers who are interested in changing this situation must first approach these topics themselves in ways that are very different from all their previous experiences with mathematics learning. They must completely reformulate their ideas about teaching the topics. . . .*
>
> *The medium for this rethinking is language. How can we think about something for which we have no words? . . .*
>
> *Multiplication of fractions is about finding multiplcative relationships between multiplicative structures. When students partition a continuous whole such as a circle, they actually find part of parts in the process. In order to create fourths, for example, a student's first create halves. The student then cuts the halves in half to create fourths. In so doing the student can verbalize that one-half of one-half is one-fourth. Later the symbolization can be connected back to the paritioning experience, first in written language and then with symbols. One-half of one-half is one-fourth;* $\frac{1}{2}$ *of* $\frac{1}{2}$ *is* $\frac{1}{4}$; $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

We witness once again an attempt to use verbal descriptions to achieve mathematical understanding.[8] An added difficulty in this case is that it is impossible to make sense of "finding multiplcative relationships between multiplicative structures." What

[8] Let me point out in passing that there is an unacceptable lack of precision in the statement "partition a continuous whole such as a circle". Is the paritioning in terms of area or shape, or something else? Look at the case "Two green triangles" on p. 86 of [Barnett] to see the kind of confusion such linguistic imprecision can lead to.

is a "multiplicative structure", and in what sense is the collection of fractions a "multiplicative structure"? What constitutes "finding multiplcative relationships" in this context? Moreover, the explanation of $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ is supposed to be valid for the product of any two fractions, but such an explanation suffers from multiple mathematical difficulties. Has it explained to the reader what *is* the product of any two fractions? And if the reader is not already predisposed towards believing the simple fact that $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, would this explanation make any sense? For example, how would the same explanation work in trying to explain $\frac{2}{19} \times \frac{83}{17} = \frac{166}{323}$?

What we need is a definition of multiplying fractions that is mathematically valid and, at the same time, sufficiently intuitive for middle school students. There is not a unique way to accomplish this, but the following will serve (see §7.3 of [Wu3]). We start with two concrete fractions $\frac{2}{3}$ and $\frac{8}{5}$. By common practice, we compute the weight of "two-thirds of eight-fifths kilograms of rice" by computing $\frac{2}{3} \times \frac{8}{5}$. On the other hand, a little reflection would reveal that what "two-thirds of eight-fifths kilograms of rice" means in the context of everyday communication is that if we divide the eight-fifths of a kilogram of rice into three equal parts by weight, then it is the weight of two of these parts. We now turn this interpretation around and adopt it as the precise definition of the product $\frac{2}{3} \times \frac{8}{5}$.

Definition of $\frac{2}{3} \times \frac{8}{5}$: $\frac{2}{3} \times \frac{8}{5}$ is the size of 2 of the parts when an object of size $\frac{8}{5}$ is divided into 3 equal parts.[9]

Please note right away that, *strictly according to this definition*, $\frac{2}{3}$ of an object of size $\frac{8}{5}$ has size equal to $\frac{2}{3} \times \frac{8}{5}$. In other words, this definition of the product of fractions is designed to capture the normal linguistic usage of *"of"*. This is distinctly different from the pedagogical fiction that one can "prove" that the preposition *"of"* carries with it the meaning of fraction multiplication, as is sometimes claimed in the professional development literature.

Now, once we agree on a definition of $\frac{2}{3} \times \frac{8}{5}$, we are bound by this definition to give an explanation of why $\frac{2}{3} \times \frac{8}{5} = \frac{2 \times 8}{3 \times 5}$ strictly on the basis of this definition. In other words, if we take 2 of the parts when an object of size $\frac{8}{5}$ is divided into 3 equal parts, then

[9] I have intentionally made the definition as conversational as possible. A more formal version would begin with a definition of a fraction as a point on the number line obtained by the procedure described in Footnote 6. Then by definition, $\frac{2}{3} \times \frac{8}{5}$ is the 2nd division point away from 0 when the segment $[0, \frac{8}{5}]$ is divided into 3 segments of equal length.

