# Fractions

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H. Wu

The main points of this presentation:

**(1) Every concept will have a definition:** Fraction, sum of two fractions, product of two fractions, percent, ratio, etc. You do not need to know more about a concept than what is contained in the definition. There are no guesses in mathematics.

**(2) Similarity between fractions and whole numbers will be emphasized throughout.** Whole number facts provide the proper guidance for what we do with fractions.

**(3) A reason will be given for every assertion.** Everything will be explained. There will be no doubts or suspicions.

**The goal:** To go through all four operations $+$, $-$, $\times$, $\div$ with fractions.

However, I will only go as fast as you can understand me. Given the limited time, there is a good chance that we won't get to division if I do *everything* systematically. So I may skip some topics to make sure we get to division. In that case, you may ask for the PDF file of these slides and read the rest on your own.

**(A) Definition of fraction** Why we need a definition of a fraction.

$(i)$ **A fraction is an abstract concept.** For the number 4, a school student sees "4 fingers". But what is $\frac{3}{11}$? Thus, if students have to add, subtract, multiply, and divide fractions, they have to know what a fraction is. They cannot work with something without knowing what it is.

$(ii)$ Reasoning in algebra and higher mathematics depends on precise definitions. Learning how to work with a precise definition of a fraction is an excellent introduction to algebra.

Comment on $(i)$: The usual way of teaching fractions does not give a definition of fraction, but insists rather that a fraction is just part of daily experience, e.g., it is like a piece of pizza.

How is a student going to think about the following problem?

> *How much is $\frac{12}{13}$ bucket of water and $\frac{7}{8}$ buckets of water together?*

What is $(\frac{12}{13} + \frac{7}{8})$ buckets of water?

The teacher only teaches adding two pieces of $pizza$ together, but this is two $buckets\ of\ water$. So what is a fraction? A piece of pizza or some bucket of water?

Comment on ($i$) (*cont.*): So the student forces herself to just think of adding pizzas.

Can she think of $\frac{12}{13}$ of a pizza? We know instinctively that we don't cut up a pizza like that. In any case, why would anyone put $\frac{12}{13}$ of a pizza and $\frac{7}{8}$ of a pizza together?

We insist to students that fractions are nothing but an extension of our daily experience and is not an abstraction, but we agree that putting $\frac{12}{13}$ of a pizza and $\frac{7}{8}$ of a pizza together is unnatural. Then it becomes difficult for a student to think about this addition.

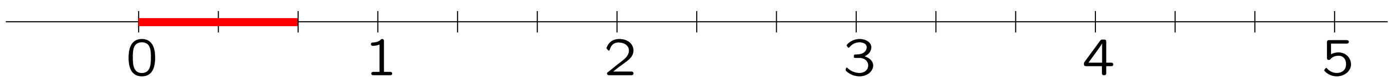
(We will have more to say about this addition later.)

On a horizontal line, let two points be singled out. Identify the point to the left with 0 and the one to the right with 1. This segment, denoted by $[\mathbf{0}, \mathbf{1}]$, is called the **unit segment** and 1 is called the **unit.**

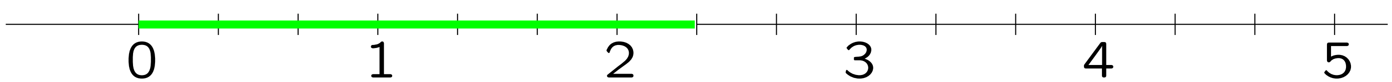

Now mark off equidistant points to the right of 1 as in a ruler, as shown, and identify the successive points with 2, 3, 4, . . . .


The line, with a sequence of equidistant points on the right identified with the whole numbers, is called the **number line**.

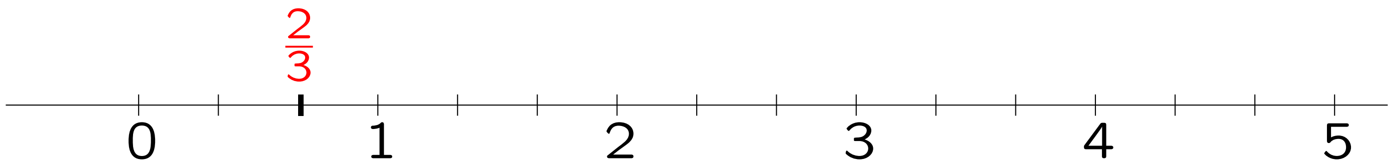
Let the unit segment $[0, 1]$ be **the whole**. Naturally, all other segments $[1, 2]$ (the segment between 1 and 2), $[2, 3]$, etc., can also be taken to be the whole. If we divide each such segment into **thirds** (three segments of equal length), then we can count the number of thirds (the "parts") by going from left to right starting from 0. Thus, the red segment comprises *two thirds*:



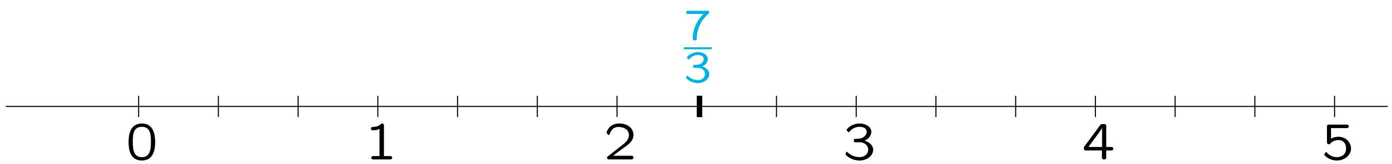

The following green segment comprises *seven thirds*:



With the left endpoint 0 understood, the red segment can be identified by its *right* endpoint, which we naturally denote by $\frac{2}{3}$:




Likewise, the green segment can be identified by its *right* endpoint, which is denoted by $\frac{7}{3}$:




Thus each "part of a whole" (in the present context of *thirds*) can be replaced by a point on the number line, so that the point that is the *7th* point to the right of 0 is denoted by $\frac{7}{3}$, and the point that is the *nth* point to the right of 0 is denoted by $\frac{n}{3}$.

Fractions with denominator equal to 5 are similarly placed on the number line: $\frac{8}{5}$ is the 8th point to the right of 0 in the sequence of *fifths*. And so on.

In general, if $n$ is a positive integer, the fraction $\frac{3}{n}$ is the third point to the right of 0 among the $n$*ths* on the number line, and if $m$ is a whole number, then the fraction $\frac{m}{n}$ is the $m$th point to the right of 0 among the $n$*ths* on the number line.

*We also agree to identify* $\frac{0}{n}$ *with* 0 *for any positive integer* $n$*.* In this way, all fractions are unambiguously placed on the number line.

For the sake of conceptual clarity as well as ease of mathematical reasoning, we will henceforth define a **fraction** to be a point on the number line as described above.

What does it mean to *define a fraction to be a point on the number line*?

It means: any time we want to explain something about fractions, we must remember that a fraction is a point on the number line and our explanation must begin with the number line. There is no longer any need to guess the right way to interpret a fraction: Just go to the number line.

*The importance of the unit.*

The meaning we assign to the number 1 determines the meaning we give to all fractions on the same number line. For example, if 1 is "the area of a piece of a given pizza" (ignoring depth), then 2 will be 2 times the area of the pizza and $\frac{12}{13}$ will be "$\frac{12}{13}$ of the area of the pizza."

