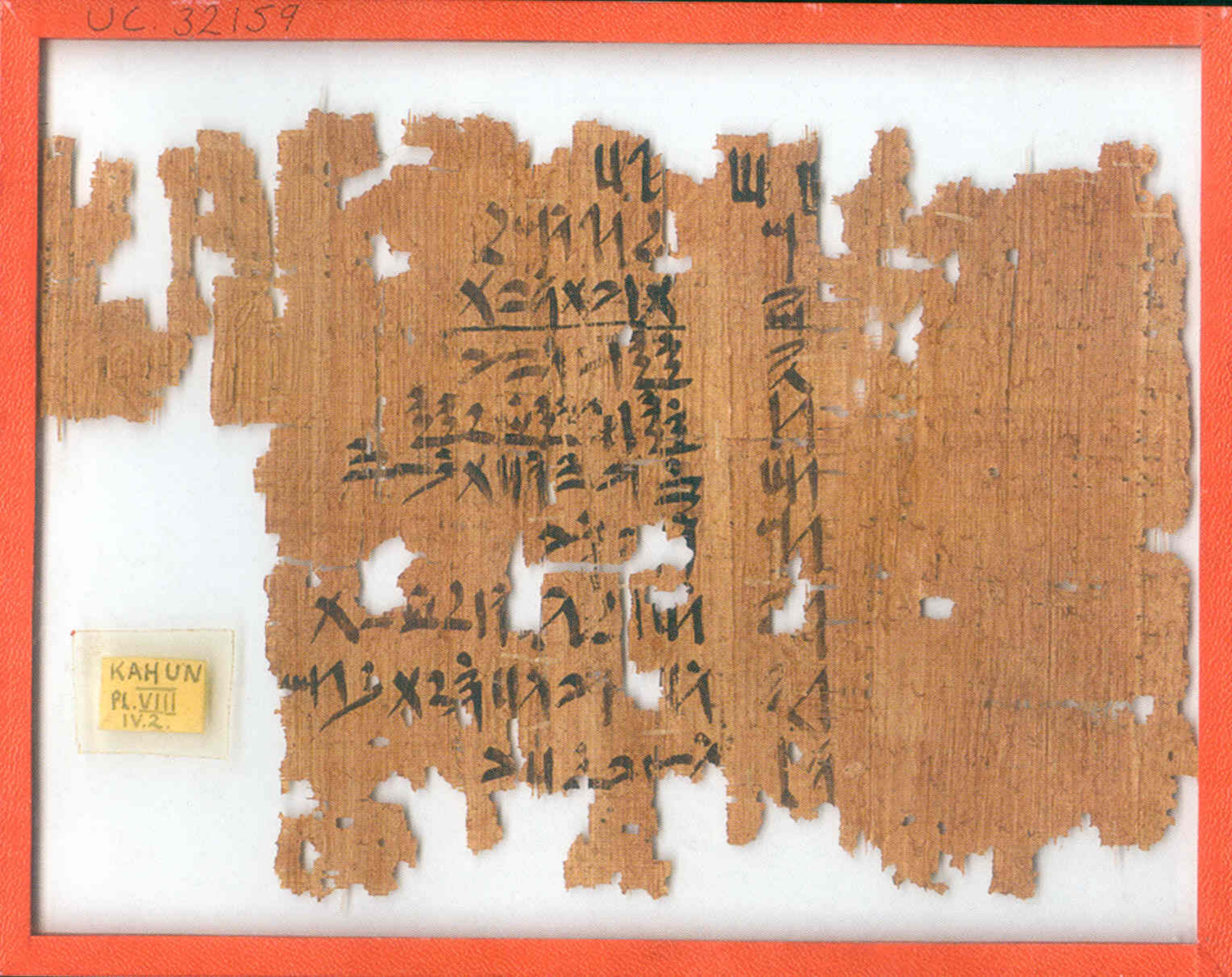
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Ancient Egyptian Mathematics: New Perspectives on Old Sources

Annette Imhausen

Pro captu lectoris
habent sua fata libelli
(Terentianus Maurus)

If books, in general, have their own special fates—which depend on their readers—the same is true for the mathematical "books" from ancient Egypt. Indeed, modern editors and subsequent readers have strongly influenced the way we view them today. And even now, readers of the third millennium can alter the fate of these early texts by their careful (or careless) reading.[1]

Sources and Early Historiography

For the past fifty years, the reputation of Egyptian mathematics has been rather poor. This has been due in part to the very limited number of available primary sources, particularly when compared with the vast collections of cuneiform mathematical texts produced in Mesopotamia. In ancient Egypt the production of mathematics (as well as literature) took place in cities. Then, as today, Egyptian cities were located along the Nile, and hence close to water. This circumstance has had significant consequences for contemporary Egyptological research. On the one hand, papyrus, the main writing material in this culture, was dependent on absolute dryness for its preservation, a condition found in the Egyptian desert where most papyrus finds were made. However, in ancient Egyptian cities, where writings concerned with the mundane affairs of daily life were discarded after use, this condition was usually not fulfilled. Therefore, most of the written evidence documenting the role of mathematics in Egyptian social, economic, and cultural life must be assumed lost forever. On the other hand, to the extent that such sources may still be retrievable some day, practical problems stand in the way. The locations of ancient Egyptian cities often coincide with those of modern urban centers. This makes it next to impossible to excavate at a number of locations where extant remains might still be found.

Among the few known (excavated) cities, the Middle Kingdom town of Lahun (also known as Illahun or Kahun) is exceptional, having yielded the richest findings of Middle Kingdom papyri so far, among which incidentally are a number of mathematical fragments.[2] The two most significant sources, however, the famous Rhind and Moscow mathematical papyri, were bought on the antiquities market, making their provenance uncertain. These and most of the other known mathematical sources were already published by 1930.

The achievements of the earliest researchers who studied these texts, especially those who worked during the first half of the twentieth century, were enormous. As editors, they managed to penetrate a foreign vocabulary of technical terms, which placed them in position to make a first attempt at understanding Egyptian mathematical methods.[3] As was common at that time, ancient sources and achievements were viewed and evaluated by means of direct comparison with modern conventions and results. In many respects, it was found that Egyptian mathematics had little in common with the methods found in modern mathematical textbooks. Nevertheless, with some effort the mathematical content of the ancient texts could be "decoded" and "translated" into modern mathematics.