we have to explain why the size of these 2 parts is

$$\frac{2 \times 8}{3 \times 5}$$

We first ask, if $\frac{8}{5}$ is divided into 3 equal parts, how big is one part? We use equivalent fractions to compute:

$$\frac{8}{5} = \frac{3 \times 8}{3 \times 5} = \frac{8}{3 \times 5} + \frac{8}{3 \times 5} + \frac{8}{3 \times 5}$$

Therefore the size of one part has been explicitly displayed as $\frac{8}{3 \times 5}$. The size of 2 of the parts is thus

$$\frac{8}{3 \times 5} + \frac{8}{3 \times 5} = \frac{2 \times 8}{3 \times 5},$$

as desired.

I hope you see that the reasoning in this special case is perfectly general, so that if we define $\boldsymbol{\frac{k}{\ell} \times \frac{m}{n}}$ to be the size of k of the parts when an object of size $\frac{m}{n}$ is divided into ℓ equal parts, then in like manner, we can show

$$\frac{k}{\ell} \times \frac{m}{n} = \frac{km}{\ell n}$$

We shall refer to this formula as the **product rule**. In most books, the product rule is the starting point of the discussion of fraction multiplication. Such discussions typically leave the *meaning* of multiplication undefined, but there would usually be some vague statement about the conceptual understanding of multiplication in a contextual situation. The inherent danger of such a presentation is that, if there is no explanation of why the product rule is true *as a result of a precise meaning of multiplication*, many students would silently entertain the notion that there is nothing special about "×" and the same formula must remain true if "+" is used instead, i.e., the formula

$$\frac{k}{\ell} + \frac{m}{n} = \frac{k+m}{\ell + n}$$

must likewise be true. As is well-known, this kind of error is common, and we should recognize it for what it is: the inevitable consequence of faulty content in the mathematics instruction.

Once the product rule is established, then we can round off the picture by explaining why *the area of a rectangle with sides of lengths* $\frac{k}{\ell}$ *and* $\frac{m}{n}$ *is* $\frac{k}{\ell} \times \frac{m}{n}$. (See §7.1 of [Wu3] for details.) Notice that the multiplication of fractions is used in everyday life principally in two contexts: in computing the area of a rectangle, and in computations of the type "how much is two-thirds of eight and half pounds of rice?" At this point, therefore, all the essential elements of fraction multiplication have been covered.

Let me make explicit the main points of this discussion. Our charge was to make sense, for middle school students, of what it means to multiply two fractions. Rather than relying on unfathomable verbal descriptions about finding "multiplcative relationships between multiplicative structures", we follow the dictates of mathematics by first formulating a precise definition of fraction multiplication. We adopted the most common linguistic interpretation of the preposition *"of"* as the official definition of this concept. Then *on the basis of this definition*, we showed — or at least gave an indication of the reasoning — why the product rule must be correct and why the area of a rectangle is given by the product of the sides. In short, we have given an explicit *mathematical* development of this concept at the level of middle school. This mathematical approach to the multiplication of fractions is superior to the linguistic approach quoted at the beginning of this section in that it can be followed logically, step-by-step, without resorting to claims about "multiplicative relationships between multiplicative structures."

It remains to answer a question that must be on the lips of some of you in the audience: is it realistic to go through the explanations of the product rule and the area formula of a rectangle for middle school students? The answer is a qualified *yes*, in the sense that explanations using concrete numbers (such as what we did with $\frac{2}{3} \times \frac{8}{5} = \frac{2\times 8}{3\times 5}$) must be given starting with grade 6. I have already indicated one reason why this must be so: teaching multiplication without explanations begs the question of why not $\frac{4}{7} + \frac{3}{11} = \frac{4+3}{7+11}$. In addition, students need explanations to firm up their knowledge of multiplication, because (as we shall see in the next section) without such a secure knowledge, they cannot hope to understand the division of fractions. As to whether the symbolic explanations of the general case of the product rule (e.g., $\frac{k}{\ell} \times \frac{m}{n} = \frac{km}{\ell n}$) and the general rectangle area formula should be given to sixth graders, that would depend on what kind of students are in the class. For seventh graders, however, such symbolic explanations should be an integral part of the instruction because seventh graders need to be exposed to the ideas of generality and abstraction in order to be prepared for algebra.

