3 The Creation of Classical Greek Mathematics

> This, therefore, is mathematics: she reminds you of the invisible form of the soul; she gives life to her own discoveries; she awakens the mind and purifies the intellect; she brings light to our intrinsic ideas; she abolishes oblivion and ignorance which are ours by birth.
>
> PROCLUS

1. *Background*

In the history of civilization the Greeks are preeminent, and in the history of mathematics the Greeks are the supreme event. Though they did borrow from the surrounding civilizations, the Greeks built a civilization and culture of their own which is the most impressive of all civilizations, the most influential in the development of modern Western culture, and decisive in founding mathematics as we understand the subject today. One of the great problems of the history of civilization is how to account for the brilliance and creativity of the ancient Greeks.

Though our knowledge of their early history is subject to correction and amplification as more archeological research is carried on, we now have reason to believe, on the basis of the *Iliad* and the *Odyssey* of Homer, the decipherment of ancient languages and scripts, and archeological investigations, that the Greek civilization dates back to 2800 B.C. The Greeks settled in Asia Minor, which may have been their original home, on the mainland of Europe in the area of modern Greece, and in southern Italy, Sicily, Crete, Rhodes, Delos, and North Africa. About 775 B.C. the Greeks replaced various hieroglyphic systems of writing with the Phoenician alphabet (which was also used by the Hebrews). With the adoption of an alphabet the Greeks became more literate, more capable of recording their history and ideas.

As the Greeks became established they visited and traded with the Egyptians and Babylonians. There are many references in classical Greek writings to the knowledge of the Egyptians, whom some Greeks erroneously considered the founders of science, particularly surveying, astronomy, and

arithmetic. Many Greeks went to Egypt to travel and study. Others visited Babylonia and learned mathematics and science there.

The influence of the Egyptians and Babylonians was almost surely felt in Miletus, a city of Ionia in Asia Minor and the birthplace of Greek philosophy, mathematics, and science. Miletus was a great and wealthy trading city on the Mediterranean. Ships from the Greek mainland, Phoenicia, and Egypt came to its harbors; Babylonia was connected by caravan routes leading eastward. Ionia fell to Persia about 540 B.C., though Miletus was allowed some independence. After an Ionian revolt against Persia in 494 B.C. was crushed, Ionia declined in importance. It became Greek again in 479 B.C. when Greece defeated Persia, but by then cultural activity had shifted to the mainland of Greece with Athens as its center.

Though the ancient Greek civilization lasted until about A.D. 600, from the standpoint of the history of mathematics it is desirable to distinguish two periods, the classical, which lasted from 600 to 300 B.C., and the Alexandrian or Hellenistic, from 300 B.C. to A.D. 600. The adoption of the alphabet, already mentioned, and the fact that papyrus became available in Greece during the seventh century B.C. may account for the blossoming of cultural activity about 600 B.C. The availability of this writing paper undoubtedly helped the spread of ideas.

2. *The General Sources*

The sources of our knowledge of Greek mathematics are, peculiarly, less authentic and less reliable than our sources for the much older Babylonian and Egyptian mathematics, because no original manuscripts of the important Greek mathematicians are extant. One reason is that papyrus is perishable; though the Egyptians also used papyrus, by luck a few of their mathematical documents did survive. Some of the voluminous Greek writings might still be available to us if their great libraries had not been destroyed.

Our chief sources for the Greek mathematical works are Byzantine Greek codices (manuscript books) written from 500 to 1500 years after the Greek works were originally composed. These codices are not literal reproductions but critical editions, so that we cannot be sure what changes may have been made by the editors. We also have Arabic translations of the Greek works and Latin versions derived from Arabic works. Here again we do not know what changes the translators may have made or how well they understood the original texts. Moreover, even the Greek texts used by the Arabic and Byzantine authors were questionable. For example, though we do not have the Alexandrian Greek Heron's manuscript, we know that he made a number of changes in Euclid's *Elements*. He gave different proofs and added new cases of the theorems and converses. Likewise Theon of Alexandria (end of 4th cent. A.D.) tells us that he altered sections of the *Elements* in his edition.

The Greek and Arabic versions we have may come from such versions of the originals. However, in one or another of these forms we do have the works of Euclid, Apollonius, Archimedes, Ptolemy, Diophantus, and other Greek authors. Many Greek texts written during the classical and Alexandrian periods did not come down to us because even in Greek times they were superseded by the writings of these men.

The Greeks wrote some histories of mathematics and science. Eudemus (4th cent. B.C.), a member of Aristotle's school, wrote a history of arithmetic, a history of geometry, and a history of astronomy. Except for fragments quoted by later writers, these histories are lost. The history of geometry dealt with the period preceding Euclid's and would be invaluable were it available. Theophrastus (*c.* 372–*c.* 287 B.C.), another disciple of Aristotle, wrote a history of physics, and this, too, except for a few fragments, is lost.

In addition to the above, we have two important commentaries. Pappus (end of 3rd cent. A.D.) wrote the *Synagoge* or *Mathematical Collection*; almost the whole of it is extant in a twelfth-century copy. This is an account of much of the work of the classical and Alexandrian Greeks from Euclid to Ptolemy, supplemented by a number of lemmas and theorems that Pappus added as an aid to understanding. Pappus had also written the *Treasury of Analysis*, a collection of the Greek works themselves. This book is lost, but in Book VII of his *Mathematical Collection* he tells us what his *Treasury* contained.

The second important commentator is Proclus (A.D. 410–485), a prolific writer. Proclus drew material from the texts of the Greek mathematicians and from prior commentaries. Of his surviving works, the *Commentary*, which treats Book I of Euclid's *Elements*, is the most valuable. Proclus apparently intended to discuss more of the *Elements*, but there is no evidence that he ever did so. The *Commentary* contains one of the three quotations traditionally credited to Eudemus' history of geometry (see sec. 10) but probably taken from a later modification. This particular extract, the longest of the three, is referred to as the Eudemian summary. Proclus also tells us something about Pappus' work. Thus, besides the later editions and versions of some of the Greek classics themselves, Pappus' *Mathematical Collection* and Proclus' *Commentary* are the two main sources of the history of Greek mathematics.

Of original wordings (though not the manuscripts) we have only a fragment concerning the lunes of Hippocrates, quoted by Simplicius (first half of 6th cent. A.D.) and taken from Eudemus' lost *History of Geometry*, and a fragment of Archytas on the duplication of the cube. And of original manuscripts we have some papyri written in Alexandrian Greek times. Related sources on Greek mathematics are also immensely valuable. For example, the Greek philosophers, especially Plato and Aristotle, had much to say about mathematics and their writings have survived somewhat in the same way as have the mathematical works.

The reconstruction of the history of Greek mathematics, based on sources

such as we have described, has been an enormous and complicated task. Despite the extensive efforts of scholars, there are gaps in our knowledge and some conclusions are arguable. Nevertheless the basic facts are clear.

3. *The Major Schools of the Classical Period*

The cream of the classical period's contributions are Euclid's *Elements* and Apollonius' *Conic Sections*. Appreciation of these works requires some knowledge of the great changes made in the very nature of mathematics and of the problems the Greeks faced and solved. Moreover, these polished works give little indication of the three hundred years of creative activity preceding them or of the issues which became vital in the subsequent history.

Classical Greek mathematics developed in several centers that succeeded one another, each building on the work of its predecessors. At each center an informal group of scholars carried on its activities under one or more great leaders. This kind of organization is common in modern times also and its reason for being is understandable. Today, when one great man locates at a particular place—generally a university—other scholars follow, to learn from the master.

The first of the schools, the Ionian, was founded by Thales (*c.* 640–*c.* 546 B.C.) in Miletus. We do not know the full extent to which Thales may have educated others, but we do know that the philosophers Anaximander (*c.* 610–*c.* 547 B.C.) and Anaximenes (*c.* 550–480 B.C.) were his pupils. Anaxagoras (*c.* 500–*c.* 428 B.C.) belonged to this school, and Pythagoras (*c.* 585–*c.* 500 B.C.) is supposed to have learned mathematics from Thales. Pythagoras then formed his own large school in southern Italy. Toward the end of the sixth century, Xenophanes of Colophon in Ionia migrated to Sicily and founded a center to which the philosophers Parmenides (5th cent. B.C.) and Zeno (5th cent. B.C.) belonged. The latter two resided in Elea in southern Italy, to which the school had moved, and so the group became known as the Eleatic school. The Sophists, active from the latter half of the fifth century onward, were concentrated mainly in Athens. The most celebrated school is the Academy of Plato in Athens, where Aristotle was a student. The Academy had unparalleled importance for Greek thought. Its pupils and associates were the greatest philosophers, mathematicians, and astronomers of their age; the school retained its pre-eminence in philosophy even after the leadership in mathematics passed to Alexandria. Eudoxus, who learned mathematics chiefly from Archytas of Tarentum (Sicily), founded his own school in Cyzicus, a city of northern Asia Minor. When Aristotle left Plato's Academy he founded another school, the Lyceum, in Athens. The Lyceum is commonly referred to as the Peripatetic school. Not all of the great mathematicians of the classical period can be identified with a school, but for the sake of coherence we shall occasionally discuss the work of a man in connection with a particular school even though his association with it was not close.