Unfortunately, this type of reading often entailed a loss of the most striking characteristics of the original sources, a drawback that was little appreciated at the time. Not surprisingly, the "achievements" of Egyptian mathematics, judged in terms of a different mathematical culture (from more than 3000 years later), looked rather crude and simple. One of the early leading authorities on ancient mathematics was Otto Neugebauer, who wrote his dissertation on Egyptian methods of cal-

Send submissions to **David E. Rowe,**
Fachbereich 17—Mathematik,
Johannes Gutenberg University,
D55099 Mainz, Germany.

culating with fractions.[4] Afterward, Neugebauer turned his attention away from Egyptian mathematics to study Mesopotamian mathematics and astronomy, which he believed was a higher level of scientific achievement. As he once expressed this:

> Egypt provides us with the exceptional case of a highly sophisticated civilization which flourished for many centuries without making a single contribution to the development of the exact sciences. [. . .] It is at this single center (Mesopotamia) that abstract mathematical thought first appeared, affecting, centuries later, neighbouring civilizations, and finally spreading like a contagious disease.[5]

It was surely in part due to the outstanding quality of the early scholarly contributions that readers accepted so readily this kind of negative assessment of Egyptian mathematics. As indicated already, this situation was compounded by the lack of new source material which—had it been there—would have required those capable of reading Egyptian texts to reflect upon the assessments of their predecessors. Thus, in the case of Mesopotamian mathematics, where new source material is still being uncovered on an almost regular basis, readers' opinions have changed significantly over time.[6] Lacking this wealth of textual material, readers of the Egyptian texts seemed to have no basis for questioning the standard views of earlier experts like Neugebauer. Indeed, once the major Egyptian mathematical papyri became available in English or German translation, various historians of mathematics began contributing new ideas based on their own readings of these first translations. Often these involved modern mathematical symbolism, leading to results that had almost nothing in common with the original source text.[7]

This once common approach has now been recognized as both anachronistic and misleading. Indeed, for the last 20 years historians of mathematics have started to take up and to rework the subject of ancient mathematics.[8] It is now generally accepted that historians of mathematics cannot work on a source text without knowing the language in which it is written or the cultural background it comes from. At the same time, it has become obvious that mathematical knowledge is not universal. It is neither independent of the cultures in which it is produced and used, nor has it developed universally from basic beginnings to more and more advanced stages of knowledge. This dependence on cultural background begins already with number systems and number concepts, as has been demonstrated by various scholars working on ethnomathematics.[9] More advanced mathematical techniques and concepts have also been shown to be dependent on the culture that created them.[10]

Current Research

From this description of past research, it follows that the editions of Egyptian mathematical sources are by now outdated. It is to be hoped that new editions can be published before the current ones reach their centenary. Likewise, older studies of Egyptian mathematics, those written more than 30 years ago, must be read with caution, bearing in mind the kind of approach past researches typically took. For an up-to-date introduction to the subject, the reader should consult the articles by Jim Ritter.[11] In the following sketch, I will attempt to give an overview of the state of current research, illustrated with selected examples from the source material.

Although there have been no spectacular new finds of mathematical papyri, extant sources, including the much-studied Rhind and Moscow papyri, still offer many clues about the role of mathematics in Egyptian life. Alongside these, the Lahun mathematical fragments have just been re-edited, including several previously unpublished fragments.[12]

Other texts are still awaiting proper publication, such as the mathematical fragments of Papyrus Berlin 6619. The earlier publications from 1900 and 1902 only contain facsimiles of the two largest fragments. Moreover, the interpretations of them then given are not without problems.[13] The Cairo wooden boards are currently available in two very small and hardly legible photos with a discussion of some of their content. While a number of demotic mathematical texts have been published, no detailed study of Egyptian mathematics in the Graeco-Roman period is available yet.[14]

Evidence from the Predynastic Period

Apart from the extant mathematical texts, however, there are further sources available throughout Egyptian history which inform us about aspects and uses of mathematics as it evolved in ancient Egypt in periods from which no mathematical texts are extant. Written evidence exists from as early as around 3000 B.C., the oldest dating from shortly before the unification of Egypt. It comes from the tomb Uj at Abydos[15] and consists of writing on pottery as well as on little tags of bone and ivory. These tags all reveal holes, suggesting they were probably once attached to some perishable goods from this grave, thus indicating their provenance and quantity.[16] The quantities were rendered using elements and style familiar from the Egyptian number system in later times, i.e., a decimal system without positional notation (see Figure 1). In this system, each power of 10 up to 1 million was represented by a different sign. In order to write any number, the respective signs, written as often as needed, were juxtaposed in a symmetric way. Note that the hieroglyphic writing, which is what most people associated with ancient Egypt, was used mostly on stone monuments. For daily life purposes, Egyptian scribes wrote with a reed (dipped in ink) on papyrus or so-called ostraca (limestone or pottery shards). The

Figure 1. Number representations on the tags from tomb Uj.

script used in this writing is more cursive and abbreviated than hieroglyphic script. Several signs can be combined to form ligatures, whereas the writing itself can vary a great deal, depending on the individual scribe (just like modern handwriting).

Mathematics in the Old Kingdom

After the unification of Egypt under a single king (around 3000 B.C.), the Old Kingdom (OK; 2686–2160 B.C.) brought forth the first period of cultural bloom in Egyptian history. Extant architectural remains, like the pyramids, as well as such artifacts as the scribal statues, demonstrate a high level of cultural attainment by this time. There can be little doubt that mathematical techniques lay at the heart of this development as a significant tool for handling organizational and administrative problems. To achieve something on the scale of the pyramids, mathematics was necessary not only for architectural planning but also for the organization of labor. The scribal statues, which depict high officials from this period, demonstrate the importance of the administrative system. Despite this, there is practically no written evidence for mathematical practices extant from this time. Many of the monumental hieroglyphic inscriptions are still extant—but these, of course, focus on eternity and tell us little about Egyptian daily life and the affairs in which mathematics played an important part. Only very few papyri from this period have survived, some in a very fragmentary state.