If 1 is the volume of a bucket of water, $\frac{1}{2}$ will be half the volume of the bucket of water, and $\frac{12}{13}$ will be "$\frac{12}{13}$ of the volume of the bucket of water."

You can see that when a fraction is identified with a point on the number line, it gains flexibility in its interpretations.

Three things are noteworthy:

($i$) The fractions with denominator 3 are *qualitatively* no different from the whole numbers: both are a sequence of equidistant points on the number line, and if we replace $\frac{1}{3}$ by 1, then the former sequence becomes the whole numbers.

($ii$) **The number line is to fractions what one's fingers are to whole numbers:** It anchors students' intuition about fractions.

($iii$) The number line is not some gimmick created specifically for the discussion of fractions. On the contrary, it is omnipresent in mathematics, being the $\boldsymbol{x}$**-axis** of the familiar coordinate system in the plane. It will follow a student everywhere as long as he or she studies anything related to mathematics.

**(B) Equivalent fractions.** The following theorem is basic.

**Theorem on equivalent fractions** *Given any two fractions* $\frac{m}{n}$ *and* $\frac{k}{\ell}$. *If there is a positive integer* $c$ *so that*

$$m = c\,k, \quad \text{and} \quad n = c\,\ell$$

*then the fractions are equal, i.e.,* $\frac{m}{n}$ *and* $\frac{k}{\ell}$ *are the same point.*

In your classroom, you wouldn't teach like this! Rather, you'd begin by saying, $\frac{3}{6}$ and $\frac{1}{2}$ are equal because

$$3 = 3 \times 1 \text{ and } 6 = 3 \times 2,$$

$\frac{10}{12}$ and $\frac{5}{6}$ are equal because

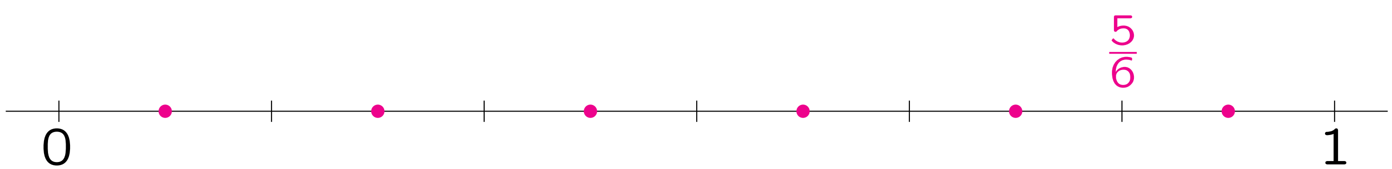
$$10 = 2 \times 5 \text{ and } 12 = 2 \times 6,$$

You'd give many examples before stating the general fact above.

Let us prove that $\frac{10}{12} = \frac{5}{6}$.

We must show that the 5th point to the right of 0 in the sequence of *sixths* is also the 10th point to the right of 0 in the sequence of *twelfths*. (*Recall: with a clear-cut definition of a fraction, there is no ambiguity about what we must prove.*)

We divide each of the sixths into 2 **equal parts** (i.e., segments of equal lengths), getting twelfths ($2 \times 6 = 12$):

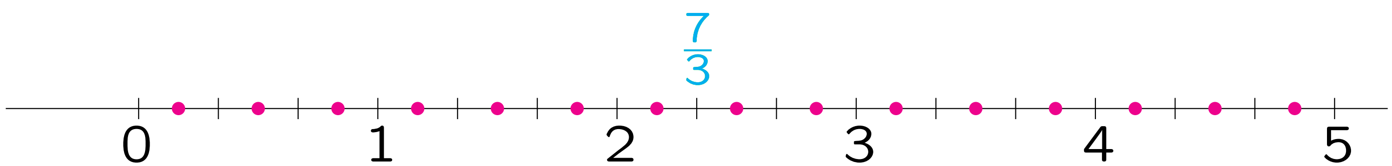



Clearly, the 5th point to the right of 0 in the sequence of *sixths* is also the 10th point to the right of 0 in the sequence of *twelfths*.

One more example: $\frac{14}{6} = \frac{7}{3}$.

We must show that the 7th point to the right of 0 in the sequence of *thirds* is also the 14th point to the right of 0 in the sequence of *sixths*.

We divide each of the thirds into 2 equal parts, getting sixths $(2 \times 3 = 6)$:




The number line now has a sequence of *sixths*, and the 14th point to the right of 0 is therefore exactly the 7th point to the right of 0 in the sequence of *thirds*, as claimed.

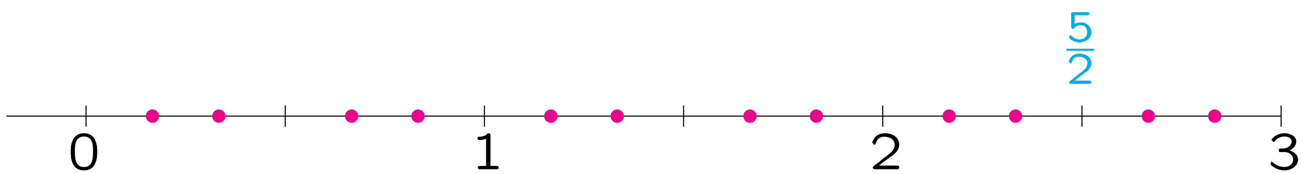
Here is a problem you can do: Explain why $\frac{5}{2} = \frac{15}{6}$.

Write down your explanation; consult with your neighbors if you wish.

*Let us try to get it done in 10 minutes.*

Solution:

Divide each of the halves into 3 equal parts, getting a sequence of sixths ($3 \times 2 = 6$):




The 5th point to the right of 0 in the sequence of *halves* is the same point as the 15th point in the sequence of *sixths*.

Thus $\dfrac{5}{2} = \dfrac{15}{6}$.

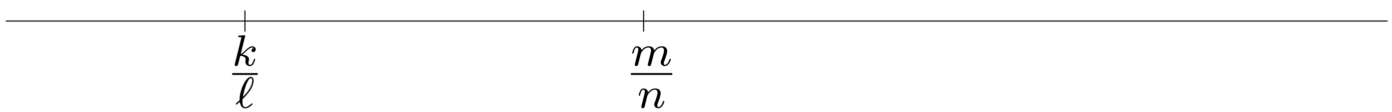
We see that the reasoning for each of $\frac{10}{12} = \frac{5}{6}$, $\frac{14}{6} = \frac{7}{3}$, and $\frac{5}{2} = \frac{15}{6}$ is the same.

The same reasoning in fact proves that $\frac{c\,k}{c\,\ell} = \frac{k}{\ell}$ for all fractions $\frac{k}{\ell}$ and for all $c > 0$.

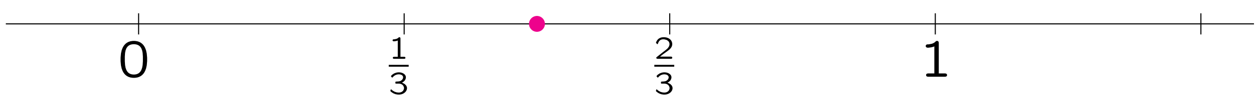
Once a theorem is proved, i.e., shown to be true, it is meant to be used. This theorem will be used often because it lies behind *every* statement about the operations of fractions. We now give some illustrations.

**(C) Comparison of fractions.** First, what does mean to say a fraction $\frac{m}{n}$ is **bigger than** another fraction $\frac{k}{\ell}$? (**Symbols:** $\frac{k}{\ell} < \frac{m}{n}$.)