4. *The Ionian School*

The leader and founder of this school was Thales. Though there is no sure knowledge about Thales' life and work, he probably was born and lived in Miletus. He traveled extensively and for a while resided in Egypt, where he carried on business activities and reportedly learned much about Egyptian mathematics. He is, incidentally, supposed to have been a shrewd businessman. During a good season for olive growing, he cornered all the olive presses in Miletus and Chios and rented them out at a high fee. Thales is said to have predicted an eclipse of the sun in 585 B.C., but this is disputed on the ground that astronomical knowledge was not adequate at that time.

He is reputed to have calculated the heights of pyramids by comparing their shadows with the shadow cast by a stick of known height at the same time. By some such use of similar triangles he is supposed to have calculated the distance of a ship from shore. He is also credited with having made mathematics abstract and with having given deductive proofs for some theorems. These last two claims, however, are dubious. Discovery of the attractive power of magnets and of static electricity is also attributed to Thales.

The Ionian school warrants only brief mention so far as contributions to mathematics proper are concerned, but its importance for philosophy and the philosophy of science in particular is unparalleled (see Chap. 7, sec. 2). The school declined in importance when the Persians conquered the area.

5. *The Pythagoreans*

The torch was picked up by Pythagoras who, supposedly having learned from Thales, founded his own school in Croton, a Greek settlement in southern Italy. There are no written works by the Pythagoreans; we know about them through the writings of others, including Plato and Herodotus. In particular we are hazy about the personal life of Pythagoras and his followers; nor can we be sure of what is to be credited to him personally or to his followers. Hence when one speaks of the work of Pythagoras one really refers to the work done by the group between 585 B.C., the reputed date of his birth, and roughly 400 B.C. Philolaus (5th cent. B.C.) and Archytas (428–347 B.C.) were prominent members of this school.

Pythagoras was born on the island of Samos, just off the coast of Asia Minor. After spending some time with Thales in Miletus, he traveled to other places, including Egypt and Babylon, where he may have picked up some mathematics and mystical doctrines. He then settled in Croton. There he

founded a religious, scientific, and philosophical brotherhood. It was a formal school, in that membership was limited and members learned from leaders. The teachings of the group were kept secret by the members, though the secrecy as to mathematics and physics is denied by some historians. The Pythagoreans were supposed to have mixed in politics; they allied themselves with the aristocratic faction and were driven out by the popular or democratic party. Pythagoras fled to nearby Metapontum and was murdered there about 497 B.C. His followers spread to other Greek centers and continued his teachings.

One of the great Greek contributions to the very concept of mathematics was the conscious recognition and emphasis of the fact that mathematical entities, numbers, and geometrical figures are abstractions, ideas entertained by the mind and sharply distinguished from physical objects or pictures. It is true that even some primitive civilizations and certainly the Egyptians and Babylonians had learned to think about numbers as divorced from physical objects. Yet there is some question as to how much they were consciously aware of the abstract nature of such thinking. Moreover, geometrical thinking in all pre-Greek civilizations was definitely tied to matter. To the Egyptians, for example, a line was no more than either a stretched rope or the edge of a field and a rectangle was the boundary of a field.

The recognition that mathematics deals with abstractions may with some confidence be attributed to the Pythagoreans. However, this may not have been true at the outset of their work. Aristotle declared that the Pythagoreans regarded numbers as the ultimate components of real, material objects.[1] Numbers did not have a detached existence apart from objects of sense. When the early Pythagoreans said that all objects were composed of (whole) numbers or that numbers were the essence of the universe, they meant it literally, because numbers to them were like atoms are to us. It is also believed that the sixth- and fifth-century Pythagoreans did not really distinguish numbers from geometrical dots. Geometrically, then, a number was an extended point or a very small sphere. However, Eudemus, as reported by Proclus, says that Pythagoras rose to higher principles (than had the Egyptians and Babylonians) and considered abstract problems for the pure intelligence. Eudemus adds that Pythagoras was the creator of pure mathematics, which he made into a liberal art.

The Pythagoreans usually depicted numbers as dots in sand or as pebbles. They classified the numbers according to the shapes made by the arrangements of the dots or pebbles. Thus the numbers 1, 3, 6, and 10 were called triangular because the corresponding dots could be arranged as triangles (Fig. 3.1). The fourth triangular number, 10, especially fascinated the Pythagoreans because it was a prized number for them, and had 4 dots on

1. *Metaphys.* I, v, 986a and 986a 21, Loeb Classical Library ed.

Figure 3.1

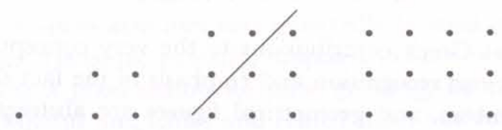

Figure 3.2

each side, 4 being another favorite number. They realized that the sums 1, 1 + 2, 1 + 2 + 3, and so forth gave the triangular numbers and that $1 + 2 + \cdots + n = (n/2)(n + 1)$.

The numbers 1, 4, 9, 16, . . . were called square numbers because as dots they could be arranged as squares (Fig. 3.2). Composite (nonprime) numbers which were not perfect squares were called oblong.

From the geometrical arrangements certain properties of the whole numbers became evident. Introducing the slash, as in the third illustration of Figure 3.2, shows that the sum of two consecutive triangular numbers is a square number. This is true generally, for as we can see, in modern notation,

$$\frac{n}{2}(n + 1) + \frac{n + 1}{2}(n + 2) = (n + 1)^2.$$

That the Pythagoreans could prove this general conclusion, however, is doubtful.

To pass from one square number to the next one, the Pythagoreans had the scheme shown in Figure 3.3. The dots to the right of and below the lines in the figure formed what they called a gnomon. Symbolically, what they saw here was that $n^2 + (2n + 1) = (n + 1)^2$. Further, if we start with 1 and

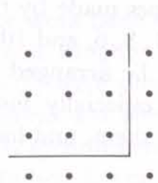

Figure 3.3

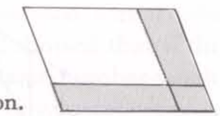

Figure 3.4. The shaded area is the gnomon.

add the gnomon 3 and then the gnomon 5, and so forth, what we have in our symbolism is

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

As to the word "gnomon," originally in Babylonia it probably meant an upright stick whose shadow was used to tell time. In Pythagoras' time it meant a carpenter's square, and this is the shape of the above gnomon. It also meant what was left over from a square when a smaller square was cut out of one corner. Later, with Euclid, it meant what was left from a parallelogram when a smaller one was cut out of one corner provided that the parallelogram in the lower right-hand corner was similar to the one cut out (Fig. 3.4).

The Pythagoreans also worked with polygonal numbers such as pentagonal, hexagonal, and higher ones. As we can see from Figure 3.5, where each dot represents a unit, the first pentagonal number is 1, the second, whose dots form the vertices of a pentagon, is 5; the third is 1 + 4 + 7, or 12, and so forth. The nth pentagonal number, in our notation, is $(3n^2 - n)/2$. Likewise the hexagonal numbers (Fig. 3.6) are 1, 6, 15, 28, . . . and generally $2n^2 - n$.

A number that equaled the sum of its divisors including 1 but not the number itself was called perfect; for example, 6, 28, and 496. Those exceeding the sum of the divisors were called excessive and those which were less were called defective. Two numbers were called amicable if each was the sum of the divisors of the other, for example, 284 and 220.

The Pythagoreans devised a rule for finding triples of integers which could be the sides of a right triangle. This rule implies knowledge of the Pythagorean theorem, about which we shall say more later. They found that when m is odd, then m, $(m^2 - 1)/2$, and $(m^2 + 1)/2$ are such a triple. However,

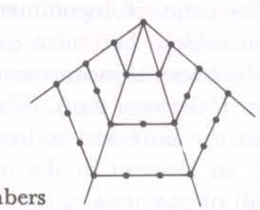

Figure 3.5. Pentagonal numbers

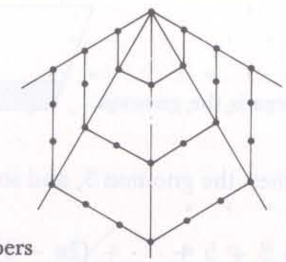

Figure 3.6. Hexagonal numbers

this rule gives only some sets of such triples. Any set of three integers which can be the sides of a right triangle is now called a Pythagorean triple.