Nevertheless, there is other direct evidence of Egyptian mathematical techniques, for example from the planning and execution of building projects such as a mastaba from Meidum (see Figure 2). Around the corners of this mastaba, beneath the ground level, four L-shaped mud-brick walls had been built. On these walls a series of diagrams can be found, which indicate the slope of the sides of the mastaba. This method of handling sloped surfaces points to the development of a concept which is well documented in the mathematical texts.[17] To express sloped surfaces, such as the sides of a pyramid, the Egyptians used the so-

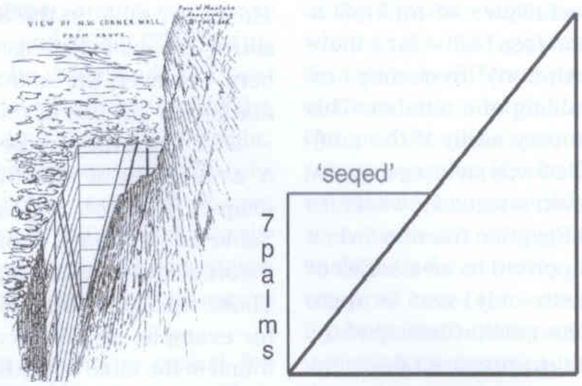


Figure 2. Indication of a sloped surface at Meidum.

called *sqd*. This Egyptian term is derived from the verb *qd*, meaning "to build." The *sqd* was used to measure the horizontal displacement of the sloped face for each vertical drop of one cubit, that is the length by which the sloped side had "moved" from the vertical at the height of one cubit. The *sqd* was always indicated in palms, and if necessary, digits. Although we have textual evidence for this concept only from the Middle Kingdom onward, sketches from the Old Kingdom indicate that it was in use during this earlier period. Note that the parallel lines drawn on the mud bricks are spaced at a distance of one cubit or seven palms.

Furthermore there is early evidence for several metrological systems. While these units can also be found in later mathematical texts, their appearance in administrative papyri as well as in the inscriptions and depictions from tombs indicates that these systems go back at least to the Old Kingdom. Some of these systems changed over time, but the sources from the Old Kingdom suffice to trace these changes.

Calculations with Unit Fractions

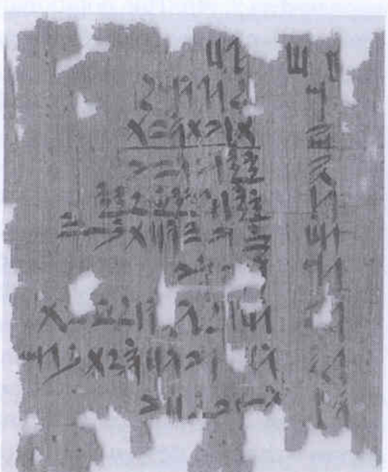
One of the most intriguing aspects of Egyptian mathematics concerns special methods for calculating fractions, which were understood in ancient Egypt as inverses of integers.[18] Hence, the Egyptian notation for fractions did not consist of a numerator and denominator, but rather a special symbol was used alongside an integer to designate the corresponding fraction. An exception was the fraction $\frac{2}{3}$, which had a special sign. The fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ were also written by using special signs (indicating that these may be older) rather than by using the general Egyptian notation.[19] In modern studies, Egyptian fractions are usually described as unit fractions, and it is often suggested that the Egyptians "restricted" themselves to calculations with fractions having a numerator of one.[20] As explained in the paragraph above, however, this is a rather anachronistic view. Moreover, seen from a modern perspective, the Egyptian system inevitably appears awkward and unnecessarily restrictive.

One of the first to study Egyptian computations with fractions was Otto Neugebauer, who devised a notational system that parallels the Egyptian notation. Fractions, as inverses of integers, are rendered by the value of the integer with an overbar: thus, $\frac{1}{5}$ would be written as $\overline{5}$, $\frac{1}{6}$ as $\overline{6}$, etc. The exceptional fraction $\frac{2}{3}$ was rendered by Neugebauer as $\overline{\overline{3}}$, whereas $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ appeared as $\overline{2}$, etc. This notational system, which closely mirrors the Egyptian concept of fraction, has become the standard way of writing Egyptian fractions in modern textbooks.

Following this concept of fractions as inverses of integers, the next step—consequently—was to express those parts that correspond to a multitude of inverses. This was done by (additive) juxtaposition of different inverses. Thus, $\frac{3}{4}$ was written in the Egyptian system as $\overline{2}\ \overline{4}$, whereas a general fraction was given as a sum of different inverses written in descending order according to their size. (Note that this notation enables one to be as accurate as necessary by considering only elements up to a certain size.)

Egyptian techniques of multiplication and division (see below for a more detailed description) frequently involved the doubling of a number. This could be done very easily if the number to be doubled was an integer or the inverse of an even integer. However, to double an odd Egyptian fraction (when the result is supposed to be a series of different inverses only) can be quite difficult to accomplish. Consequently, it proved useful to prepare tables giving the results for doubling the inverses of odd numbers. These can be found in the so-called $2 \div N$ tables still extant in two sources: at the beginning of the Rhind Mathematical Papyrus (for odd $N = 1 - 101$) and in the Lahun fragment UC 32159 (for odd $N = 1 - 21$).

Figure 3 shows the fragment UC 32159 in which the numbers are arranged in two columns. The first column shows (what we call) the divisor N, except for the first entry which shows both the dividend 2 and the divisor 3. This is followed by a second column that alternatingly shows fractions of the divisor and their value (as a series of inverses). Thus, the second line starts with the divisor 5 in the first column: it is $2 \div 5$ that shall be expressed as unit fractions. This is followed in the second column by $\overline{3}$, $1\ \overline{\overline{3}}$, $\overline{15}$, and $\overline{3}$. This has to be read as $\overline{3}$ of 5 is $1\ \overline{\overline{3}}$, and $\overline{15}$ of 5 is $\overline{3}$. Since $1\ \overline{\overline{3}}$ and $\overline{3}$ added equal 2, the series of unit fractions needed to represent $2 \div 5$ is $\overline{3}\ \overline{15}$.