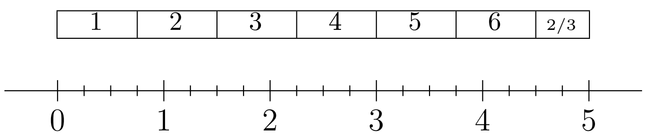
5 Division of Fractions

"Invert and multiply" may well be the phrase that inspires the most loathing in elementary and middle schools. The soul-searching in the mathematics education literature on how to make this skill more palatable tends to be pedagogical in nature, with little or no attention given to the mathematical content of this procedure. Usually one tries to over-simplify or short-circuit some aspects of the imbedded mathematics. I will now show you why neither over-simplification nor short-circuiting is necessary if the concept of division is clearly understood, *starting with the division of whole numbers.*

Here is the pedagogical recommendation on the teaching of fraction division in *middle school* taken from [PSSM] (p. 219):

> *A common way of formally justifying the "invert and multiply" procedure is to use sophisticated arguments involving the manipulation of algebraic rational expressions — arguments beyond the reach of many middle-grade students. This process can seem very remote and mysterious to many students. Lacking an understanding of the underlying rationale, many students are therefore unable to repair their errors and clear up their confusions about division of fractions on their own. An alternative approach involves helping students understand the division of fractions by building on what they know about the division of whole numbers. If students understand the meaning of division as repeated subtraction, they can recognize that* $24 \div 6$ *can be interpreted as "How many sets of 6 are there in a set of 24?" This view of division can also be applied to fractions, as seen in [the following figure]. To solve this problem, students can visualize repeatedly cutting of* $3/4$ *yard of ribbon. The 5 yards of ribbon would provide enough for 6 complete bows, with a remainder of* $2/4$*, or* $1/2$*, yard of ribbon, which is enough for only* $2/3$ *of a bow. Carefully sequenced experiences with problems such as these can help students build an understanding of division of fractions.*

If 5 yards of ribbon are cut into pieces that are each $3/4$ yard long to make bows, how many bows can be made?



Let me isolate a few keys points made in the preceding passage:

(a) To understand "invert and multiply", one needs to understand sophisticated arguments involving rational algebraic expressions.

(b) Understanding the division of whole numbers would lead us to understand the division of fractions as repeated subtraction.

(c) One can see easily that 1/2 yard of ribbon is 2/3 of a bow of length equal to 3/4 of a yard.

I will proceed to show you why the first two points, (a) and (b), are based on an erroneous understanding of the concept of division. To this end, I too will go over the division of whole numbers.

What does it mean when we say $36 \div 9 = 4$? It means there are 4 groups of 9 in 36. So $36 = 9 + 9 + 9 + 9$, which is the same as $36 = 4 \times 9$. Of course, if we start with $36 = 4 \times 9$ instead, then 36 is already given as 4 groups of 9 so that, according to the usual presentation of division in elementary school, we would have $36 \div 9 = 4$. We may therefore summarize this discussion by saying that

$36 \div 9 = 4$ is exactly the same as $36 = 4 \times 9$.

This seemingly trite statement actually expresses a mathematical truth that is, unfortunately, not emphasized or not even recognized in elementary school mathematics. It is this: when we teach students the concept of *division* after they have learned the multiplication of whole numbers, all we do is nothing more than teaching them *an alternate way of expressing multiplication.*[10] In other words, in mathematical terms, there is nothing new in division beyond rewriting multiplication.

The same consideration then extends to the division of arbitrary whole numbers. In general, if a, b, c are whole numbers, $c \neq 0$, then replacing the division symbol $\div$ by the fraction bar and omitting the $\times$ symbol between letters (we are in the context of middle school mathematics), we have:

$\frac{a}{c} = b$ is exactly the same as $a = bc$.