The Pythagoreans studied prime numbers, progressions, and those ratios and proportions they regarded as beautiful. Thus if p and q are two numbers, the arithmetic mean A is $(p + q)/2$, the geometric mean G is $\sqrt{pq}$, and the harmonic mean H, which is the reciprocal of the arithmetic mean of $1/p$ and $1/q$, is $2pq/(p + q)$. Now G is seen to be the geometric mean of A and H. The proportion $A/G = G/H$ was called the perfect proportion and the proportion $p:(p + q)/2 = 2pq/(p + q):q$ was called the musical proportion.

Numbers to the Pythagoreans meant whole numbers only. A ratio of two whole numbers was not a fraction and therefore another kind of number, as it is in modern times. Actual fractions, expressing parts of a monetary unit or a measure, were employed in commerce, but such commercial uses of arithmetic were outside the pale of Greek mathematics proper. Hence the Pythagoreans were startled and disturbed by the discovery that some ratios—for example, the ratio of the hypotenuse of an isosceles right triangle to an arm or the ratio of a diagonal to a side of a square—cannot be expressed by whole numbers. Since the Pythagoreans had concerned themselves with whole-number triples that could be the sides of a right triangle, it is most likely that they discovered these new ratios in this work. They called ratios expressed by whole numbers commensurable ratios, which means that the two quantities are measured by a common unit, and they called ratios not so expressible, incommensurable ratios. Thus what we express as $\sqrt{2}/2$ is an incommensurable ratio. The ratio of incommensurable magnitudes was called αλογος (*alogos*, inexpressible). The term αρρητος (*arratos*, not having a ratio) was also used. The discovery of incommensurable ratios is attributed to Hippasus of Metapontum (5th cent. B.C.). The Pythagoreans were supposed to have been at sea at the time and to have thrown Hippasus overboard for having produced an element in the universe which denied the Pythagorean doctrine that all phenomena in the universe can be reduced to whole numbers or their ratios.

The proof that $\sqrt{2}$ is incommensurable with 1 was given by the Pythagoreans. According to Aristotle, their method was a *reductio ad absurdum*—that is, the indirect method. The proof showed that if the hypotenuse were commensurable with an arm then the same number would be both odd and even. It runs as follows: Let the ratio of hypotenuse to arm of an isosceles right triangle be $\alpha:\beta$ and let this ratio be expressed in the smallest numbers. Then $\alpha^2 = 2\beta^2$ by the Pythagorean theorem. Since α^2 is even, α must be even, for the square of any odd number is odd.[2] Now the ratio $\alpha:\beta$ is in its lowest terms. Hence β must be odd. Since α is even, let $\alpha = 2\gamma$. Then $\alpha^2 = 4\gamma^2 = 2\beta^2$. Hence $\beta^2 = 2\gamma^2$ and so β^2 is even. Then β is even. But β is also odd and so there is a contradiction.

This proof, which is of course the same as the modern one that $\sqrt{2}$ is irrational, was included in older editions of Euclid's *Elements* as Proposition 117 of Book X. However, it was most likely not in Euclid's original text and so is omitted in modern editions.

Incommensurable ratios are expressed in modern mathematics by irrational numbers. But the Pythagoreans would not accept such numbers. The Babylonians did work with such numbers by approximating them, though they probably did not know that their sexagesimal fractional approximations could never be made exact. Nor did the Egyptians recognize the distinctive nature of irrationals. The Pythagoreans did at least recognize that incommensurable ratios are entirely different in character from commensurable ones.

This discovery posed a problem that was central in Greek mathematics. The Pythagoreans had, up to this point, identified number with geometry. But the existence of incommensurable ratios shattered this identification. They did not cease to consider all kinds of lengths, areas, and ratios in geometry, but they restricted the consideration of numerical ratios to commensurable ones. The theory of proportions for incommensurable ratios and all kinds of magnitudes was provided by Eudoxus, whose work we shall consider shortly.

Some geometrical results are also credited to the Pythagoreans. The most famous is the Pythagorean theorem itself, a key theorem of Euclidean geometry. The Pythagoreans are also supposed to have discovered what we learn as theorems about triangles, parallel lines, polygons, circles, spheres, and the regular polyhedra. They knew in particular that the sum of the angles of a triangle is 180°. A limited theory of similar figures and the fact that a plane can be filled out with equilateral triangles, squares, and regular hexagons are included among their results.

The Pythagoreans started work on a class of problems known as

2. Any odd whole number can be expressed as $2n + 1$ for some n. Then $(2n + 1)^2 = 4n^2 + 4n + 1$, and this is necessarily odd.

application of areas. The simplest of these was to construct a polygon equal in area to a given polygon and similar to another given one. Another was to construct a specified figure with an area exceeding or falling short of another by a given area. The most important form of the problem of application of areas is: Given a line segment, construct on part of it or on the line segment extended a parallelogram equal to a given rectilinear figure in area and falling short (in the first case) or exceeding (in the second case) by a parallelogram similar to a given parallelogram. We shall discuss application of areas when we study Euclid's work.

The most vital contribution of the Greeks to mathematics is the insistence that all mathematical results be established deductively on the basis of explicit axioms. Hence the question arises as to whether the Pythagoreans proved their geometric results. No unequivocal answer can be given, but it is very doubtful that deductive proof on any kind of axiomatic basis, explicit or implicit, was a requirement in the early or middle period of Pythagorean mathematics. Proclus does affirm that they proved the angle sum theorem; this may have been done by the late Pythagoreans. The question of whether they proved the Pythagorean theorem has been extensively pursued, and the answer is that they probably did not. It is relatively easy to prove it by using facts about similar triangles, but the Pythagoreans did not have a complete theory of similar figures. The proof given in Proposition 47 of Book I of Euclid's *Elements* (Chap. 4, sec. 4) is a difficult one because it does not use the theory of similar figures, and this proof was credited by Proclus to Euclid himself. The most likely conclusion about proof in Pythagorean geometry is that during most of the life of the school the members affirmed results on the basis of special cases, much as they did in their arithmetic. However, by the time of the late Pythagoreans, that is, about 400 B.C., the status of proof had changed because of other developments; so these latter-day members of the brotherhood may have given legitimate proofs.

6. *The Eleatic School*

The Pythagorean discovery of incommensurable ratios brought to the fore a difficulty that preoccupied all the Greeks, namely, the relation of the discrete to the continuous. Whole numbers represent discrete objects, and a commensurable ratio represents a relation between two collections of discrete objects, or two lengths that have a common unit measure so that each length is a discrete collection of units. However, lengths in general are not discrete collections of units; this is why ratios of incommensurable lengths appear. Lengths, areas, volumes, time, and other quantities are, in other words, continuous. We would say that line segments, for example, can have irrational as well as rational lengths in terms of some unit. But the Greeks had not attained this view.

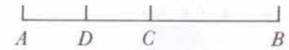

Figure 3.7

The problem of the relation of the discrete to the continuous was brought into the limelight by Zeno, who lived in the southern Italian city of Elea. Born some time between 495 and 480 B.C., Zeno was a philosopher rather than a mathematician, and like his master Parmenides was said to have been a Pythagorean originally. He proposed a number of paradoxes, of which four deal with motion. His purpose in posing these paradoxes is not clear because not enough of the history of Greek philosophy is known. He was said to be defending Parmenides, who had argued that motion or change is impossible. He was also attacking the Pythagoreans, who believed in extended but indivisible units, the points of geometry. We do not know precisely what Zeno said but must rely upon quotations from Aristotle, who cites Zeno in order to criticize him, and from Simplicius, who lived in the sixth century A.D. and based his statements on Aristotle's writings.

The four paradoxes on motion are distinct, but the import of all four taken together was probably intended to be the significant argument. Two opposing views of space and time were held in Zeno's day: one, that space and time are infinitely divisible, in which case motion is continuous and smooth; and the other, that space and time are made up of indivisible small intervals (like a movie), in which case motion is a succession of minute jerks. Zeno's arguments are directed against both theories, the first two paradoxes being against the first theory and the latter two against the second theory. The first paradox of each pair considers the motion of a single body and the second considers the relative motion of bodies.

Aristotle in his *Physics* states the first paradox, called the Dichotomy, as follows: "The first asserts the nonexistence of motion on the ground that that which is in motion must arrive at the half-way stage before it arrives at the goal." This means that to traverse AB (Fig. 3.7), one must first arrive at C; to arrive at C one must first arrive at D; and so forth. In other words, on the assumption that space is infinitely divisible and therefore that a finite length contains an infinite number of points, it is impossible to cover even a finite length in a finite time.