By definition, it means: on the number line, $\frac{m}{n}$ is to the right of $\frac{k}{\ell}$. (*Textbooks don't give any definition.*)




We must make sure that this definition is consistent with our intuition: Check that $2 < 5$, $\frac{2}{3} < 1$, $\frac{1}{3} < \frac{1}{2}$, etc.



Here is a standard problem:

> *Which of* $\frac{19}{54}$ *and* $\frac{6}{17}$ *is bigger?*

Once we have agreed on "lying further to the right on the number line" as the *definition* of "bigger", the only way we can answer this question is to determine which of $\frac{19}{54}$ and $\frac{6}{17}$ lies to the right of the other.

It must be recognized that the difficulty lies in having to compare the 19th point (to the right of 0) in the sequence of *54ths* with the 6th point (to the right of 0) in the sequence of *17ths*. We simply don't know at this juncture how to compare a **54th** with a **17th**, i.e., $\frac{1}{54}$ with $\frac{1}{17}$.

Consider, for example, the analogous problem:

*Which is longer, 19 feet or 6 meters?*

You try to find *a common unit*. In this case, *cm* is good:

$$19 \text{ ft.} = 19 \times 30.48 \text{ cm} = 579.12 \text{ cm}$$

Since 6 meters is 600 cm, we see that 6 meters is longer.

Thus, faced with comparing 19 **54ths** and 6 **17ths**, we try to find a common unit for $\frac{1}{54}$ and $\frac{1}{17}$. The Theorem says

$$\frac{1}{54} = \frac{17}{54 \times 17} \quad \text{and} \quad \frac{1}{17} = \frac{54}{54 \times 17}$$

So $\frac{1}{54 \times 17}$ will serve as a common unit for $\frac{1}{54}$ and $\frac{1}{17}$.

Now we apply the Theorem twice to get:

$$\frac{19}{54} = \frac{19 \times 17}{54 \times 17} = \frac{323}{54 \times 17}$$

$$\frac{6}{17} = \frac{54 \times 6}{54 \times 17} = \frac{324}{54 \times 17}$$

In terms of $\dfrac{1}{54 \times 17}$

*the 324th point to the right of 0 is clearly to the right of the 323rd point.*

We therefore **conclude** that $\frac{6}{17}$ is bigger than $\frac{19}{54}$.

You may wonder why I spent so much time emphasizing something obvious: why not just say that because $323 < 324$, therefore

$$\frac{323}{54 \times 17} < \frac{324}{54 \times 17}\,?$$

Because, what do you say to a student who claims:

$$\frac{7}{12} < \frac{7}{13} \quad \text{because} \quad 12 < 13?$$

Get your students into the habit of thinking *precisely* by dedicating sufficient time to precise and clear reasoning.

The basic idea of the above can be abstracted: given two fractions $\frac{m}{n}$ and $\frac{k}{\ell}$, the Theorem says we can *always* rewrite them as two fractions with equal denominators, e.g.,

$$\frac{\ell m}{\ell n} \quad \text{and} \quad \frac{kn}{\ell n}$$

We just did that: we rewrote $\frac{19}{54}$ and $\frac{6}{17}$ as

$$\frac{19 \times 17}{54 \times 17} \quad \text{and} \quad \frac{54 \times 6}{54 \times 17}$$

*However*, sometimes this is not the only way to rewrite two fractions so that they have the same denominator!

Suppose we are given $\frac{7}{10}$ and $\frac{69}{100}$. We could rewrite them as

$$\frac{7 \times 100}{10 \times 100} \quad \text{and} \quad \frac{10 \times 69}{10 \times 100}$$

But common sense tells you that, *in this special case*, we can rewrite them as

$$\frac{7 \times 10}{10 \times 10} \quad \text{and} \quad \frac{69}{100}$$

**Summary: Fundamental Fact on Fraction Pairs (FFFP)**
Any two fractions may be regarded as two fractions with the same denominator.

FFFP has far reaching consequences. For example:

**Cross-Multiplication Algorithm (CMA)** : *Given any two fractions* $\frac{k}{\ell}$ *and* $\frac{m}{n}$,

$$\frac{k}{\ell} = \frac{m}{n} \quad \text{if and only if} \quad kn = \ell m$$

$$\frac{k}{\ell} < \frac{m}{n} \quad \text{if and only if} \quad kn < \ell m$$

This is quite clear once we write $\frac{k}{\ell}$ and $\frac{m}{n}$ as

$$\frac{kn}{\ell n} \qquad \text{and} \qquad \frac{\ell m}{\ell n}$$

then the equality of the denominators allows us to compare the fractions by comparing the numerators.

**(D) Comparison of decimals.** By definition, a **decimal** is a fraction whose denominator is a power of 10 written in the special notation introduced by the German Jesuit astronomer C. Clavius (1538-1612):

$$\frac{235}{100} \; (= \frac{235}{10^2}) \quad \text{we write as} \quad 2.35;$$

$$\text{write} \quad \frac{57}{10000} \; (= \frac{57}{10^4}) \quad \text{we write as} \quad 0.0057$$

**A decimal is a fraction**. The alternate definition of 2.35 as $2 + \frac{3}{10} + \frac{5}{100}$ (i.e., 2 and 3 tenths and 5 hundredths) requires the addition of fractions to be defined first before the alternate definition can make sense.

Comparing decimals is therefore a special case of comparing fractions, but with one important advantage. If we want to compare, let us say, 0.12 with 0.098, there is no doubt as to what common denominator to use. For example, rewrite

$$\frac{12}{100} \text{ and } \frac{98}{1000}$$

as

$$\frac{120}{1000} \text{ and } \frac{98}{1000},$$

respectively. Clearly 0.12 is bigger.

Here is a problem for you: Which is bigger, 0.59 or $\frac{16}{27}$?

Solution:

By definition, $0.59 = \frac{59}{100}$. To compare $\frac{59}{100}$ with $\frac{16}{27}$, we use the **CMA**:

$$59 \times 27 = 1593 < 1600 = 100 \times 16$$

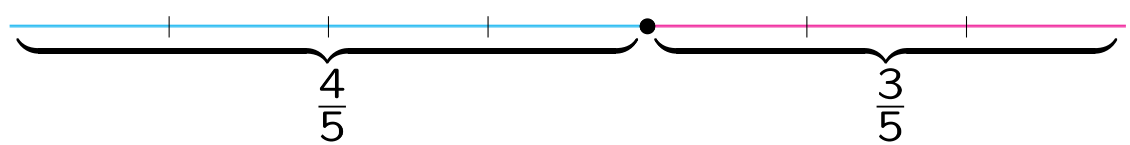
So $\frac{59}{100} < \frac{16}{27}$.

**(E) Addition of fractions.** First, how do we add whole numbers? $4+3$ is obtained by "combining 4 and 3". Thus it is the length of the **concatenation** of a segment of length 4 and a segment of length 3. The meaning of "concatenation" is clear from the following picture:


In other words, the "concatenation of two segments" is the segment obtained by joining an endpoint of one segment to an endpoint of the other and putting them on a straight line.

Now, because whole numbers are also fractions, the meaning of $\frac{4}{5} + \frac{3}{5}$ should not be different from the addition of whole numbers.