We emphasize once more that, in mathematics, division has no independent existence as it is only a rewriting of multiplication.

We should mention in passing that, once we are in possession of this correct understanding of division, we can explain two vexing issues connected with division in

[10]I am obliged to point out that what this says is *not* the same as the glib statement that "division is the inverse of multiplication". The latter does not make sense, literally, because both division and multiplication are binary operations and there is thus no possibility of one being the inverse of the other. It is a matter of precision: what we say here is that the two statements $36 \div 9 = 4$ and $36 = 4 \times 9$ are interchangeable so that knowing either one is exactly the same as knowing the other.

elementary grades. The first is why *among whole number*, we cannot write down expressions such as $7 \div 5$. This is because there is no multiplicative statement of the kind $7 = A \times 5$ for a *whole number* A. More generally, if a whole number a is not a multiple of another whole number c, then we cannot write down $a \div c$ because there is no multiplicative statement of the kind $a = bc$ for a *whole number* b. This explanation deserves to be stressed in grades 2 to 4, but it usually is not. A second issue is why one cannot divide by 0 (notice that in the preceding statement about $\frac{a}{c} = b$, we stipulated that $c \neq 0$). Indeed, to be able to write, for example, $36 \div 0 = b$ for some whole number b, we must be able to write $36 = b \times 0$ to begin with. But this is impossible because 36 is not 0, whereas $b \times 0$ is always 0. Of course, this in turn begs the question of why not consider $0 \div 0$. Let us say $0 \div 0 = b$ for some whole number b, then this means $0 = b \times 0$. But *any* number b would make this equality valid, for instance $0 = 1 \times 0$ and $0 = 2 \times 0$. Again *by our understanding of division*, these two equalities would mean $0 \div 0 = 1$ and $0 \div 0 = 2$. In fact, the same reasoning shows that $0 \div 0 = x$ for *any* number x. This is an absurd situation that must be avoided. So we rule out division by 0.

From this perspective on division, we come to understand that the usual division-with-remainder algorithm is actually a misnomer: it is not about division in the sense described above at all (except for the special case of remainder 0) but about something different and something special to the whole numbers.[11] The interpretation of division among whole numbers as repeated subtraction comes from treating division from the point of view of division-with-remainder. As we have emphasized, this is an incorrect way to look at division on a conceptual level. There is in addition another level of logical difficulty with this way of looking at the division of fractions, as we proceed to point out.

Consider the case of dividing 23 by 4 (allowing fractions as answers now); the answer is of course $5\frac{3}{4}$. But we can obtain $5\frac{3}{4}$ this way: write out the division-with-remainder of 23 divided by 4: $23 = (5 \times 4) + 3$, then from the quotient 5 and remainder 3, we can construct the answer: $5 + \frac{3}{4}$, which is of course $5\frac{3}{4}$. We may paraphrase this process by saying that there are *5* copies of 4 in 23, with a *3* left over, so that the answer of 23 divided by 4 is $5 + \frac{3}{4}$. What the above passage from [PSSM] does is to mimic this process in the context of the division of fractions. Thus, to divide $\frac{5}{4}$ by $\frac{2}{3}$, we perform something like the division-with-remainder by writing down

$$\frac{5}{4} = \left(1 \times \frac{2}{3}\right) + \frac{7}{12}$$

[11] Or something like whole numbers. In mathematics, we have a name for these objects: *Euclidean domains*.

That is to say, there is a maximum of *1* copy of $\frac{2}{3}$ in $\frac{5}{4}$, with $\frac{7}{12}$ left over. *Now, to imitate the case of whole numbers*, we are going to declare that the division of $\frac{5}{4}$ by $\frac{2}{3}$ is equal to $1 + \frac{7/12}{2/3}$. Unfortunately, this process gets stuck at this point because, whereas in the preceding case of whole numbers we could rely on our knowledge of fractions to make sense of the $\frac{3}{4}$ in $5 + \frac{3}{4}$, now we cannot make sense of the $\frac{7/12}{2/3}$ in $1 + \frac{7/12}{2/3}$ because what we set out to do was precisely to make sense of divisions such as $\frac{7/12}{2/3}$. In short, the writers of [PSSM] used circular reasoning to explain the division of fractions.