Aristotle, refuting Zeno, says there are two senses in which a thing may be infinite: in divisibility or in extent. In a finite time one can come into contact with things infinite in respect to divisibility, for in this sense time is also infinite; and so a finite extent of time can suffice to cover a finite length. Zeno's argument has been construed by others to mean that to go a finite length one must cover an infinite number of points and so must get to the end of something that has no end.

The second paradox is called Achilles and the Tortoise. According to

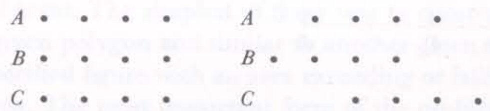

Figure 3.8

Aristotle: "It says that the slowest moving object cannot be overtaken by the fastest since the pursuer must first arrive at the point from which the pursued started so that necessarily the slower one is always ahead. The argument is similar to that of the Dichotomy, but the difference is that we are not dividing in halves the distances which have to be passed over." Aristotle then says that if the slowly moving object covers a finite distance, it can be overtaken for the same reason he gives in answering the first paradox.

The next two paradoxes are directed against "cinematographic" motion. The third paradox, called the Arrow, is given by Aristotle as follows: "The third paradox he [Zeno] spoke about, is that a moving arrow is at a standstill. This he concludes from the assumption that time is made up of instants. If it would not be for this supposition, there would be no such conclusion." According to Aristotle, Zeno means that at any instant during its motion the arrow occupies a definite position and so is at rest. Hence it cannot be in motion. Aristotle says that this paradox fails if we do not grant indivisible units of time.

The fourth paradox, called the Stadium or the Moving Rows, is put by Aristotle in these words: "The fourth is the argument about a set of bodies moving on a race-course and passing another set of bodies equal in number and moving in the opposite direction, the one starting from the end, the other from the middle and both moving at equal speed; he [Zeno] concluded that it follows that half the time is equal to double the time. The mistake is to assume that two bodies moving at equal speeds take equal times in passing, the one a body which is in motion, and the other a body of equal size which is at rest, an assumption which is false."

The probable point of Zeno's fourth paradox can be stated as follows: Suppose that there are three rows of soldiers, A, B, and C (Fig. 3.8), and that in the smallest unit of time B moves one position to the left, while in that time C moves one position to the right. Then relative to B, C has moved two positions. Hence there must have been a smaller unit of time in which C was one position to the right of B or else half the unit of time equals the unit of time.

It is possible that Zeno merely intended to point out that speed is relative. C's speed relative to B is not C's speed relative to A. Or he may have meant there is no absolute space to which to refer speeds. Aristotle says that Zeno's fallacy consists in supposing that things that move with the same speed past a moving object and past a fixed object take the same time. Neither Zeno's argument nor Aristotle's answer is clear. But if we think of

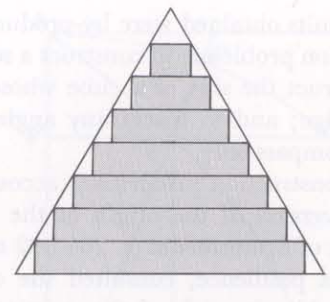
Figure 3.9

this paradox as attacking indivisible smallest intervals of time and indivisible smallest segments of space, which Zeno was attacking, then his argument makes sense.

We may include with the Eleatics Democritus (*c.* 460–*c.* 370 B.C.) of Abdera in Thrace. He is reputed to have been a man of great wisdom who worked in many fields, including astronomy. Since Democritus belonged to the school of Leucippus and the latter was a pupil of Zeno, many of the mathematical questions Democritus considered must have been suggested by Zeno's ideas. He wrote works on geometry, on number, and on continuous lines and solids. The works on geometry could very well have been significant predecessors of Euclid's *Elements*.

Archimedes says Democritus discovered that the volumes of a cone and a pyramid are 1/3 of the volumes of the cylinder and prism having the same base and height, but that the proofs were made by Eudoxus. Democritus regarded the cone as a series of thin indivisible layers (Fig. 3.9), but was troubled by the fact that if the layers were equal they should yield a cylinder and if unequal the cone could not be smooth.

7. *The Sophist School*

After the final defeat of the Persians at Mycale in 479 B.C., Athens became the major city in a league of Greek cities and a commercial center. The wealth acquired through trading, which made Athens the richest city of its time, was used by the famous leader Pericles to build up and adorn the city. Ionians, Pythagoreans, and intellectuals generally were attracted to Athens. Here emphasis was given to abstract reasoning and the goal of extending the domain of reason over the whole of nature and man was set.

The first Athenian school, the Sophist, embraced learned teachers of grammar, rhetoric, dialectics, eloquence, morals, and—what is of interest to us—geometry, astronomy, and philosophy. One of their chief pursuits was the use of mathematics to understand the functioning of the universe.

Many of the mathematical results obtained were by-products of efforts to solve the three famous construction problems: to construct a square equal in area to a given circle; to construct the side of a cube whose volume is double that of a cube of given edge; and to trisect any angle—all to be performed with straightedge and compass only.

The origin of these famous construction problems is accounted for in various ways. For example, one version of the origin of the problem of doubling the cube, found in a work of Eratosthenes (*c.* 284–192 B.C.), relates that the Delians, suffering from a pestilence, consulted the oracle, who advised constructing an altar double the size of the existing one. The Delians realized that doubling the side would not double the volume and turned to Plato, who told them that the god of the oracle had not so answered because he wanted or needed a doubled altar, but in order to censure the Greeks for their indifference to mathematics and their lack of respect for geometry. Plutarch also gives this story.

Actually, these construction problems are extensions of problems already solved by the Greeks. Since any angle could be bisected, it was natural to consider trisection. Since the diagonal of a square is the side of a square double in area to that of the original square, the corresponding problem for the cube becomes relevant. The problem of squaring the circle is typical of many Greek problems of constructing a figure of prescribed shape equal in area to a given figure. Another problem not quite so famous was to construct regular polygons of seven and more sides. Here, too, the construction of the square, regular pentagon, and regular hexagon suggested the next step.

Various explanations of the restriction to straightedge and compass have been given. The straight line and the circle were, in the Greek view, the basic figures, and the straightedge and compass are their physical analogues. Hence constructions with these tools were preferable. The reason is also given that Plato objected to other mechanical instruments because they involved too much of the world of the senses rather than the world of ideas, which he regarded as primary. It is very likely, however, that in the fifth century the restriction to straightedge and compass was not rigid. But, as we shall see, constructions played a vital role in Greek geometry and Euclid's axioms did limit constructions to those made with straightedge and compass. Hence from his time on, this restriction may have been taken more seriously. Pappus, for example, says that if a construction can be carried out with straightedge and compass, a solution using other means is not satisfactory.

The earliest known attempt to solve any of the three famous problems was made by the Ionian Anaxagoras, who is supposed to have worked on squaring the circle while in prison. We do not know any more about this work. One of the most famous attempts is due to Hippias of Elis, a city in the Peloponnesus. Hippias, a leading Sophist, was born about 460 B.C. and was a contemporary of Socrates.

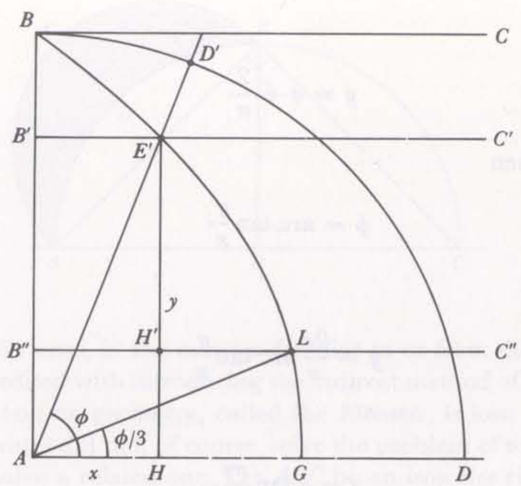

Figure 3.10

In his attempts to trisect an angle, Hippias invented a new curve, which, unfortunately, is not itself constructible with straightedge and compass. His curve is called the quadratrix and is generated as follows: Let AB (Fig. 3.10) rotate clockwise about A at a constant speed to the position AD. In the *same time* let BC move downward parallel to itself at a uniform speed to AD. Suppose AB reaches AD' as BC reaches $B'C'$. Let E' be the intersection of AD' and $B'C'$. Then E' is a typical point on the quadratrix $BE'G$. G is the final point on the quadratrix.[3]