2 3	$\overline{\overline{3}}$ 2
5	$\overline{3}$ 1 $\overline{\overline{3}}$ $\overline{15}$ $\overline{3}$
7	$\overline{4}$ 1 $\overline{2}$ $\overline{4}$ $\overline{28}$ $\overline{4}$
9	$\overline{6}$ 1 $\overline{2}$ $\overline{18}$ $\overline{2}$
11	$\overline{6}$ 1 $\overline{\overline{3}}$ $\overline{6}$ $\overline{66}$ $\overline{6}$
13	$\overline{8}$ 1 $\overline{2}$ $\overline{8}$ $\overline{52}$ $\overline{4}$ $\overline{104}$ $\overline{8}$
15	$\overline{10}$ 1 $\overline{2}$ $\overline{30}$ $\overline{2}$
17	$\overline{12}$ 1 $\overline{3}$ $\overline{12}$ $\overline{51}$ $\overline{3}$ $\overline{69}$ $\overline{4}$
19	$\overline{12}$ 1 $\overline{2}$ $\overline{12}$ $\overline{75}$ $\overline{4}$ $\overline{114}$ [$\overline{6}$]
21	$\overline{14}$ 1 $\overline{2}$ $\overline{42}$ $\overline{2}$

Figure 3. Fragment UC 32159: $2 \div N$ table (Copyright Petrie Museum of Egyptian Archaeology, University College London).

The $2 \div N$ table in the Rhind papyrus shows the same arrangement of numbers; however, the solutions there are marked by the use of red ink.

Obviously, the representation of $2 \div N$ as a series of unit fractions is not unique. However, the Egyptian $2 \div N$ Table uses for each N only one of the theoretically possible representations. Those we find in the Lahun fragment, for example, are identical to the ones found in the table of the Rhind papyrus. And whenever an odd fraction is doubled within the mathematical texts, it is this same representation that we find used.

This circumstance has fascinated a number of experts on additive number theory. In fact, there have been several attempts to crack the puzzle posed by the $2 \div N$ Table by finding the criteria that led the Egyptians to employ just these particular representations. Yet, while it is possible to describe some of the general tendencies—e.g., representations with fewer elements are favored as are also representations with larger inverses, etc.—it has not been possible to establish strict mathematical rules that explain the choices the Egyptians mathematicians made. Rather than criticizing them for their lack of insight—or blaming them for not having followed strict rules that would comply with a different mathematical concept of fractions devised by another culture several thousand years later—it seems more appropriate to recognize that mathematics is, indeed, culturally dependent; our modern point of view may not afford us the best picture of past achievements. Thus, instead of trying to concoct an explanation of the Egyptian solutions by using modern mathematics, it may be more rewarding simply to "accept" the Egyptian table and examine its use and usefulness within the mathematical environment that employed it.

Mathematical Problem Texts from the Middle Kingdom

Apart from tables, the mathematical texts also include special procedures articulated within problem texts. As these names indicate, such texts set out a problem and then give instructions showing how to solve it. Procedure texts derive from an educational setting. They may have been written by a teacher, who was compiling a handbook, or perhaps by a student engaged in practicing mathematical techniques. An appreciation of this context is important for understanding these texts, which were intended to prepare scribes for the mathematical tasks they would later have to execute as part of their daily work.[21] Given that these texts were written for this type of mathematical education, it should not be expected that we can learn *how* the Egyptians developed their mathematical knowledge from sources of this nature.

The extant hieratic mathematical texts contain roughly one hundred problems. Furthermore, in the largest of these texts, the Rhind Mathematical Papyrus (see Figure 4), we can discern an arrangement of these problems according to their rising level of difficulty. This is not to be judged by purely mathematical aspects alone but also by additional knowledge (often from a practical background) which is necessary to solve the problems. This can be seen, for example, in pRhind, problems 31–34 and those immediately following, problems 35–38. Mathematically, both groups teach a procedure for determining an "unknown" number if its sum with fractions of itself is given. The procedure for solving the problems in both groups is roughly the same. However, in the second group (pRhind, problems 35–38), the "unknown" number is not an abstract number but a quantity of grain. Therefore the result, which is determined in the same way as in the preceding problems, needs to be transformed afterwards into the respective metrological units.[22]

The style of Egyptian mathematical problem texts can best be appreciated by looking at an actual example, like problem 56 of the Rhind Mathematical Papyrus:

Method of calculating a pyramid, 360 is its base, 250 is its height.
You shall let me know its inclination.
You calculate half of 360. It results as 180.
You divide 180 by 250. $\overline{2}\ \overline{5}\ \overline{50}$ of a cubit results.
1 cubit is 7 palms.[23]
You multiply with 7.

	·	7
\	$\overline{2}$	$3\ \overline{2}$
\	$\overline{5}$	$1\ \overline{3}\ \overline{15}$
\	$\overline{50}$	$\overline{10}\ \overline{25}$ Its inclination: $5\ \overline{25}$ palms

Problem 56, like the other four examples of pyramid problems found in the Rhind Papyrus (nos. 57, 58, 59, and 59b), teaches the relation between the base, height, and inclination of the sides. This example complements the OK sketch found on the walls around the mastaba with sloping sides, which was discussed above. In fact, the technical term *sqd*—the number of palms the slope of a slanted plane recedes per vertical difference of one cubit—is explicitly indicated in the problem text. Thus, the base, height, and inclination of a pyramid are linked by the relation:

$$\text{inclination} = 7 \text{ palms} \times \frac{{}^{1}\!/_{2}\ \textit{base}}{\textit{height}}$$

The problem above presents a pyramid with base (360) and height (250); its inclination is to be calculated. The procedure calls for calculating half of the base and dividing this by the height. The result is then multiplied by 7 to obtain the inclination in palms. Having grasped "what is going on" in this problem, let us now take a second, closer look at the Egyptian text and its means of structure.