The circularity would have been more prominently exposed had we considered a division such as $\frac{1}{4}$ divided by $\frac{2}{3}$, because in that case there would have been no repeated subtraction to begin with, and this attempted explanation of fraction division would have run aground at the outset.

Let us now do the division of fractions correctly. The key point is that, so far as division is concerned, there should be no conceptual difference between division among whole numbers and division among fractions because both whole numbers and fractions are part of the real numbers. (This is an example of the so-called *longitudinal coherence of the curriculum* in [Wu1], where one may find a more extensive discussion of this circle of ideas.) Therefore the meaning of division between fractions should be entirely analogous to that among whole numbers, i.e., we can simply imitate what we have just learned about the division of whole numbers. Life then becomes very simple: if A, B, C are fractions and $C \neq 0$, then **by definition**

$\frac{A}{C} = B$ is exactly the same as $A = BC$.

At the risk of being redundant, let me repeat: in order to find out if it is true that $\frac{A}{C} = B$, *by definition*, all one has to do is to check if $A = BC$ is true. For example, we know

$$\frac{\frac{14}{15}}{\frac{7}{3}} = \frac{2}{5}$$

must be true because we can easily check that the corresponding multiplicative statement

$$\frac{14}{15} = \frac{2}{5} \times \frac{7}{3}$$

is true.

More generally, if a, b, c, d are whole numbers and b, c, d are all nonzero, then *by definition*, we get

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c}$$

because the corresponding multiplicative statement

$$\frac{a}{b} = \left(\frac{a}{b} \times \frac{d}{c}\right) \times \frac{c}{d}$$

is trivially true. Notice that we have just explained the **invert and multiply rule**. *This is all there is to the meaning of invert and multiply.* With reference to item (a) above, this explanation has nothing to do with sophisticated arguments about algebraic rational expressions. In particular — and this is a point worthy of emphasis — it is not beyond middle school students, at least not those who will take algebra in grade 8.

One reason we want to achieve a clear understanding of concepts (in contrast to ambiguous verbal explanations) is that it always lead to greater power in problem solving. Armed with this new understanding of what invert and multiply is about, let us begin by redoing the preceding problem concerning ribbon and bows. If we can make N bows of $\frac{3}{4}$ of a yard from a 5 yard ribbon, then N of $\frac{3}{4}$'s is equal to 5. By our definition of fraction multiplication in the last section, this is expressed as

$$5 = N \times \frac{3}{4}$$

By the definition of division, this is the same as

$$N = \frac{5}{\frac{3}{4}}$$

and the invert-and-multiply rule gives immediately

$$N = \frac{20}{3} = 6\frac{2}{3}.$$

How to interpret this number $6\frac{2}{3}$? The quoted passage implicitly floats the idea that the use of invert and multiply is too mysterious for students to gain any "understanding" of the number $6\frac{2}{3}$ whereas the method of repeated subtraction helps them to do that. I am going to show you that the quoted passage is wrong on both counts, i.e., invert and multiply gives the correct interpretation of the answer without any extracurricular interventions, whereas the method of repeated subtraction would fail in general to yield the correct answer to such problems. We start with $6\frac{2}{3}$. Putting the original equation $5 = N \times \frac{3}{4}$ and the answer $N = 6\frac{2}{3}$ together, we get

$$5 = 6\frac{2}{3} \times \frac{3}{4} = \left(6 + \frac{2}{3}\right) \times \frac{3}{4}$$

By the distributive law, this leads to

$$5 = \left(6 \times \frac{3}{4}\right) + \left(\frac{2}{3} \times \frac{3}{4}\right)$$

This is then the *explicit* statement that 5 yards of ribbon is equal to 6 of the bows of $\frac{3}{4}$ yard and an extra $\frac{2}{3}$ of a bow of $\frac{3}{4}$ yard combined. (Once more, we see the importance of knowing multipliction before approaching division, as we have used the meaning of fraction multiplication yet again to interpret $\frac{2}{3} \times \frac{3}{4}$.)