The equation of the quadratrix in terms of rectangular Cartesian coordinates can be obtained as follows: Let AD' reach AD in some fraction t/T of the total time T that AB takes to reach AD. Since AD' and $B'C'$ move at constant speeds, $B'C'$ covers that part $E'H$ of BA in the same fraction of the total time. Hence

$$\frac{\phi}{\pi/2} = \frac{E'H}{BA}.$$

If we denote $E'H$ by y and BA by a, then

(1) $$\frac{\phi}{\pi/2} = \frac{y}{a}$$

3. The point G cannot be obtained directly from the definition of the curve because AB reaches AD at the same instant as BC reaches AD and so there is no point of intersection of the rotating line and the horizontal line. G can be obtained only as the limit of preceding points of the quadratrix. By using the calculus we can show that $AG = 2a/\pi$ where $a = AB$.

or

$$y = a \cdot \phi \cdot \frac{2}{\pi}.$$

But if $AH = x$, then

$$\phi = \text{arc tan}\,\frac{y}{x}.$$

Hence

$$y = \frac{2a}{\pi}\,\text{arc tan}\,\frac{y}{x}$$

or

$$y = x \tan \frac{\pi y}{2a}.$$

The curve, if constructible, could be used to trisect any acute angle. Let ϕ be such an angle. Then trisect y so that $E'H' = 2H'H$. Draw $B''C''$ through H' and let it cut the quadratrix in L. Draw AL. Then $\sphericalangle LAD = \phi/3$, for, by the argument which led to (1),

$$\frac{\sphericalangle LAD}{\pi/2} = \frac{H'H}{a}$$

or

$$\frac{\sphericalangle LAD}{\pi/2} = \frac{y/3}{a}.$$

But by (1)

$$\frac{\phi}{\pi/2} = \frac{y}{a}.$$

Hence

$$\sphericalangle LAD = \frac{\phi}{3}.$$

Another famous discovery that resulted from the work on the construction problems was made by Hippocrates of Chios (5th cent. B.C.), the most famous mathematician of his century, who is to be distinguished from his contemporary Hippocrates of Cos, the father of Greek medicine. The mathematician Hippocrates flourished in Athens during the second half of the century; he was not a Sophist, but most likely a Pythagorean. He is credited with the idea of arranging theorems so that later ones can be proven on the

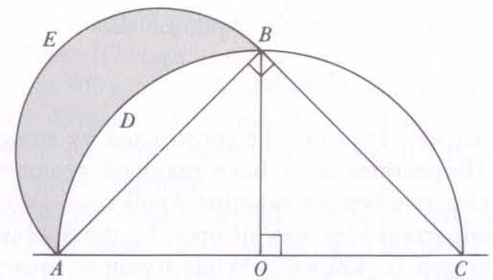

Figure 3.11

basis of earlier ones, in the manner familiar to us from the study of Euclid. He is also credited with introducing the indirect method of proof into mathematics. His text on geometry, called the *Elements*, is lost.

Hippocrates did not, of course, solve the problem of squaring the circle, but he did solve a related one. Let ABC be an isosceles right triangle (Fig. 3.11) and let it be inscribed in the semicircle with center at O. Let AEB be the semicircle with AB as diamter. Then

$$\frac{\text{Area semicircle } ABC}{\text{Area semicircle } AEB} = \frac{AC^2}{AB^2} = \frac{2}{1}.$$

Hence the area $OADB$ equals the area of the semicircle AEB. Now subtract the area ADB common to both. Then the area of the lune (shaded area) equals the area of triangle AOB. Thus the area of the lune, an area bounded by arcs, equals the area of a rectilinear figure; or, a curvilinear figure is reduced to a rectilinear one. This result is called a quadrature; that is, a curvilinear area has been computed in effect because it is equal to an area bounded by straight lines and the latter can be computed.

In this proof Hippocrates had to use the fact that the areas of two circles are to each other as the squares of their diameters. It is doubtful that Hippocrates really had a proof of this fact because the proof depends upon the method of exhaustion invented later by Eudoxus.

Hippocrates also squared three other lunes. This work on the lunes is known to us through the writings of Simplicius and is the only sizable fragment of *classical* Greek mathematics that we have as originally written.

Hippocrates also showed that the problem of doubling the cube can be reduced to finding two mean proportionals between the given side and one twice as long. In our algebraic notation, let x and y be such that

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}.$$

Then

$$x^2 = ay \quad \text{and} \quad y^2 = 2ax.$$

Since $y = x^2/a$ we obtain from the second equation that

$$x^3 = 2a^3.$$

This x is the desired answer. It cannot be constructed by straightedge and compass. Of course Hippocrates must have reasoned geometrically, in a manner that will be clearer when we examine Apollonius' *Conic Sections*.

One more very important idea was hit upon by the Sophists Antiphon (5th cent. B.C.) and Bryson (*c.* 450 B.C.). While trying to square the circle, Antiphon conceived the idea of approaching the area of a circle by inscribing polygons of more and more sides. Bryson added to this the idea of using circumscribed polygons. Antiphon further suggested that the circle be considered a polygon of an infinite number of sides. We shall see how these ideas were taken up by Eudoxus in the method of exhaustion (Chap. 4, sec. 9).

8. *The Platonic School*

The Platonic school succeeded the Sophists in the leadership of mathematical activity. Its forerunners, Theodorus of Cyrene in North Africa (born *c.* 470 B.C.) and Archytas of Tarentum in southern Italy (428–347 B.C.), were Pythagoreans and both taught Plato. Their teachings may have produced the strong Pythagorean influence in the entire Platonic school.

Theodorus is noted for having proved that the ratios that we represent as $\sqrt{3}, \sqrt{5}, \sqrt{7}, \ldots, \sqrt{17}$ are incommensurable with a unit. Archytas introduced the idea of regarding a curve as generated by a moving point and a surface as generated by a moving curve. He solved the duplication of the cube problem by finding two mean proportionals between two given quantities. These mean proportionals were constructed geometrically by finding the intersection of three surfaces: a circle rotated about a tangent, a cone, and a cylinder. The construction is quite detailed and does not warrant space here. Archytas also wrote on mathematical mechanics, designed machines, studied sound, and contributed inventions and some theory on musical scales.

The Platonic school was headed by Plato and included Menaechmus (4th cent. B.C.), his brother Dinostratus (4th cent. B.C.), and Theaetetus (*c.* 415–*c.* 369 B.C.). Many other members are known to us only by name.

Plato (427–347 B.C.) was born of a distinguished family and early in life had political ambitions. But the fate of Socrates convinced him there was no place in politics for a man of conscience. He traveled in Egypt and among the Pythagoreans in lower Italy; the Pythagorean influence may have been generated by these contacts. About 387 B.C. Plato founded in Athens his Academy, which in most respects was like a modern university. The Academy had grounds, buildings, students, and formal courses taught by Plato and his aides. During the classical period the study of mathematics and philosophy was favored there. Though the main center for mathematics shifted to Alexandria about 300 B.C., the Academy remained preeminent in philosophy throughout the Alexandrian period. It lasted nine hundred years until it was closed by the Christian emperor Justinian in A.D. 529 because it taught "pagan and perverse learning."

Plato, one of the most informed men of his day, was not a mathematician; but his enthusiasm for the subject and his belief in its importance for philosophy and for the understanding of the universe encouraged mathematicians to pursue it. It is noteworthy that almost all of the important mathematical work of the fourth century was done by the friends and pupils of Plato. Plato himself seems to have been more concerned to improve and perfect what was known.

Though we may not be sure to what extent the concepts of mathematics were treated as abstractions prior to Plato's time, there is no question that Plato and his successors did so regard them. Plato says that numbers and geometrical concepts have nothing material in them and are distinct from physical things. The concepts of mathematics are independent of experience and have a reality of their own. They are discovered, not invented or fashioned. This distinction between abstractions and material objects may have come from Socrates.

A quotation from Plato's *Republic* may serve to illustrate the contemporary view of the mathematical concepts. Socrates addresses Glaucon:

> And all arithmetic and calculation have to do with number.
>
> Yes....
>
> Then this is knowledge of the kind for which we are seeking, having a double use, military and philosophical; for the man of war must learn the art of number or he will not know how to array his troops, and the philosopher also, because he has to rise out of the sea of change and lay hold of true being, and therefore he must be an arithmetician.... Then this is a kind of knowledge which legislation may fitly prescribe; and we must endeavour to persuade those who are to be the principal men of our State to go and learn arithmetic, not as amateurs, but they must carry on the study until they see the nature of numbers with the mind only; nor again, like merchants or retail-traders, with a view to buying or selling, but for the sake of their military use, and of the soul herself; and because this will be the easiest way for her to pass from becoming to truth and being.... I mean, as I was saying, that arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number, and rebelling against the introduction of visible or tangible objects into the argument....[4]

4. Book VII, sec. 525; in B. Jowett, *The Dialogues of Plato*, Clarendon Press, 1953, Vol. 2.

Another quotation[5] discusses the concepts of geometry. Speaking about mathematicians, Plato says: "And do you not know also that although they further make use of the visible forms and reason about them, they are thinking not of these, but of the ideals which they resemble . . . but they are really seeking to behold the things themselves, which can be seen only with the eye of the mind."