We define $\mathbf{\frac{4}{5}} + \mathbf{\frac{3}{5}}$ to be the length of the concatenation of one segment of length $\frac{4}{5}$ followed by a second segment of length $\frac{3}{5}$:



*In terms of a segment of length* $\frac{1}{5}$, $\frac{4}{5} + \frac{3}{5}$ is just the concatenation of 4 such segments and 3 such segments, and is therefore exactly $4 + 3$ such segments. *Essentially the same as* $4 + 3$.

Thus, by definition,

$$\frac{4}{5} + \frac{3}{5} \;=\; \frac{4+3}{5}$$

Next, we consider something more complicated: $\frac{4}{7} + \frac{2}{5}$.

We define $\mathbf{\frac{4}{7}} + \mathbf{\frac{2}{5}}$ in exactly the same way: it is the length of the concatenation of one segment of length $\frac{4}{7}$ followed by another segment of length $\frac{2}{5}$:

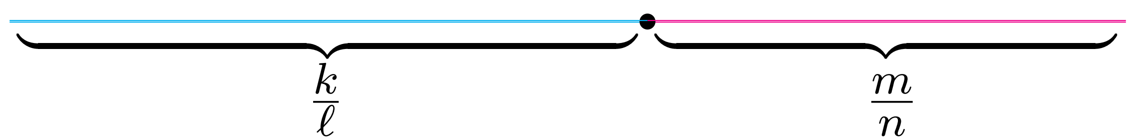
By definition, $\frac{4}{7} + \frac{2}{5}$ is the total length of 4 of the $\frac{1}{7}$'s and 2 of the $\frac{1}{5}$'s.

This is like adding 4 feet and 2 meters; we cannot find its exact value until we can find *a common unit for feet and meters*.

The same with fractions. FFFP tells us what to do: use $\frac{1}{7\times 5}$ as the common unit.

$$\frac{4}{7}+\frac{2}{5} \;=\; \frac{4\times 5}{7\times 5}+\frac{7\times 2}{7\times 5} \;=\; \frac{34}{35}$$

In general, we define the addition of $\frac{k}{\ell}$ and $\frac{m}{n}$ in exactly the same way: $\boldsymbol{\frac{k}{\ell}} + \boldsymbol{\frac{m}{n}}$ is the length of the concatenation of one segment of length $\frac{k}{\ell}$ and another of length $\frac{m}{n}$:




By FFFP,

$$\frac{k}{\ell} + \frac{m}{n} = \frac{kn}{\ell n} + \frac{\ell m}{\ell n} = \frac{kn + \ell m}{\ell n}$$

By definition, addition of fractions is *commutative*, i.e.,

$$\frac{k}{\ell} + \frac{m}{n} \; = \; \frac{m}{n} + \frac{k}{\ell}$$

because the length of a concatenated segment is independent of the order of the segments being concatenated.

Commutativity of addition will be useful when we consider mixed numbers.

Note that we have added fractions **without** once considering Least Common Denominator. The LCD is a grave distraction to learning fractions but contributes nothing to the understanding of fraction addition.

*The continuity from whole numbers to fractions is of critical importance for the learning of fractions.*

At the moment, students learn about the addition of whole numbers as "combining things", but the addition of fractions is presented as a completely different concept: getting the Least Common Denominator and rewriting the numerators. **Nothing about "combining things".**

The discontinuity leads to unnecessary mental disorientation and unwillingness to learn mathematics.

Our definition of fraction addition above shows clearly why adding fractions still means "combining things".

**A 1978 NAEP** (National Assessment of Educational Progress in the U.S.) **question on the eighth grade test:**

Estimate $\dfrac{12}{13} + \dfrac{7}{8}$.

(1) 1

(2) 19

(3) 21

(4) I don't know

(5) 2

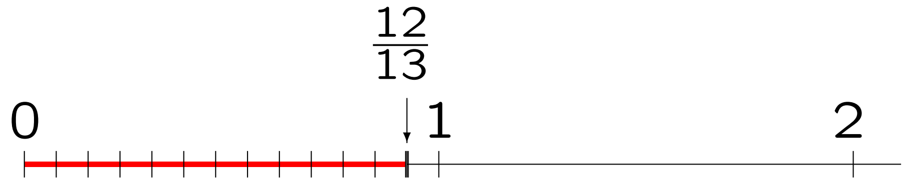
(Recall: We encountered this addition at the beginning of the workshop.)

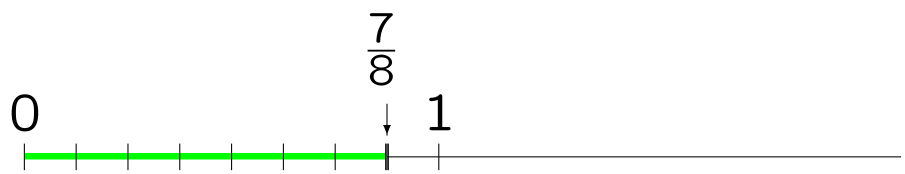
The statistics:

- 7% chose "1".
- 28% chose "19".
- 27% chose "21".
- 14% chose "I don't know".
- 24% chose "2" (the correct answer).

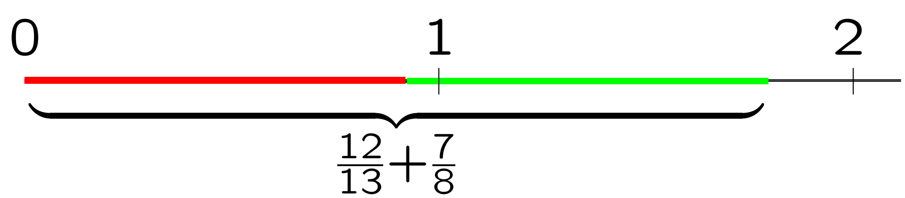
**Do we only blame the students?**

Students need a mental image of a fraction as intuitive as the mental image of a whole number given by the fingers on their hands.




Direct concatenation gives:



**(F) Mixed numbers.** It is customary to shorten a sum such as $3 + \frac{2}{7}$ to just $3\frac{2}{7}$; in mathematics, we love abbreviations!

In general, we call a fraction $\frac{m}{n}$ a **proper fraction** if it is smaller than 1, or what is the same, $m < n$. Then the notation $\boldsymbol{q\frac{m}{n}}$, where $q$ is a nonzero whole number and $\frac{m}{n}$ is a proper fraction, is called a **mixed number**.

It is important that you do not introduce the concept of a "mixed number" until you have defined the addition of fractions. Students have to realize that a mixed number is a *sum* of a nonzero whole number and a proper fraction.

The *fear of mixed numbers* comes mainly from (at least in America) introducing mixed numbers as soon as fractions are mentioned, *before* the addition of fractions is discussed.

The advantage of knowing the correct definition of mixed numbers is that it is clear how to convert a mixed number to a fraction. For example:

$$3\frac{2}{7} = \frac{3}{1}+\frac{2}{7} = \frac{3\times 7}{7}+\frac{2}{7} = \frac{23}{7}$$

There is no need to memorize any formulas!