Moral: the correct interpretation of the answer obtained by invert and multiply is imbedded in the mathematics itself. All we have to do is to look inside the mathematics for an understanding of the calculation.

So why is the interpretation of division as repeated subtraction not good enough? Because in item (c) above, the fact that 1/2 yard of ribbon is 2/3 of a bow of length 3/4 of a yard was really the result of guesswork; no explanation was given and repeated subtraction had no say in the matter. In mathematics, guesswork is simply not good enough. To bring home this point, suppose we ask

> how many bows of length $\frac{37}{64}$ of a yard are there in a ribbon of length $\frac{5}{18}$ yard?

Then there is no repeated subtraction to perform. But if we use invert and multiply, then it is easy: If there are N bows of length $\frac{37}{64}$ of a yard in $\frac{5}{18}$ yard, then

$$\frac{5}{18} = N \times \frac{37}{64}$$

so that (invert and multiply!)

$$N = \frac{\frac{5}{18}}{\frac{37}{64}} = \frac{320}{666} = \frac{160}{333}.$$

The reasoning that we just went through (which we now skip) says explicitly that there are $\frac{160}{333}$ of a bow of length $\frac{37}{64}$ yard in a ribbon of $\frac{5}{18}$ yard.

The limitation of viewing division as repeated subtraction is shown up in a different problem:

> How many $\frac{8}{13}$'s are there in $43\frac{2}{7}$?

No one would dream of doing this problem by the tedious method of repeated subtraction, I believe. Using invert and multiply, this is easy (I will suppress the by-now familiar details): from

$$\frac{43\frac{2}{7}}{\frac{8}{13}} = 70\frac{19}{56},$$

we conclude that there are 70 and $\frac{19}{56}$ of $\frac{8}{13}$'s in $43\frac{2}{7}$.

Invert and multiply furnishes a classic example of the confrontation between content and pedagogy. Because division is, by comparison with the other arithmetic operations, a subtle concept, the difficulty of teaching invert and multiply has led many to stop looking into the mathematics and look for pedagogical solutions outside of mathematics instead. The quoted passage from [PSSM] is but one example of how the purely pedagogical approach to invert and multiply can go awry. By taking the mathematics seriously, we arrive at a different proposal for teaching invert and multiply. This alternate proposal, one that is based solidly on an understanding of the content, is not a quick fix because it requires a mathematically sound preparation for students in the early grades on the division of whole numbers, and of course an equally sound teaching of the concept of division in middle school. Rarely does a quick fix exist for any substantive issue in mathematics education. Nothing is easy, but it is better to meet the difficulty head on than teach incorrect mathematics.

6 Epilogue

For ease of discussion, I have intentionally oversimplified the issue of teaching by separating it into the pedagogical and content components. In reality, such a pure separation does not exist. What is true is that there is usually an emphasis on one or the other. If my talk is at all successful, then you will agree that the present preoccupation with pedagogical techniques independent of content cannot go far. The central issue of mathematics education, at least in grades 5–12, is still mathematics. I hope I have given you some illustrations of this point of view by showing how a deeper understanding of the content can fundamentally alter a well-intentioned pedagogical approach to a topic. It is my considered opinion that sound pedagogy can only be launched from the platform of mathematical competence. Viable pedagogical options are only visible through the lens of true mathematical understanding.

Let me conclude with the general observation that when mathematical difficulties are not removed from lessons, discussions of pedagogical improvements are meaningless. Great pedagogy lavished on incorrect mathematics makes bad education. Students do not learn mathematics when they are taught incorrect mathematics.

Content dictates pedagogy in mathematics education.