It is clear from these quotations that Plato and other Greeks for whom he speaks valued abstract ideas and preferred mathematical ideas as a preparation for philosophy. The abstract ideas with which mathematics deals are akin to others such as goodness and justice, the understanding of which is the goal of Plato's philosophy. Mathematics is the preparation for knowledge about the ideal universe.

Why did the Greeks prefer and stress the abstract concepts of mathematics? We cannot answer the question, but we should note that the early Greek mathematicians were philosophers and philosophers generally exerted a formative influence in the development of Greek mathematics. Philosophers are interested in ideas and typically show their propensity for abstractions in many domains. Thus the Greek philosophers thought about truth, goodness, charity, and intelligence. They speculated about the ideal society and the perfect state. The later Pythagoreans and the Platonists distinguished sharply between the world of ideas and the world of things. Relationships in the material world were subject to change and hence did not represent ultimate truth, but relationships in the ideal world were unchanging and absolute truths; these were the proper concern of the philosopher.

Plato in particular believed that the perfect ideals of physical objects are the reality. The world of ideals and relationships among them is permanent, ageless, incorruptible, and universal. The physical world is an imperfect realization of the ideal world and is subject also to decay. Hence the ideal world alone is worthy of study. Infallible knowledge can be obtained only about pure intelligible forms. About the physical world we can have only opinions; and physical science is sunk in the dregs of a sensuous world.

We are not sure whether the Platonists contributed the deductive structure of mathematics. They were concerned with proof and with the methodology of reasoning. Proclus and Diogenes Laertius (3rd cent. A.D.) credit the Platonists with two types of methodology. The first is the method of analysis, where what is to be established is regarded as known and the consequences deduced until a known truth or a contradiction is reached. If a contradiction is reached then the desired conclusion is false. If a known truth is reached then the steps are reversed, if possible, and the proof is made. The second is the method of *reductio ad absurdum* or the indirect method. The first method was probably not new with Plato, but perhaps he emphasized

5. *Republic*, Book VI, sec. 510.

the necessity for the subsequent synthesis. The indirect method, as already noted, is also attributed to Hippocrates.

The status of deductive structure with Plato is best indicated by a passage in *The Republic.*[6] He says

> You are aware that students of geometry, arithmetic and the kindred sciences assume the odd and the even and the figures and three kinds of angles and the like in their several branches of science; these are their hypotheses, which they and everybody are supposed to know, and therefore they do not deign to give any account of them either to themselves or others; but they begin with them, and go on until they arrive at last, and in a consistent manner, at their conclusion.

If this passage is indeed descriptive of the mathematics of the time, then proofs were certainly made, but the axiomatic basis was implicit or may have varied somewhat from one mathematician to another.

Plato did affirm the desirability of a deductive organization of knowledge. The task of science was to discover the structure of (ideal) nature and to give it an articulation in a deductive system. Plato was the first to systematize the rules of rigorous demonstration, and his followers are supposed to have arranged theorems in logical order. Also, we know that in Plato's Academy the question arose whether a given problem can be solved at all on the basis of the known truths and the hypotheses given in the problem. Whether or not mathematics was actually deductively organized on the basis of explicit axioms by the Platonists, there is no question that deductive proof from some accepted principles was required from at least Plato's time onward. By insisting on this form of proof the Greeks were discarding all rules, procedures, and facts that had been accepted in the body of mathematics for thousands of years preceding the Greek period.

Why did the Greeks insist on deductive proof? Since induction, observation, and experimentation were and still are vital sources of knowledge heavily and advantageously employed by the sciences, why did the Greeks prefer in mathematics deductive reasoning to the exclusion of all other methods? We know that the Greeks, the philosophical geometers as they were called, liked reasoning and speculation, as evidenced by their great contributions to philosophy, logic, and theoretical science. Moreover, philosophers are interested in obtaining truths. Whereas induction, experimentation, and generalizations based on experience yield only probable knowledge, deduction gives absolutely certain results if the premises are correct. Mathematics in the classical Greek world was part of the body of truths philosophers sought and accordingly had to be deductive.

Still another reason for the Greek preference for deduction may be found in the contempt shown by the educated class of the classical Greek

6. Book VI, sec. 510.

period toward practical affairs. Though Athens was a commercial center, business as well as such professions as medicine were carried on by the slave class. Plato contended that the engagement of freemen in trade should be punished as a crime, and Aristotle said that in the perfect state no citizen (as opposed to slaves) should practice any mechanical art. To thinkers in such a society, experimentation and observation would be alien. Hence no results, scientific or mathematical, would be derived from such sources.

There is, incidentally, evidence that in the sixth and fifth centuries B.C. the Greek attitude toward work, trade, and technical skills had been quite different and that mathematics had been applied to the practical arts. Thales used his mathematics to improve navigation. Solon, a ruler of the sixth century, invested the crafts with honor and inventors were esteemed. *Sophia*, the Greek word usually taken to mean wisdom and abstract thought, at that time meant technical skill. It was the Pythagoreans, Proclus says, who "transformed mathematics into a free education," that is, an education for free men rather than a skill for slaves.

Plutarch in his life of Marcellus substantiates the change in attitude toward such devices as mechanical instruments:

> Eudoxus and Archytas had been the first originators of this far-famed and highly-prized art of mechanics, which they employed as an elegant illustration of geometrical truths, and as means of sustaining experimentally, to the satisfaction of the senses, conclusions too intricate for proof by words and diagrams. As, for example, to solve the problem, so often required in constructing geometrical figures, given the two extremes, to find the two mean lines of a proportion, both of these mathematicians had recourse to the aid of instruments, adapting to their purpose certain curves and sections of lines. But what with Plato's indignation at it, and his invectives against it as the mere corruption and annihilation of the one good of geometry, which was thus shamefully turning its back upon the unembodied objects of pure intelligence to recur to sensation, and to ask help (not to be obtained without base supervisions and deprivation) from matter, so it was that mechanics came to be separated from geometry, and, repudiated and neglected by philosophers, took its place as a military art.

This accounts for the poor development of experimental science and the science of mechanics in the classical Greek period.

Whether or not historical research has isolated the relevant factors to explain the Greek preference for deductive reasoning, we do know that they were the first to insist on deductive reasoning as the sole method of *proof* in mathematics. This requirement has been characteristic of mathematics ever since and has distinguished mathematics from all other fields of knowledge or investigation. However, we have yet to see to what extent later mathematicians remained true to this principle.

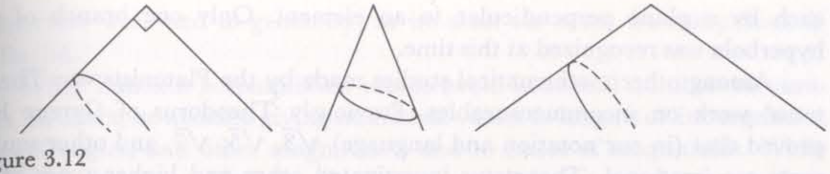

Figure 3.12

As far as the content of mathematics is concerned, Plato and his school improved the definitions and are also supposed to have proved new theorems of plane geometry. Further, they gave impetus to solid geometry. In Book VII, section 528 of *The Republic*, Plato says that before one can consider astronomy, which treats solids in motion, one needs a science of such solids. But this science, he says, has been neglected. He complains that the investigators of solid figures have not received due support from the state. Plato and his associates proceeded to study solid geometry and are supposed to have proved new theorems. They studied the properties of the prism, pyramid, cylinder, and cone; and they knew that there can be at most five regular polyhedra. The Pythagoreans undoubtedly knew that one can form three of these solids with 4, 8, and 20 equilateral triangles, the cube with squares, and the dodecahedron with 12 pentagons, but the proof that there can be no more than five is probably due to Theaetetus.

The Platonic school's most significant discovery was the conic sections. The discovery is attributed by the Alexandrian Eratosthenes to Menaechmus, a geometer and astronomer, who was a pupil of Eudoxus but a member of Plato's Academy. While it is not known for certain what led to the discovery of the conic sections, a common belief is that it resulted from the work on the famous construction problems. We know that Hippocrates of Chios solved the problem of doubling the cube by finding an x and y such that

$$a:x = x:y = y:2a.$$

But these equations say that

$$x^2 = ay, \quad y^2 = 2ax, \quad \text{and} \quad xy = 2a^2.$$

Hence we can see through coordinate geometry that x and y are the coordinates of the point of intersection of two parabolas or a parabola and a hyperbola. Menaechmus worked on the problem and saw both ways of solving it through pure geometry. According to the mathematical historian Otto Neugebauer (1899–), the conic sections might have originated in work on the construction of sundials.