To convert a fraction to a mixed number, use *division-with-remainder* on the numerator with the denominator as the divisor:

$$\frac{37}{7} = \frac{(5\times 7)+2}{7} = \frac{5\times 7}{7}+\frac{2}{7} = 5+\frac{2}{7} = 5\frac{2}{7}$$

To compute with mixed numbers, remember that addition is commutative, e.g.,

$$1\frac{2}{3} + 7\frac{4}{5} \;=\; 1 + \frac{2}{3} + 7 + \frac{4}{5} \;=\; (1+7) + (\frac{2}{3} + \frac{4}{5})$$

We know that

$$\frac{2}{3} + \frac{4}{5} \;=\; \frac{2 \times 5}{3 \times 5} + \frac{3 \times 4}{3 \times 5} \;=\; \frac{22}{15} \;=\; \frac{1 \times 15 + 7}{15} \;=\; 1 + \frac{7}{15}$$

Thus

$$1\frac{2}{3} + 7\frac{4}{5} \;=\; 8 + 1 + \frac{7}{15} \;=\; 9\frac{7}{15}$$

Here is a problem for you: do the same addition by first converting $1\frac{2}{3}$ and $7\frac{4}{5}$ to fractions, and then add.

Solution:

$$1\frac{2}{3}+7\frac{4}{5} = \frac{5}{3}+\frac{39}{5}$$
$$= \frac{5\times 5}{3\times 5} + \frac{3\times 39}{3\times 5}$$
$$= \frac{25+117}{15} = \frac{142}{15}$$

This is a perfectly good answer. However, if you wish, we can convert it to a mixed number:

$$1\frac{2}{3}+7\frac{4}{5} = \frac{(9\times 15)+7}{15} = 9\frac{7}{15}$$

**(G) Subtraction.** It will be the same as addition, with one element of subtlety.

Once we have negative numbers, we can subtract a fraction from *any* fraction. But in terms of student learning, it may be better to break up subtraction into two steps:

(1) $\frac{k}{\ell} - \frac{m}{n}$, with $\frac{k}{\ell} > \frac{m}{n}$, so that the answer is a fraction rather than a negative fraction.

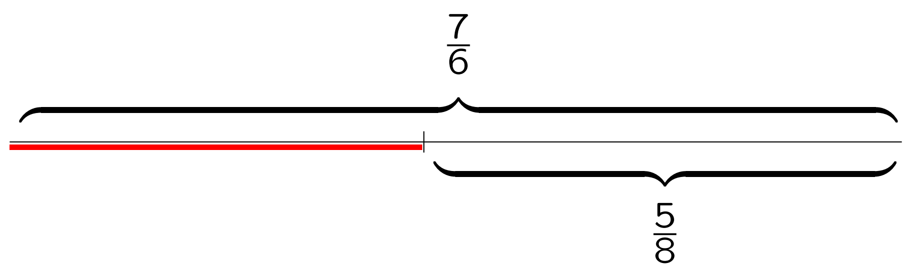
(2) $\frac{k}{\ell} - \frac{m}{n}$, with no restrictions on $\frac{k}{\ell}$ or $\frac{m}{n}$.

We only deal with step (1) here.

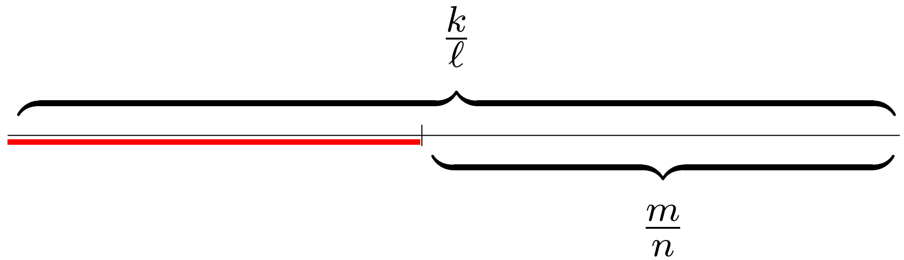
Consider $\dfrac{7}{6} - \dfrac{5}{8}$.

The CMA tells us that $\frac{7}{6} > \frac{5}{8}$. The meaning of this subtraction is then the same as the case of whole numbers: “taking $\frac{5}{8}$ away from $\frac{7}{6}$.”

*More formally:* $\boldsymbol{\frac{7}{6} - \frac{5}{8}}$ is the length of the segment that remains (the red segment below) after a segment of length $\frac{5}{8}$ has been removed from one end of a segment of length $\frac{7}{6}$.



In general, if $\frac{m}{n} < \frac{k}{\ell}$, then we define $\boldsymbol{\frac{k}{\ell} - \frac{m}{n}}$ to be the length of the remaining segment when a segment of length $\frac{m}{n}$ is removed from one end of a segment of length $\frac{k}{\ell}$.




We use FFFP to compute:

$$\frac{k}{\ell} - \frac{m}{n} = \frac{kn}{\color{blue}{\ell n}} - \frac{\ell m}{\color{blue}{\ell n}} = \frac{kn - \ell m}{\ell n}$$

Because of the CMA, we know $kn > \ell m$ so that the whole number subtraction $kn - \ell m$ gives a *whole number* rather than a negative integer.

*Compute* $7\frac{1}{3} - 2.94$.

$$\begin{aligned} 7\frac{1}{3} - 2.94 &= \frac{22}{3} - \frac{294}{100} = \frac{22 \times 100}{300} - \frac{3 \times 294}{300} \\ &= \frac{2200 - 882}{300} \\ &= \frac{1318}{300} \end{aligned}$$

This is a good answer, but if we wish, we can get a mixed number instead:

$$\frac{1318}{300} = \frac{4 \times 300 + 118}{300} = 4\frac{118}{300}$$

**(H) Multiplication.** The product $3 \times 5$ means "3 copies of 5".

So $\frac{2}{3} \times \frac{5}{4}$ should mean, at least *intuitively*, "two-thirds of a copy of $\frac{5}{4}$.

How to make precise the concept of "two-thirds of a copy of $\frac{5}{4}$"? Imagine buying $1\frac{1}{4}$ kg ($= \frac{5}{4}$ kg) of cheese, and you want to give two-thirds of it to your mother. What you do is divide the cheese into 3 equal parts (by weight) and give your mother 2 of the parts.

This suggests a way to define $\frac{2}{3} \times \frac{5}{4}$ *precisely.*

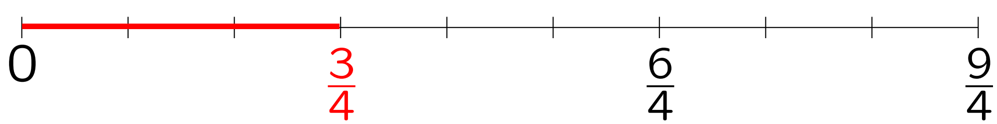
(Yes, we need a precise definition.)

**Definition** $\frac{2}{3} \times \frac{5}{4}$ is the length of 2 concatenated segments when $[0, \frac{5}{4}]$ is divided into 3 segments of equal length.

To compute $\frac{2}{3} \times \frac{5}{4}$, it suffices to find out the length of one segment when $[0, \frac{5}{4}]$ is divided into 3 segments of equal length, i.e., it suffices to compute

$$\frac{1}{3} \times \frac{5}{4}$$

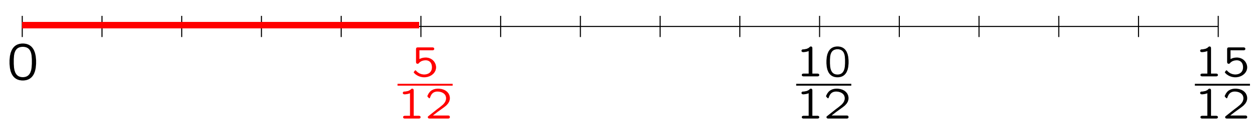
We begin with an easier calculation: $\frac{1}{3} \times \frac{9}{4}$. It is $\frac{3}{4}$.