References

[Ball] D.L. Ball, Research on teaching mathematics: Making subject matter knowledge part of the equation, in *Advances in Research in Teaching*, Volume 2, J. Brophy, ed., JAI Press, 1991, 1-48.

[Barnett] C. Barnett, D. Goldstein, and B. Jackson, *Fractions, Decimals, & Percents*, Mathematics Teaching Cases and Facilitator's Discussion Guide, Heinemann, 1994.

[Beckmann] Sybilla Beckmann, *Mathematics for Elementary Teachers*, Addison-Wesley, 2005.

[B-W] Mary Burmester and H. Wu, Some lessons from California, November 29, 2001; Revised, May 25, 2004. http://math.berkeley.edu/∼wu/

[Car-Rom] T. P. Carpenter and T. A. Romberg, *Power Practices in Mathematics and Science*, National Center for Improving Student Learning and Achievement in Mathematics and Science, University of Wisconsin System, 2004. (Distributed by Learning Point Associates.)

[Confrey-Stohl] Jere Confrey and Vicki Stohl, eds., *On Evaluating Curricular Effectiveness*, National Academy Press, Washington DC, 2004.

[H-R-B] Heather Hill, Brian Rowan, and Deborah L. Ball, Effects of teachers' mathematical knowledge for teaching on student achievement, http://www-personal.umich.edu/∼dball/BallSelectPapersTechnicalR.html

[Jensen] Gary R. Jensen, *Arithmetic for Teachers*, American Mathematical Society, 2003.

[Kilpatrick] J. Kilpatrick, J. Swafford, and B. Findell, eds., *Adding It Up*, National Academy Press, Washington DC, 2001.

[Ma] Liping Ma, *Knowing and Teaching Elementary Mathematics*, Erlbaum, 1999.

[Merseth] K. K. Merseth, *Windows on Teaching Math*, Case Book and Facilitator's Guide, Teachers College, 2003.

[MET] *The Mathematical Education of Teachers*, CBMS Issues in Mathematics Education, Volume 11, American Mathematical Society, 2001.

[P-B] Thomas H. Parker and Scott Baldridge, *Elementary Mathematics for Teachers*, Sefton-Ash Publishing, 2003. Available from http://www.singaporemath.com

[PSSM] *Principles and Standards for School Mathematics*, National Council of Teachers of Mathematics, Reston, 2000.

[Rosen] Frederic Rosen (transl.), *The Algebra of Muhammed ben Musa*, London: Oriental Translation Fund, 1831.

[Schoenfeld] A. H. Schoenfeld, When good teaching leads to bad results: The disasters of "well-taught" mathematics courses, *Education Psychologist*, 23 (1988) 145–166.

[Shulman] Lee S. Shulman, Those who understand: Knowledge growth in teaching, *Educational Researcher*, 15 (1986), 4-14.

[Stein] M. K. Stein, M. S. Smith, M. A. Henningsen, and E. A. Silver, *Implementing Standards-Based Mathematics Instruction*, Teachers College, 2000.

[Viadero] Debra Viadero, Teaching Mathematics Requires Special Set of Skills, *Education Week*, October 13, 2004. http://www.edweek.org/agentk-12/articles/2004/10/13/07mathteach.h24.html

[Wu1] H. Wu, What is so difficult about the preparation of mathematics teachers? (November 29, 2001; revised March 6, 2002). http://math.berkeley.edu/~wu/

[Wu2] H. Wu, Chapter 1: Whole Numbers (Draft) (July 15, 2000; revised, September 3, 2002) http://math.berkeley.edu/~wu/

[Wu3] H. Wu, Chapter 2: Fractions (Draft) (June 20, 2001; revised September 3, 2002). http://math.berkeley.edu/~wu/

[Wu4] H. Wu, Geometry: Our Cultural Heritage (book review), *Notices Amer. Math. Soc.* 51 (2004), 529-537. http://math.berkeley.edu/~wu/

[Wu5] H. Wu, Assessment in school algebra, *available upon request from the author.*

[Wu6] H. Wu, The role of mathematics in a mathematics lesson — An examination of some case books, to appear.