The text begins—as is typical for mathematical problem texts—with a title "**Method of calculating** a pyramid." Note that the beginning of the title is written in red ink (rendered in my translation in **bold**). This use of red ink helps the reader recognize at a glance the beginnings of individual problems. The title of mathematical problems is very often given as "Method of . . ." followed by a key word which indicates the type of problem. In our example, the key word is the Egyptian *mr*, "pyramid."

After this title, the given data are introduced, and they are always specific numerical values. This statement of the data is generally followed by a question or command, outlining the problem that the scribe shall solve. In this example: "You shall let me know its inclination." Next, we see a sequence of instructions, followed by intermediate results. This procedure then leads to the numerical solution of the problem. Each instruction usually consists of one arithmetic operation. The Egyptian mathematical language distinguishes addition, subtraction, multiplication, division, halving, inverting, squaring, and the extraction of square roots. These individual mathematical operations are expressed without any use of mathematical symbols. The instructions themselves are always given as complete sentences.

Furthermore, in this part of the text, a special verb form is used, the so-called ***sḏm.ḥr=f***. The name consists of the Egyptian verb "to hear" (***sḏm***), which is used in Egyptian grammars to demonstrate different conjugations, its characteristic morphological element (***ḥr***) and the suffix pronoun of the third person singular (***f***). Its function is to express a "general truth" which results as a necessary sequence from previously stated conditions.[24] In the mathematical texts, the ***sḏm.ḥr=f*** is used for both instructions and announcing intermediate results. As for the latter, the verb form expresses "mathematical facts"—if 2 and 2 are added, the result will necessarily be 4. The use of the ***sḏm.ḥr=f*** in the instructions underlines the specific procedural character of the text: the sequence of instructions necessarily has to be followed to solve the problem. The last instruction given, the multiplication of ($\overline{2}\ \overline{5}\ \overline{50}$) by 7 is followed by a scheme of numbers. This carries out the actual multiplication in the Egyptian manner, which may now be described.

Multiplication (and division) are executed following a scheme that uses two columns of numbers.[25] Each multiplication begins with the initialization which is found in the first line of the scheme: a dot is placed in the first column and the number to be multiplied in the second column. The multiplication is carried out by subsequent operations in both columns using a variety of techniques, depending on the numerical values of the numbers that shall be multiplied. The aim is to find the multiplier as a combination of entries in lines of the first column. The respective lines of the second column will then be the result of the multiplication.

Problem 56 of the Rhind Papyrus shows the notation used to compute $7 \times \overline{2}\ \overline{5}\ \overline{50}$. The initialization is followed by three more lines, each of which indicates one of the required fractional parts ($\overline{2}$, $\overline{5}$, $\overline{50}$) of the multiplier. How the individual entires of the second column were found is not obvious. It is possible that there may have been tables for fractional parts of 7, as this was a number that leads to complicated calculations, but which came up frequently due to the metrological conventions.[26]

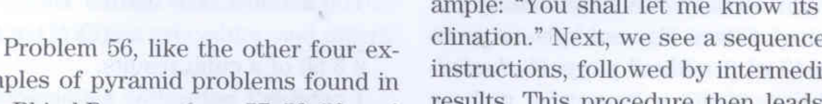


Figure 4. Rhind Mathematical Papyrus, No. 56 Copyright The British Museum.

Finally, the result of the problem is announced. Next to the text of the problem there is a sketch indicating characteristic measurements for this problem, i.e., the values of base and height (see Figure 5). This step-by-step layout can be found in virtually all Egyptian problem texts. This being the case, one can easily see that the formal aspect of phrasing mathematics in the form of procedures will be completely lost if a problem is "translated" into a modern algebraic equation (in this case: inclination = ($\frac{1}{2}$ *base/height*) × 7 palms). While this formula has the advantage of informing a modern reader at a single glance how an ancient measure was defined, it conveys nothing whatsoever about the procedural character of Egyptian mathematics. Moreover, algebraic formulae played no part in Egyptian mathematics so that the above formulation for the *sqd* is anachronistic, at best, as it is foreign to the methods actually found in Egyptian texts.

Analyzing Egyptian Problem Texts

As it happens, a closer analysis of the problem texts reveals many hitherto unnoticed methodological features of Egyptian mathematics. Indeed, the procedural format can be used as a key to analyze not only individual problems but also various types of problems as found in the mathematical papyri. To get beyond a superficial understanding of Egyptian mathematics, however, a method was needed that enabled a reader to analyze and compare the Egyptian procedures. Such an approach was first proposed by Jim Ritter.[27] In my dissertation I have adapted this method to analyze the various procedures used in all hieratic mathematical problems.[28]

The analysis of a specific problem text can be carried out by rewriting it in two stages. In the first, one keeps the numerical values indicated in the source text but rewrites the instructions by replacing the rhetoric formulations with modern symbols that indicate the respective arithmetic operations. The data are noted at the beginning of the scheme by their numerical values. Thus, for the example cited above (pRhind, problem 56), the text would be rewritten as follows:

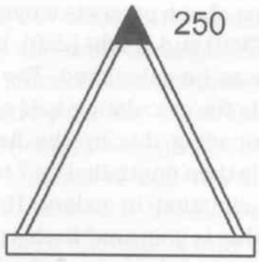


Figure 5. Sketch at the end of Rhind Mathematical Papyrus, No. 56.

Method of calculating a pyramid,
360 is its base, 360
250 is its height. 250
You shall let me know
its inclination.
You calculate half of 360.
(1) $\overline{2} \times 360$
It results as 180. = 180
You divide 180 by 250.
(2) $180 \div 250$
$\overline{2}\ \overline{5}\ \overline{50}$ of a cubit results.
= $\overline{2}\ \overline{5}\ \overline{50}$
1 cubit is 7 palms.
You multiply with 7.
(3) $\overline{2}\ \overline{5}\ \overline{50} \times 7$

The result allows one to see at a glance whether the arithmetic operations to be carried out were simplified by the choice of data. For example, in problem 43 of the Rhind papyrus, the calculation of the volume of a granary with circular base, the diameter of the granary is given as 9. This greatly facilitates the calculational procedure, the first step of which is to determine $\frac{1}{9}$ of the diameter. In the values of our problem, the given data were 360 and 250. While the first step, halving 360, is fairly straightforward, the second, the division of the result of the first step by the second datum results in a fraction of three parts, which then has to be multiplied by 7. Thus, by comparison, the data in problems 58 and 59 result in easier calculations.