We could compute $\frac{1}{3} \times \frac{9}{4} = \frac{3}{4}$, because it is easy to divide 9 segments of equal length $(= \frac{1}{4})$ into 3 parts of equal length: each part has three segments.

Now we want $\frac{1}{3} \times \frac{5}{4}$. How to divide *5* segments of equal length $(= \frac{1}{4})$ into *3* parts of equal length?

However, by equivalent fractions, $\frac{1}{3} \times \frac{5}{4} = \frac{1}{3} \times \frac{3 \times 5}{3 \times 4}$. We know how to divide 15 segments of equal length $(= \frac{1}{12})$ into 3 parts of equal length: each part has five segments.




$\frac{1}{3} \times \frac{5}{4} = \frac{5}{12}$.

The equality $\frac{1}{3} \times \frac{5}{4} = \frac{5}{12}$ means if we divide $[0, \frac{5}{4}]$ into 3 equal parts, then the length of one part is $\frac{5}{12}$. Since $\frac{2}{3} \times \frac{5}{4}$ is the length of 2 concatenated parts when $[0, \frac{5}{4}]$ is divided into 3 equal parts, we see that

$$\frac{2}{3} \times \frac{5}{4} \;=\; \frac{5}{12} + \frac{5}{12} \;=\; \frac{10}{12}$$

We recognize that this says:

$$\frac{2}{3} \times \frac{5}{4} \;=\; \frac{2 \times 5}{3 \times 4}$$

In general, we **define** $\boldsymbol{\frac{k}{\ell} \times \frac{m}{n}}$ to be the length of $k$ (concatenated) parts when $[0, \frac{m}{n}]$ is divided into $\ell$ equal parts.

This leads us to guess that, in general, for any two fractions $\frac{k}{\ell}$ and $\frac{m}{n}$,

$$\frac{k}{\ell} \times \frac{m}{n} = \frac{km}{\ell n}$$

We call this the **Product Formula**.

**Proof of the Product Formula:**

$\frac{k}{\ell} \times \frac{m}{n}$ is the length of $k$ parts, each part of length

$$\frac{1}{\ell} \times \frac{m}{n}$$

We first compute the length of $\frac{1}{\ell} \times \frac{m}{n}$.

$$\frac{1}{\ell} \times \frac{m}{n} \;=\; \frac{1}{\ell} \times \frac{\ell m}{\ell n} \;=\; \frac{m}{\ell n}$$

Now $\frac{k}{\ell} \times \frac{m}{n}$ is the length of $k$ concatenated copies of $\frac{m}{\ell}$, which is

$$\underbrace{\frac{m}{\ell n} + \cdots + \frac{m}{\ell n}}_{k} \;=\; \frac{km}{\ell n}$$

So the Product Formula is true.

The Product Formula shows that fraction multiplication is commutative in general:

$$\frac{k}{\ell} \times \frac{m}{n} \; = \; \frac{m}{n} \times \frac{k}{\ell}$$

If you find this formula boring, do the following problem:

Which is heavier, $\frac{7}{9}$ of $\frac{11}{4}$ kg of sand, or

$\frac{11}{4}$ of $\frac{7}{9}$ kg of sand?

(By definition, the first is the totality of 7 parts when $\frac{11}{4}$ is divided into 9 equal parts, while the latter is 11 parts when $\frac{7}{9}$ is divided into 4 equal parts. Are they equal?)

Fraction Multiplication is also associative and distributive in general:

$$\left(\frac{k}{\ell} \times \frac{m}{n}\right) \times \frac{a}{b} \;=\; \frac{k}{\ell} \times \left(\frac{m}{n} \times \frac{a}{b}\right)$$

and

$$\frac{k}{\ell} \times \left(\frac{m}{n} + \frac{a}{b}\right) \;=\; \left(\frac{k}{\ell} \times \frac{m}{n}\right) \;+\; \left(\frac{k}{\ell} \times \frac{a}{b}\right)$$

The verification using the Product Formula is routine (and somewhat tedious).

We next explore the many consequences of the Product Formula.

A first consequence of the Product Formula is the **cancellation phenomenon**, e.g.,

$$\frac{\cancel{8} \times 5}{\cancel{9} \times 13} \times \frac{7 \times \cancel{9}}{\cancel{8} \times 11} = \frac{5}{13} \times \frac{7}{11}$$

i.e., *we cancelled the 8 in top and bottom, and cancelled the 9 in top and bottom.*

We could do that because, by the Product Formula and the theorem on equivalent fractions, we have:

$$\frac{8 \times 5}{9 \times 13} \times \frac{7 \times 9}{8 \times 11} = \frac{(8 \times 9) \times (5 \times 7)}{(8 \times 9) \times (13 \times 11)} = \frac{5 \times 7}{13 \times 11}$$

Obviously $\frac{5}{13} \times \frac{7}{11}$ is also equal to $\frac{5 \times 7}{13 \times 11}$.

By cancellation, any nonzero fraction $\frac{m}{n}$ satisfies

$$\frac{n}{m} \times \frac{m}{n} \;=\; 1$$

Now take an arbitrary fraction $\frac{k}{\ell}$ and multiply both sides of the above equality by $\frac{k}{\ell}$. Then we get:

$$\frac{k}{\ell} \times \left(\frac{n}{m} \times \frac{m}{n}\right) \;=\; \frac{k}{\ell}$$

By the associative law, we have

$$\left(\frac{k}{\ell} \times \frac{n}{m}\right) \times \frac{m}{n} \;=\; \frac{k}{\ell}$$

Denoting the fraction $\frac{k}{\ell} \times \frac{n}{m}$ by $Q$, this means:

> Given any nonzero fraction $\frac{m}{n}$ and any fraction $\frac{k}{\ell}$, there is always a fraction $Q$ so that $\frac{k}{\ell} \;=\; Q \times \frac{m}{n}$.

The fact that for any fractions $\frac{m}{n}$ and $\frac{k}{\ell}$, $(\frac{m}{n} \neq 0)$, there is always a fraction $Q$ so that $\frac{k}{\ell} = Q \times \frac{m}{n}$, will be basic to the discussion of fraction division.

We now give a second interpretation of $\frac{k}{\ell} \times \frac{m}{n}$:

$\dfrac{k}{\ell} \times \dfrac{m}{n}$ is $\frac{k}{\ell}$ copies of $\frac{m}{n}$, in the sense of everyday language.

**If $\frac{k}{\ell}$ is a whole number**, e.g., 5, then by the Product Formula,

$$5 \times \frac{m}{n} \;=\; \frac{5}{1} \times \frac{m}{n} \;=\; \frac{5m}{n} \;=\; \underbrace{\frac{m}{n} + \cdots + \frac{m}{n}}_{5},$$

which displays "5 copies of $\frac{m}{n}$".

**If $\frac{k}{\ell}$ is a proper fraction**, e.g., $\frac{3}{7}$, then by the definition of fraction multiplication, $\frac{3}{7} \times \frac{m}{n}$ is exactly "$\frac{3}{7}$ copies of $\frac{m}{n}$".

**Finally, if $\frac{k}{\ell}$ is not a proper fraction**, e.g., $\frac{35}{4}$, then we first write it as a mixed number, $8\frac{3}{4}$. Suppose the capacity of a bucket is $\frac{m}{n}$ liters. Does $(8\frac{3}{4} \times \frac{m}{n})$ liters have the meaning of "8 and $\frac{3}{4}$ buckets"?