Menaechmus introduced the conic sections by using three types of cones (Fig. 3.12), right-angled, acute-angled, and obtuse-angled, and cutting each by a plane perpendicular to an element. Only one branch of the hyperbola was recognized at this time.

Among other mathematical studies made by the Platonists was Theaetetus' work on incommensurables. Previously Theodorus of Cyrene had proved that (in our notation and language) $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, and other square roots are irrational. Theaetetus investigated other and higher types of irrationals and classified them. We shall note these types when we study Book X of Euclid's *Elements*. In this work of Theaetetus we see how the number system was being extended to more irrationals, but only those incommensurable ratios were studied which arose from geometrical thinking and could be constructed geometrically as lengths. Dinostratus, another Platonist, showed how to use the quadratrix of Hippias to square the circle. Aristaeus the Elder (*c.* 320 B.C.) is said by Pappus to have written a work in five books called *Elements of Conic Sections*.

9. *The School of Eudoxus*

The greatest of the classical Greek mathematicians and second only to Archimedes in all antiquity was Eudoxus. Eratosthenes called him "godlike." He was born in Cnidos in Asia Minor about 408 B.C., studied under Archytas in Tarentum, traveled in Egypt, where he learned some astronomy, and then founded a school at Cyzicus in northern Asia Minor. About 368 B.C. he and his followers joined Plato. Some years later he returned to Cnidos and died there about 355 B.C. An astronomer, physician, geometer, legislator, and geographer, he is most noted for the creation of the first astronomical theory of the heavenly motions (Chap. 7).

His first great contribution to mathematics was a new theory of proportion. The discovery of more and more irrationals (incommensurable ratios) made it necessary for the Greeks to face these numbers. Were they in fact numbers? They occurred in geometrical arguments, whereas the whole numbers and ratios of whole numbers occurred both in geometry and in the general study of quantity. Moreover, how could proofs of geometry which had been made for commensurable lengths, areas, and volumes be extended to incommensurable ones?

Eudoxus introduced the notion of a magnitude (Chap. 4, sec. 5). It was not a number but stood for entities such as line segments, angles, areas, volumes, and time which could vary, as we would say, continuously. Magnitudes were opposed to numbers, which jumped from one value to another, as from 4 to 5. No quantitative values were assigned to magnitudes. Eudoxus then defined a ratio of magnitudes and a proportion, that is, an equality of two ratios, to cover commensurable and incommensurable ratios. However, again, no numbers were used to express such ratios. The concepts of ratio and

proportion were tied to geometry, as we shall see when we study Book V of Euclid.

What Eudoxus accomplished was to avoid irrational numbers as numbers. In effect, he avoided giving numerical values to lengths of line segments, sizes of angles, and other magnitudes, and to ratios of magnitudes. While Eudoxus' theory enabled the Greek mathematicians to make tremendous progress in geometry by supplying the necessary logical foundation for incommensurable ratios, it had several unfortunate consequences.

For one thing, it forced a sharp separation between number and geometry, for only geometry could handle incommensurable ratios. It also drove mathematicians into the ranks of the geometers, and geometry became the basis for almost all rigorous mathematics for the next two thousand years. We still speak of x^2 as x square and x^3 as x cube instead of x second or x third, say, because the magnitudes x^2 and x^3 had only geometric meaning to the Greeks.

The Eudoxian solution to the problem of treating incommensurable lengths or the irrational number actually reversed the emphasis of previous Greek mathematics. The early Pythagoreans had certainly emphasized number as the fundamental concept, and Archytas of Tarentum, Eudoxus' teacher, stated that arithmetic alone, not geometry, could supply satisfactory proofs. However, in turning to geometry to handle irrational numbers, the classical Greeks abandoned algebra and irrational numbers as such. What did they do about solving quadratic equations, where the solutions can indeed be irrational numbers? And what did they do about the simple problem of finding the area of a rectangle whose sides are incommensurable? The answer is that they converted most of algebra to geometry, in a manner we shall examine in the next chapter. The geometric representation of irrationals and of operations with irrationals was, of course, not practical. It might be logically satisfactory to think of $\sqrt{2}\cdot\sqrt{3}$ as an area of a rectangle, but if one needed to know the product in order to buy floor covering, he would not have it.

Though the Greeks devoted their deepest efforts in mathematics to geometry, we must keep in mind that whole numbers and ratios of whole numbers were still acceptable concepts. This area of mathematics, as we shall see, was built up deductively in Books VII, VIII, and IX of Euclid's *Elements*. The material is essentially what we call the theory of numbers or the properties of integers.

The question also arises: What did the classical Greeks do about the need for numbers in scientific work and in commerce and other practical affairs? Classical Greek science, as we shall see, was qualitative. As for the practical uses of numbers, we mentioned earlier that the intellectuals of that period confined themselves to philosophical and scientific activities and took no hand in commerce or the trades; educated people did not concern themselves with practical problems. But one could think about all rectangles in geometry without concerning himself in the least with the actual dimensions of even one rectangle. Mathematical thought was thus separated from practical needs, and there was no compulsion for the mathematicians to improve arithmetical and algebraic techniques. When the barrier between the cultured and slave classes was breached in the Alexandrian period (300 B.C. to about A.D. 600) and educated men interested themselves in practical affairs, the emphasis shifted to quantitative knowledge and the development of arithmetic and algebra.

To return to the contributions of Eudoxus, the powerful Greek method of establishing the areas and volumes of curved figures, now called the method of exhaustion, is also due to him. We shall examine the method and its use, as given by Euclid, later. It is really the first step in the calculus but does not use an explicit theory of limits. With it Eudoxus proved, for example, that the areas of two circles are to each other as the squares of their radii, the volumes of two spheres are to each other as the cubes of their radii, the volume of a pyramid is one-third the volume of a prism of the same base and altitude, and the volume of a cone is one-third the volume of the corresponding cylinder.

Some authority can be found to credit every school from Thales' onward with having introduced the deductive organization of mathematics. There is no question, however, that the work of Eudoxus established the deductive organization on *the basis of explicit axioms*. The necessity for understanding and operating with incommensurable ratios is undoubtedly the reason for this step. Since Eudoxus undertook to provide the precise logical basis for these ratios, he most likely saw the need to formulate axioms and deduce consequences one by one so that no mistakes would be made with these unfamiliar and troublesome magnitudes. This need to work with incommensurable ratios also undoubtedly reinforced the earlier decision to rely only on deductive reasoning for proof.

Because the Greeks sought truths and had decided on deductive proof, they had to obtain axioms that were themselves truths. They did find statements whose truth was self-evident to them, though the justifications given for accepting the axioms as indisputable truths varied. Almost all Greeks believed that the mind was capable of recognizing truths. Plato applied his theory of anamnesis, that we have had direct experience of truths in a period of existence as souls in another world before coming to earth, and we have but to recall this experience to know that these truths included the axioms of geometry. No experience on earth is necessary. Some historians read into statements by Plato and Proclus the idea that there can be some arbitrariness in the axioms, provided only that they are clear and true in the mind of the individual. The important thing is to reason deductively on the basis of the ones chosen. Aristotle had a good deal to say about axioms and we shall note his views in a moment.

10. *Aristotle and His School*

Aristotle (384–322 B.C.) was born in Stageira, a city in Macedonia. For twenty years he was a pupil and colleague of Plato, and for three years, from 343 to 340 B.C., he was the tutor of Alexander the Great. In 335 B.C. he founded his own school, the Lyceum. It had a garden, a lecture room, and an altar to the Muses.

Aristotle wrote on mechanics, physics, mathematics, logic, meteorology, botany, psychology, zoology, ethics, literature, metaphysics, economics, and many other fields. There is no one book on mathematics but discussions of the subject occur in a variety of places, and he uses it to illustrate a number of points.

He viewed the sciences as falling into three types—theoretical, productive, and practical. The theoretical ones, which seek truth, are mathematics, physics (optics, harmonics, and astronomy), and metaphysics; of these mathematics is the most exact. The productive sciences are the arts; and the practical ones, for example ethics and politics, seek to regulate human actions. In the theoretical sciences, logic is preliminary to the several subjects included there, and the metaphysician discusses and explains what the mathematician and natural philosopher (scientist) take for granted, for example, the being or reality of the subject matter and the nature of axioms.

Though Aristotle did not contribute significant new mathematical results (a few theorems in Euclid are his), his views on the nature of mathematics and its relation to the physical world were highly influential. Whereas Plato believed that there was an independent, eternally existing world of ideas which constituted the reality of the universe and that mathematical concepts were part of this world, Aristotle favored concrete matter or substance. However, he too arrived at an *emphasis* on ideas, namely, the universal essences of physical objects, such as hardness, softness, heaviness, lightness, sphericity, coldness, and warmth. Numbers and geometrical forms, too, were properties of real objects; they were recognized by abstraction but belonged to the objects. Thus mathematics deals with abstract concepts, which are derived from properties of physical bodies.