This first stage of rewriting is especially helpful when dealing with a corrupt text, as the modern reader is forced to follow the source text and identify the procedure in a step-by-step fashion. It then becomes immediately apparent where specific difficulties arise in the source.

To further analyze the text so as to reveal how its procedures are related to those used in other problems, it is necessary to distinguish between different types of numbers that can appear throughout the procedure. The first numbers a reader encounters are the data of the given problem. From the second instruction on, three types of numbers are possible: data, intermediate results, and constants. To distinguish these, and also to get a clearer view of the structure of the procedure, a second stage of rewriting is required. In this stage the data are indicated by symbols D_i, whereas intermediate results are specified by a number in parentheses (x) which specifies the step in the procedure that leads to the given result. The only actual numbers that now appear in the rewritten text are constants. Thus, for our example, the result of this second rewriting is as follows:

Method of calculating a pyramid,
360 is its base, D1
250 is its height. D2
You shall let me know
its inclination.
You calculate half of 360.
(1) $\overline{2} \times$ D1
It results as 180.
You divide 180 by 250.
(2) (1) ÷ D2
$\overline{2}\ \overline{5}\ \overline{50}$ of a cubit results.
1 cubit is 7 palms.
You multiply with 7. (3) (2) × 7

In my dissertation I have analyzed the procedures of all hieratic mathematical problems by rewriting the procedure in the form of a symbolic algorithm. This makes it possible to compare the various procedures used and analyze their respective complexity. The analysis of problems by means of their procedures or algorithms thus constitutes a powerful tool for comparing the structure of individual mathematical problem texts. From this, one can learn a great deal about Egyptian mathematical techniques. Within the Rhind Mathematical Papyrus, for example, one finds groups of problems with similar procedures (pRhind, No. 24–27), as well as a progression within one group from basic procedures to more elaborate ones (pRhind, No. 69–78).

Identifying an unambiguous symbolic algorithm can sometimes be straightforward, as in the example above. Unfortunately this is not the case with all problems. Individual instructions may be missing—sometimes they are replaced by a written calculation, or several steps are summarized in one instruction only. These types of difficulties can sometimes be overcome by taking into account all of the available source material. If—as in the Rhind Papyrus—several problems of the same kind are available and their procedures are identical insofar as they are explicitly stated, then those problems which lack certain instructions can occasionally be reconstructed by means of the more detailed problems.

I would like to stress in this context that both types of rewriting are merely tools for analyzing specific aspects of the procedures found in the problem texts, whereas the source texts themselves remain central and should never be neglected in any analysis. Taking the three versions of the procedure together, however, enables one to form a more complete analysis that includes not only the various procedures but also technical mathematical vocabulary, as well as the relation of drawings and calculations carried out in writing connected with the procedure, and others.

Mathematics within the Context of Egyptian Culture

Another integral part of the reassessment of Egyptian mathematics concerns its role within Egyptian culture. Mathematics was one of the key elements of scribal training in pharaonic Egypt. It provided the scribes with a crucial tool they needed to fulfil their administrative tasks as well as to plan and carry out construction projects. Consequently, many of the mathematical problems they dealt with were related to practical matters, e.g., the distribution of rations, the volume of granaries, or the amount of produce to be delivered by a worker. Our understanding of mathematical problems of this kind is at least partially dependent on our appreciation for these larger contexts.

This can be demonstrated with the so-called bread and beer problems, which appear against the background of economic activity, baking and brewing, under the control of a local authority (state or temple). A quantity of grain is taken from a granary and then given to workers who produce bread and/or beer from it. Obviously it was necessary to ascertain the quantity—in loaves of bread or vessels of beer—of a given quality (in this case measured by grain content) that was equivalent to the amount of grain initially given to the workers. The mathematical side of this control is represented by the bread and beer problems.[29] The terminology used in these problems is taken from the respective technological language. Thus the bread and beer problems evolve around the ***psw***, a unit which measures how many loaves of bread have been made from one ***ḥḳ3.t*** of grain. Apart from the ***psw***, there are two additional standard phrases indicating the use of specific kinds of grain products and their quantities. Obviously, this has further consequences for the respective calculations. Similar observations can be made for other groups of practical problems as well. These generally involve not only the "basic" mathematical terminology but also further knowledge related to the technological or administrative background. This usually makes them not only more difficult to understand but also less likely to be "mirrored" by a familiar problem in modern mathematics. Thus, early historical research often neglected this area of Egyptian mathematics.

However, as is obvious from the ordering of the problems found in the Rhind Papyrus, it was precisely these practical problems that were considered more advanced. After all, the aim of the mathematical handbooks was to prepare scribes for their daily administrative work. So if we want to obtain insights into Egyptian mathematics, we must consider these problems and try to understand them. The setting of the individual problem may help to point to further sources (not only textual) which may be useful to understand the additional terminology and practice. Furthermore, it is this type of problem that indicates other possibilities of gaining knowledge about Egyptian mathematics apart from the restricted corpus of mathematical texts. The actual output of the scribes in doing their daily work provides us with numerous documents that prove the use of mathematical techniques. Thus Michel Guillemot has used a ration text from Kahun to analyze mathematical practices.[30] These can be linked to techniques taught in mathematical papyri.[31] It is to be hoped that this example can be followed for other texts as well.

The most promising sources still to be explored in this respect are the Reisner Papyri. This set of four papyrus rolls contains calculations for the building of a sanctuary, including ration tables, actual building calculations, as well as the administration of workshops.[32] They not only enlarge the meagre set of seven problems related to architecture which are known from the Moscow (problem 14) and Rhind (problems 56–60) papyri, but they also demonstrate that the amount of work done was linked to a specific number of workers (and rations) per day.

Evidence of Mathematics in the New Kingdom

While the mathematical texts date almost exclusively from the Middle Kingdom, other sources are available from all periods of Egyptian history. The Wilbour Papyrus, a text from the New Kingdom, is an official record of measurements and assessments of fields over a distance of 90 miles along the Nile. The fields are given by localization and acreage, their assessments referring to taxes specified in amounts of grain.