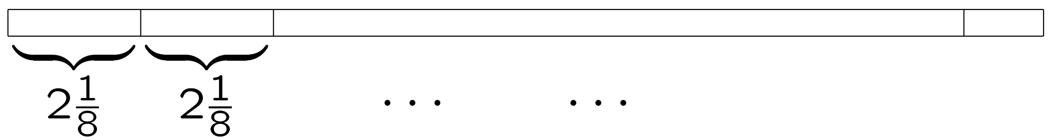
By the distributive law, $8\frac{3}{4} \times \frac{m}{n}$ liters is equal to

$$8\frac{3}{4} \times \frac{m}{n} \;=\; (8 + \frac{3}{4}) \times \frac{m}{n} \;=\; (8 \times \frac{m}{n}) + (\frac{3}{4} \times \frac{m}{n})$$

Now $8 \times \frac{m}{n}$ liters is "8 buckets", and $\frac{3}{4} \times \frac{m}{n}$ liters is (by definition) $\frac{3}{4}$ of the bucket.

Thus $8\frac{3}{4} \times \frac{m}{n}$ liters is "$8\frac{3}{4}$ buckets" (if the capacity of the bucket is $\frac{m}{n}$ liters).

**Example.** *A rod* $15\frac{5}{7}$ *meters long is cut into short pieces which are* $2\frac{1}{8}$ *meters long. How many short pieces are there?*




Students are taught that the way to do such problems is to *divide*. In other words, the answer is

$$\frac{15\frac{5}{7}}{2\frac{1}{8}}$$

But why? How to explain this to students?

Solution: We do the problem by multiplication, and return to it at the end after we have done division.

Let us say $\frac{a}{b}$ short pieces make up the rod. By what we just did, this says

$$\frac{a}{b} \times 2\frac{1}{8} \;=\; 15\frac{5}{7}$$

But $2\frac{1}{8} = \frac{17}{8}$ and $15\frac{5}{7} = \frac{110}{7}$, so

$$\frac{a}{b} \times \frac{17}{8} \;=\; \frac{110}{7}$$

Therefore,

$$\frac{a}{b} \;=\; \frac{a}{b} \times \frac{17}{8} \times \frac{8}{17} \;=\; \frac{110}{7} \times \frac{8}{17} \;=\; \frac{880}{119} \;=\; 7\frac{47}{119}$$

Comment on $7\frac{47}{119}$.

We know the answer is: "$7\frac{47}{119}$ short pieces equal the whole rod", i.e.,

$$7\frac{47}{119} \times 2\frac{1}{8} \;=\; 15\frac{5}{7}$$

What is the meaning of $\frac{47}{119}$? The equation above says

$$15\frac{5}{7} \;=\; (7 + \frac{47}{119}) \times 2\frac{1}{8} \;=\; (7 \times 2\frac{1}{8}) + (\frac{47}{119} \times 2\frac{1}{8})$$

This says, *explicitly*, that the whole rod consists of 7 short pieces, each $2\frac{1}{8}$ meters long, plus $\frac{47}{119}$ *of a short piece*.

Why bother with such an elaborate definition of multiplication? Why not just define multiplication by the Product Formula

$$\frac{k}{\ell} \times \frac{m}{n} \;=\; \frac{km}{\ell n}\,?$$

Because: $(i)$ This definition of multiplication immediately raises the question: why not define addition as

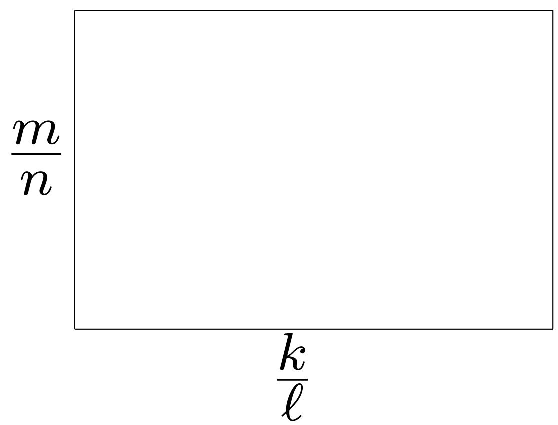
$$\frac{k}{\ell} + \frac{m}{n} \;=\; \frac{k+m}{\ell+n}$$

It may not easy to explain to school students why not.

$(ii)$ Problems such as the one about the rod cannot be done if multiplication is defined by the product formula with no other meaning.

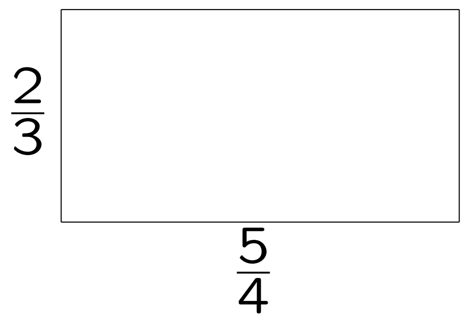
**(I) Multiplication of fractions as area.** The following theorem complements our understanding of what fraction multiplication means.

$$\boldsymbol{\frac{m}{n} \times \frac{k}{\ell}} \;=\; \textit{the area of a rectangle with sides } \frac{m}{n} \textit{ and } \frac{k}{\ell}$$

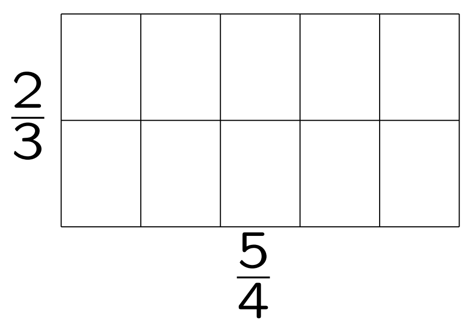



To prove the theorem, we first prove that the area of a rectangle with sides $\frac{m}{n}$ and $\frac{k}{\ell}$ is $\frac{mk}{n\ell}$. Then we use the Product Formula to conclude that the latter is $\frac{m}{n} \times \frac{k}{\ell}$.

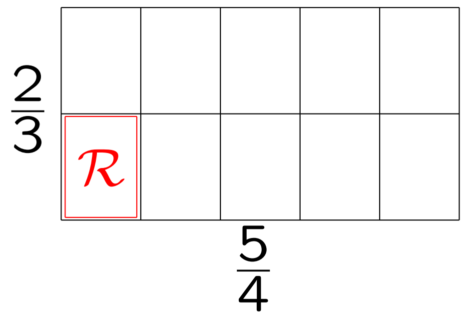
It suffices to give the proof of the theorem for the special case of a rectangle with sides $\frac{2}{3}$ and $\frac{5}{4}$, because the reasoning in the general case is no different. Thus we have:



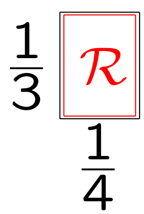

Recall that $\frac{2}{3}$ is 2 copies of $\frac{1}{3}$, and $\frac{5}{4}$ is 5 copies of $\frac{1}{4}$, as shown:



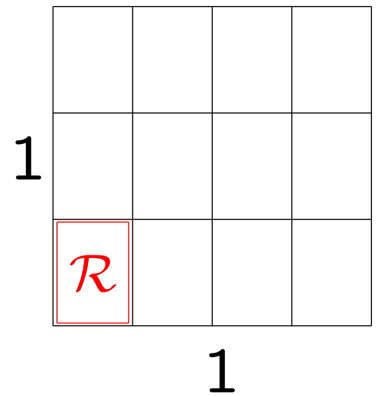
If we can find out the area of any of the smaller rectangles, such as the red one $\mathcal{R}$ below, then the area of the big rectangle would just be the sum of $(2 \times 5)$ of the area of $\mathcal{R}$.