Aristotle discusses definition. His notion of definition is modern; he calls it a name for a collection of words. He also points out that definition must be in terms of something prior to the thing defined. Thus he criticizes the definition, "a point is that which has no part," because the words "that which" do not say what they refer to, except possibly "point" and so the definition is not proper. He grants the need for undefined terms, since there must be a starting point for the series of definitions, but later mathematicians lost sight of this need until the end of the nineteenth century.

He also notes (as Plato, according to Plutarch, did earlier) that a definition tells us what a thing is but not that the thing exists. The existence of defined things has to be proved except in the case of a few primary things such as point and line, whose existence is assumed along with the first principles or axioms. Thus one can define a square, but such a figure may not exist; that is, the properties demanded in the definition may be incompatible. Leibniz gave the example of a regular polyhedron with ten faces; one can define such a figure but it does not exist. If one did not realize that this figure did not exist, and proceeded to prove theorems about it, his results would be nonsensical. The method of proving existence that Aristotle and Euclid adopted was construction. The first three axioms in Euclid's *Elements* grant the construction of straight lines and circles; all other mathematical concepts must be constructed to establish their existence. Thus angle trisectors, though definable, are not constructible with straight lines and circles and so could not be considered in Greek geometry.

Aristotle also treats the basic principles of mathematics. He distinguishes between the axioms or common notions, which are truths common to all sciences, and the postulates, which are acceptable first principles for any one science. Among axioms he includes logical principles, such as the law of contradiction, the law of excluded middle, the axiom that if equals are added or subtracted from equals the results are equal, and other such principles. The postulates need not be self-evident but their truth must then be attested to by the consequences derived from them. The collection of axioms and postulates should be of the fewest possible, provided they enable all the results to be proved. Though, as we shall see, Euclid uses Aristotle's distinction between common notions and postulates, all mathematicians up to the late nineteenth century overlooked this distinction and treated axioms and postulates as equally self-evident. According to Aristotle, the axioms are obtained from the observation of physical objects. They are immediately apprehended generalizations. He and his followers gave many definitions and axioms or improved earlier ones. Some of the Aristotelian versions were taken up by Euclid.

Aristotle discusses the fundamental problem of how points and lines can be related. A point, he says, is indivisible and has position. But then no accumulation of points, however far it may be carried, can give us anything divisible, whereas of course a line is a divisible magnitude. Hence points cannot make up anything continuous like a line, for point cannot be continuous with point. A point, he says, is like the now in time; now is indivisible and not a part of time. A point may be an extremity, beginning, or divider of a line but is not a part of it or of magnitude. It is only by *motion* that a point can generate a line and thus be the origin of magnitude. He also argues that a point has no length and so if a line were composed of points, it would have no length. Similarly, if time were composed of instants there would be no interval of time. His definition of continuity, which a line possesses, is: A thing is continuous when the limits at which any two successive

parts touch are one and the same and are, as the word continuous implies, held together. Actually Aristotle makes many statements about continuous magnitudes which are not in agreement with each other. The substance of his doctrine, nevertheless, is that points and numbers are discrete quantities and must be distinguished from the continuous magnitudes of geometry. There is no continuum in arithmetic. As to the relation of the two fields, he considers arithmetic—that is, the theory of numbers—more accurate, because numbers lend themselves to abstraction more readily than the geometric concepts. He also considers arithmetic to be prior to geometry because the number three is needed to consider a triangle.

In discussing infinity he makes a distinction, which is important today, between the potentially infinite and the actually infinite. The age of the earth, if it had a sudden beginning, is potentially infinite but is never at any time actually infinite. According to Aristotle, only the potentially infinite exists. The positive integers, he grants, are potentially infinite in that we can always add 1 to any number and get a new one, but the infinite set as such does not exist. Further, most magnitudes cannot be even potentially infinite, because if they were continually added to they could exceed the bounds of the universe. Space, however, is potentially infinite, in that it can be repeatedly subdivided, and time is potentially infinite in both ways.

A major achievement of Aristotle was the founding of the science of logic. In producing correct laws of mathematical reasoning the Greeks had laid the groundwork for logic, but it took Aristotle to codify and systematize these laws into a separate discipline. Aristotle's writings make it abundantly clear that he derived logic from mathematics. His basic principles of logic—the law of contradiction, which asserts that a proposition cannot be both true and false, and the law of excluded middle, which maintains that a proposition must be either true or false—are the heart of the indirect method of mathematical proof. Further, Aristotle used mathematical examples taken from contemporary texts to illustrate his principles of reasoning. Aristotelian logic remained unchallenged until the nineteenth century.

Though the science of logic was derived from mathematics, logic eventually came to be considered independent of and prior to mathematics and applicable to all reasoning. Even Aristotle, as already noted, regarded logic as preliminary to science and philosophy. In mathematics he emphasized deductive proof as the sole basis for establishing facts. For Plato, who believed that mathematical truths preexist or exist in a world independent of man, reasoning was not the guarantee of the correctness of theorems; the logical powers played only a secondary role. They made explicit, so to speak, what was already known to be true.

One member of Aristotle's school especially worthy of note is Eudemus of Rhodes, who lived in the last part of the fourth century B.C. and was the author of the Eudemian summary quoted by Proclus and Simplicius. As we noted earlier, Eudemus wrote histories of arithmetic, geometry, and astronomy. He is the first historian of science on record. But what is more significant is that the knowledge already existing in his time should have been sufficiently extensive to warrant histories.

The last of the men of the classical period we shall mention here is Autolycus of Pitane, an astronomer and geometer, who flourished about 310 B.C. He was not a member of Plato's or Aristotle's schools, though he did teach one of the leaders who succeeded Plato. Of three books he wrote, two have come down to us; they are the earliest Greek books we have intact, though only through manuscripts that presumably are copies of Autolycus' work. These books, *On the Moving Sphere* and *On Risings and Settings*, were eventually included in a collection called the *Little Astronomy* (as distinguished from Ptolemy's later *Great Collection*, or the *Almagest*). *On the Moving Sphere* treats meridian circles, great circles generally, and what we would call parallels of latitude, as well as the visible and invisible areas produced by a distant light source shining on a rotating sphere, as the sun does on the earth. The book presupposes theorems of spherical geometry which must, therefore, have been known to the Greeks of that time. Autolycus' second book, on the rising and setting of stars, belongs to observational astronomy.

The form of the book on moving spheres is significant. Letters denote points on diagrams. The propositions are logically ordered. Each proposition is first stated generally, then repeated, but with explicit reference to the figure; finally, the proof is given. This is the style Euclid uses.

Bibliography

Apostle, H. G.: *Aristotle's Philosophy of Mathematics*, University of Chicago Press, 1952.

Ball, W. W. Rouse: *A Short Account of the History of Mathematics*, Dover (reprint), 1960, Chaps. 2–3.

Boyer, Carl B.: *A History of Mathematics*, John Wiley and Sons, 1968, Chaps. 4–6.

Brumbaugh, Robert S.: *Plato's Mathematical Imagination*, Indiana University Press, 1954.

Gomperz, Theodor: *Greek Thinkers*, 4 vols., John Murray, 1920.

Guthrie, W. K. C.: *A History of Greek Philosophy*, Cambridge University Press, 1962 and 1965, Vols. 1 and 2.

Hamilton, Edith: *The Greek Way to Western Civilization*, New American Library, 1948.

Heath, Thomas L.: *A History of Greek Mathematics*, Oxford University Press, 1921, Vol. 1.

———: *A Manual of Greek Mathematics*, Dover (reprint), 1963, Chaps. 4–9.

———: *Mathematics in Aristotle*, Oxford University Press, 1949.

Jaeger, Werner: *Paideia*, 3 vols., Oxford University Press, 1939–44.

Lasserre, François: *The Birth of Mathematics in the Age of Plato*, American Research Council, 1964.

Maziarz, Edward A., and Thomas Greenwood: *Greek Mathematical Philosophy*, F. Unger, 1968.

Sarton, George: *A History of Science*, Harvard University Press, 1952, Vol. 1, Chaps. 7, 11, 16, 17, and 20.

Scott, J. F.: *A History of Mathematics*, Taylor and Francis, 1958, Chap. 2.

Smith, David Eugene: *History of Mathematics*, Dover (reprint), 1958, Vol. 1, Chap. 3.

van der Waerden, B. L.: *Science Awakening*, P. Noordhoff, 1954, Chaps. 4–6.

Wedberg, Anders: *Plato's Philosophy of Mathematics*, Almqvist and Wiksell, 1955.