Another major opportunity to find relevant sources of mathematics for the New Kingdom is provided by the excavation of Deir el Medina. Deir el Medina is the modern name of an ancient Egyptian village on the West Bank of the Nile opposite Luxor. The village was inhabited by workmen who were responsible for the construction and decoration of the tombs in the Valley of the Kings. Deir el Medina has yielded a huge quantity of artifacts and texts relating to daily life in the New Kingdom—similar to the findings at Lahun for the Middle Kingdom. Among the sources are ration texts, building plans, as well as texts for the education of scribes. The ostracon in Figure 6 shows a fragment of an exercise in the multiplicative writing of large numbers. It shows in the first column

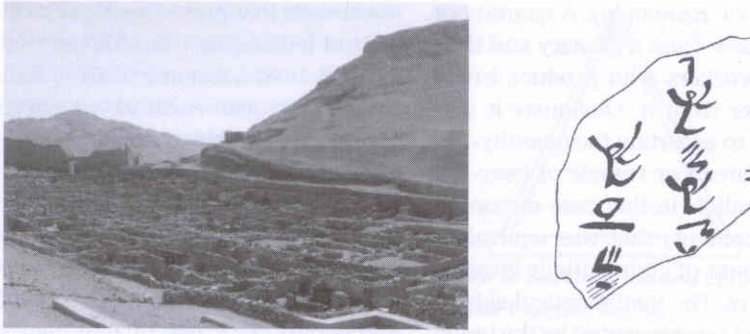


Figure 6. Deir el Medina: Remains and Ostracon with Number Exercise.

(on the right) the numbers 600,000, 700,000, and 800,000 and in the second (middle) column the numbers 5,000,000, 6,000,000, and 7,000,000 written by the sign for the number 100,000 (or 1,000,000) with the respective multiplicative factors (6, 7, and 8 and 5, 6, and 7) below. The third column (left) shows again the sign for 1,000,000 and two illegible signs below.

Conclusions

Although Egyptian mathematics will probably never have the vast number of sources that still can be found in other cultures like India or Mesopotamia, there is more available than has been used so far.[33] The analysis of all the available mathematical texts, taken along with the additional material from administrative economic and literary contexts related to Egyptian mathematics, is certain to provide a better foundation for understanding its role within Egyptian culture. This integrated approach represents an important advance beyond the early studies that relied exclusively on an internal analysis of a small corpus of mathematical texts, which served for several decades as the sole basis for assessing nearly three millennia of mathematical life in ancient Egypt. By carefully rereading these classical mathematical texts while according the new sources a serious first reading, we may anticipate that the fate of Egyptian mathematics faces an exciting future.

NOTES

1. I thank David Rowe for his comments on previous versions of this article and for his corrections of my English. I also thank Richard Parkinson of the British Museum and Stephen Quirke of the Petrie Museum for permission to include photographs of sources.
2. See Annette Imhausen and Jim Ritter, "Mathematical Papyri," in: Mark Collier and Stephen Quirke (eds.), *The UCL Lahun Papyri: Religious, Literary, Legal, Mathematical and Medical*, Oxford: Arcaheopress 2004.
3. Among the early editions, the most noteworthy are still Thomas E. Peet, *The Rhind Mathematical Papyrus. British Museum 10057 and 10058*, London: Hodder and Stoughton 1923, and Wasili W. Struve, *Mathematischer Papyrus des Staatlichen Museums der Schönen Künste in Moskau* (Quellen und Studien zur Geschichte der Mathematik, Abteilung A: Quellen, Vol. 1), Heidelberg: Springer 1930.
4. Otto Neugebauer, *Die Grundlagen der ägyptischen Bruchrechnung*, Berlin: Julius Springer 1926.
5. Otto Neugebauer, *A History of Ancient Mathematical Astronomy* (Part Two). Berlin, Heidelberg, New York: Springer 1975: 559.
6. See for example the interpretations of Plimpton 322, e.g., compare Joran Friberg, "Methods and traditions of Babylonian mathematics: Plimpton 322, Pythagorean triples and the Babylonian triangle parameter equations," *Historia Mathematica* 8 (1981): 277–318 and the recent reassessment by Eleanor Robson (Eleanor Robson, "Neither Sherlock Holmes nor Babylon: a reassessment of Plimpton 322," *Historia Mathematica* 28 (2001); 167–206 and Eleanor Robson, "Words and pictures: new light on Plimpton 322," *American Mathematical Monthly* 109 (2002): 105–120).
7. An extreme example of this is Richard Gillings, "The Volume of a Truncated Pyramid in Ancient Egyptian Papyri," *The Mathematics Teacher* 57 (1964): 552–555.
8. For Egyptian mathematics, see for example James Ritter, "Chacun sa vérité: les mathématiques en Égypte et en Mésopotamie," in: Michel Serres (ed.), *Éléments d'histoire des sciences*: 39–61, Paris: Bordas 1989; James Ritter, "Egyptian Mathematics," in: Helaine Selin (ed.), *Mathematics Across Cultures: The History of Non-Western Mathematics*: 115–136, Dordrecht, Boston, London: Kluwer 2000, as well as Annette Imhausen, *Ägyptische Algorithmen: Eine Untersuchung zu den mittelägyptischen mathematischen Aufgabentexten*, Wiesbaden: Otto Harrassowitz 2003. For Greek Mathematics, cf. Serafina Cuomo, *Ancient Mathematics*, London, New York: Routledge 2001, Michael N. Fried and Sabetai Unguru, *Apollonius of Perga's Conica. Text, Context, Subtext*, Leiden: Brill 2001, as well as David Fowler, *The Mathematics of Plato's Academy: A New Reconstruction* (Second Edition), Oxford: Clarendon Press 1999, and Reviel Netz, *The Shaping of Deduction in Greek Mathematics: A Study of Cognitive History* (Ideas in Context 51), Cambridge: Cambridge University Press 1999. For Mesopotamian mathematics, see most recently Jens Høyrup, *Lengths, Widths, Surfaces. A Portrait of Old Babylonian Algebra and its Kin*, New York: Springer 2002, and Eleanor Robson, *Mesopotamian Mathematics, 2100–1600 BC: Technical Constants in Bureaucracy and Education* (Oxford Editions of Cuneiform Texts XIV), Oxford: Clarendon Press 1999.
9. See Gary Urton, *The Social Life of Numbers. A Quechua Ontology of Numbers and Philosophy of Arithmetic*, Austin, Texas: University of Texas Press 1997, and Marcia Ascher, *Mathematics Elsewhere. An Exploration of Ideas across Cultures*, Princeton, N.J.: Princeton University Press 2002.
10. See, for example, for Mesopotamia Jens Høyrup, *Lengths, Widths, Surfaces. A Portrait of Old Babylonian Algebra and its Kin*, New York: Springer 2002.
11 See note 8.
12. See Annette Imhausen and Jim Ritter, "Mathematical Papyri," in: Mark Collier and Stephen Quirke (eds.), *The UCL Lahun Papyri: Religious, Literary, Legal, Mathematical and Medical*, Oxford: Arcaheopress 2004. Another mathematical fragment will be published in the next volume of that series.
13. See Oleg Berlev, "Review of William Kelly Simpson: Papyrus Reisner III: The Records of a Building Project in the Early Twelfth Dynasty, Boston: Museum of Fine Arts 1969," *Bibliotheca Orientalis* 28 (1971): 324–326, esp. p. 325.