We will prove that the area of $\mathcal{R}$ is $\frac{1}{3\times 4}$. Recall $\mathcal{R}$ has sides $\frac{1}{3}$ and $\frac{1}{4}$.

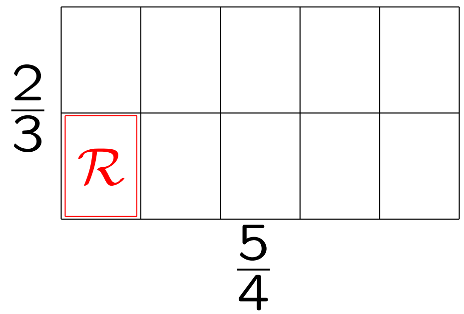


Look at a **unit square**, i.e., a square all of whose sides have length 1. Then the unit square is *paved* by $3 \times 4 = 12$ rectangles all of which are congruent to $\mathcal{R}$, as shown.




On the number line, let the unit 1 be the area of the unit square. The areas of these 12 rectangles provide a division of the unit 1 into 12 equal parts, so the area of any one of these 12 rectangles is $\frac{1}{12}$. In particular, the area of $\mathcal{R}$ is $\frac{1}{12}$.

Now back to the original rectangle with side lengths $\frac{2}{3}$ and $\frac{5}{4}$. It is paved by $2 \times 5 = 10$ rectangles each congruent to $\mathcal{R}$, as shown.




Since the area of each of the 10 rectangles is $\frac{1}{12}$, as shown above, the total area of the big rectangle is therefore $\frac{10}{12} = \frac{2\times5}{3\times4}$. By the Product Formula,

$$\text{area of rectangle of } \tfrac{2}{3} \text{ by } \tfrac{5}{4} \;=\; \frac{2 \times 5}{3 \times 4}$$

This completes the proof.

**(J) Division.** To uncover the meaning of the division of fractions, again we look to whole numbers for guidance because whole numbers are themselves fractions and their division cannot be different from the division of arbitrary fractions.

We tell students that $\frac{24}{6}$ (the preferred notation for "$24 \div 6$") is 4 because $4 \times 6 = 24$, that $\frac{48}{3} = 16$ because $16 \times 3 = 48$, that $\frac{54}{18} = 3$ because $3 \times 18 = 54$, etc.

We can summarize as follows. If $m$, $n$, $q$ are whole numbers ($n \neq 0$ and $m$ is a multiple of $n$), then we say

$$\frac{m}{n} = q \quad \text{if} \quad m = qn$$

**Comments** $(i)$ Division among whole numbers is nothing more than a different way of writing a multiplication fact.

$(ii)$ A division $\frac{m}{n}$ among whole numbers $m$, $n$ cannot be carried out unless $m$ is a multiple of $n$. For example, $\frac{37}{16}$ is not a division among whole numbers.

$(iii)$ Among whole numbers, be careful to distinguish between **division** and **division-with-remainder**. *Division* (like addition, subtractions and multiplication) is a **binary operation**, in the sense that it send two numbers (e.g., 48 and 3) to a third number (16 in this case). *Division-with-remainder* is not a binary operation as it sends 37 and 16 to two numbers, 2 (quotient) and 5 (remainder).

For whole numbers $m$, $n$ ($n \neq 0$), we say $m$ divided by $n$ is the whole number $q$ if we already know $m = qn$.

For the division of a fraction $M$ by another fraction $N$ ($N \neq 0$), we simply *imitate*:

Given fractions $M$, $N$, ($N \neq 0$), we define **$\boldsymbol{M}$ divided by $\boldsymbol{N}$ to be the fraction $\boldsymbol{Q}$** if we already know $M = QN$. In symbols,
$$\frac{\boldsymbol{M}}{\boldsymbol{N}} = \boldsymbol{Q} \quad \text{if} \quad M = QN.$$

Recall an earlier fact: *Given any nonzero fraction* $\frac{m}{n}$ *and any fraction* $\frac{k}{\ell}$*, there is always a fraction* $Q$ *so that* $\frac{k}{\ell} = Q \times \frac{m}{n}$.

Thus given fractions $M$ and $N$ as above, *there is always a fraction* $Q$ *so that* $\dfrac{M}{N} = Q$.

Conclusion: Unlike whole numbers, it makes sense to divide any fraction $M$ by a nonzero fraction $N$, i.e.,$\dfrac{M}{N}$, because there is always a fraction $Q$ so that $M = QN$. (So $\dfrac{M}{N} = Q$).

We repeat: the statement that the division of a fraction $M$ by a nonzero fraction $N$ is equal to $Q$, i.e., $\dfrac{M}{N} = Q$, is merely a different way of writing the multiplicative fact that $M = QN$ for a fraction $Q$. The fact that there is always such a fraction $Q$ is guaranteed.

Let $M = \frac{k}{\ell}$ and $N = \frac{m}{n}$. If $\dfrac{\frac{k}{\ell}}{\frac{m}{n}} = Q$, then $\frac{k}{\ell} = Q \times \frac{m}{n}$.

Multiplying both sides by $\frac{n}{m}$, we get

$$\frac{k}{\ell} \times \frac{n}{m} = Q \times \frac{m}{n} \times \frac{n}{m}$$

Thus $Q = \dfrac{k}{\ell} \times \dfrac{n}{m}$.

To recapitulate: if $Q$ is the division of a fraction $\frac{k}{\ell}$ by $\frac{m}{n}$, then

$$Q = \frac{k}{\ell} \times \frac{n}{m}$$

In other words, "to divide, you invert and multiply", i.e., invert $\frac{m}{n}$ to get $\frac{n}{m}$ and then use it to multiply $\frac{k}{\ell}$.

In America, this used to be considered totally incomprehensible. Even now (2010), some mathematics educators still try to avoid teaching the invert-and-multiply rule.

With the concept of *division* clearly defined, we see why one should invert and multiply.

Let us revisit an earlier problem: *A rod* $15\frac{5}{7}$ *meters long is cut into short pieces which are* $2\frac{1}{8}$ *meters long. How many short pieces are there?*

The detailed discussion of multiplication makes sense of the fact that if there are $Q$ short pieces in the rod, then $15\frac{5}{7} = Q \times 2\frac{1}{8}$. By the definition of division, this means

$$Q = \frac{15\frac{5}{7}}{2\frac{1}{8}} = \frac{\frac{110}{7}}{\frac{17}{8}} = \frac{110}{7} \times \frac{8}{7} = \frac{880}{119} = 7\frac{47}{119}$$

where we have used the invert and multiply rule to compute the division.

Notice that we have **explained** why one should **divide** $15\frac{5}{7}$ by $2\frac{1}{8}$.

**Comments:** The discussion of division is heavily dependent on a solid knowledge of multiplication. First, the fact that $\frac{M}{N}$ always makes sense depends on a fact proved about multiplication. In the solution of word problems, such as the last problem with the rod, the possibility of reasoning with the problem to get a solution again depends on a solid grounding in multiplication.

In mathematics, foundational knowledge is always critical.