14. Richard Parker, "A Demotic Mathematical Papyrus Fragment," *Journal of Near Eastern Studies* 18 (1959): 275–279; Richard Parker, *Demotic Mathematical Papyri*, Providence, R.I.: Brown University Press 1972; Richard Parker, "A Mathematical Exercise–P. Dem. Heidelberg 663," *Journal of Egyptian Archaeology* 61 (1975): 189–196. A list of Demotic mathematical ostraca can be found in Jim Ritter, "Egyptian Mathematics," in: Helaine Selin (ed.), *Mathematics across Cultures. The History of Non-Western Mathematics*, Dordrecht: Kluwer 2000: 134, note 27.
15. See Günter Dreyer, *Umm el-Qaab I. Das prädynastische Königsgrab U-j und seine frühen Schriftzeugnisse*, Mainz: Von Zabern 1998.
16. For a discussion of the inscriptions on these tags, see Günter Dreyer, *Umm el-Qaab I. Das prädynastische Königsgrab U-j und seine frühen Schriftzeugnisse*, Mainz: Von Zabern 1998, pp. 137–145, and John Baines, "The Earliest Egyptian Writing: Development, Context, Purpose," in: Stephen D. Houston, *The First Writing. Script Invention as History and Process*, Cambridge: Cambridge University Press 2004: 150–189.
17. See problems 56–60 of the Rhind Mathematical Papyrus.
18. Jim Ritter, "Mathematics in Egypt," in: Helaine Selin (ed.), *Encyclopedia of the History of Science, Technology and Medicine in Non-Western Cultures*, Dordrecht, Boston, London: Kluwer 1997, p. 631.
19. For the prehistory of Egyptian fractions and their development see Jim Ritter, "Metrology and the Prehistory of Fractions," in: Paul Benoit, Karine Chemla, Jim Ritter (eds.), *Histoire de fractions, fractions d'histoire*: 19–34, Basel, Boston, Berlin: Birkhäuser 1992.
20. See, for example, the description of Couchoud: " . . . il ne semble avoir connu que les fractions unitaires, c'est à dire celles dans lesquelles le numérateur est toujours équivalent à l'unité, . . ." (Sylvia Couchoud, *Mathématiques Égyptiennes. Recherches sur les connaissances mathématiques de l'Égypte pharaonique*, Paris: Le Léopard d'Or 1993, p. 21) or that of Gillings: "When the Egyptian scribe needed to compute with fractions he was confronted with many difficulties arising from the restriction of his notation. His method of writing numbers did not allow him to write such simple fractions as $\frac{3}{5}$ or $\frac{5}{9}$ because all fractions had to have unity for their numerators (with one exception)." (Richard J. Gillings, *Mathematics in the Time of the Pharaohs*, Cambridge, Mass.: MIT Press 1972, p. 20).
21. See Jim Ritter, "Egyptian Mathematics," in: Helaine Selin (ed.), *Mathematics across Cultures. The History of Non-Western Mathematics*, Dordrecht, Boston, London: Kluwer 2000, p. 120.
22. For a discussion of the use of an abstract number system and conversions into metrological systems, see Jim Ritter, "Egyptian Mathematics," in: Helaine Selin (ed.), *Mathematics across Cultures. The History of Non-Western Mathematics*, Dordrecht, Boston, London: Kluwer 2000, pp. 121–122.
23. The cubit was the Egyptian standard measure of length. 1 cubit consisted of 7 palms; each palm, of 4 digits.
24. An English example for its use would be the statement "If I have a stone in my hand, and let it drop, then the stone falls to the ground." The last part of this statement "then the stone falls to the ground" is where the ***sḏm.ḫr=f*** is used in an Egyptian text.
25. Examples can be found in Annette Imhausen and Jim Ritter, "Mathematical Fragments: UC32114, UC32118, UC32134, UC32159–UC32162," in: Mark Collier and Stephen Quirke, *The UCL Lahun Papyri: Religious, Literary, Legal, Mathematical and Medical* (British Archaeological Reports International Series 1209): 71–96, Oxford; Archaeopress 2004, esp. pp. 85–86.
26. The New Kingdom Ostracon Senmut 153 may be interpreted as a table of $\frac{1}{7}$ in this way, see David Fowler, *The Mathematics of Plato's Academy: A New Reconstruction* (Second Edition), Oxford: Clarendon Press 1999, p. 269.
27. Jim Ritter, "Chacun sa vérité: les mathématiques en Égypte et en Mésopotamie," in: Michel Serres (ed.), *Élements d'histoire des sciences*: 39–61, Paris: Bordas 1989 (English edition: Jim Ritter, "Measure for Measure: Mathematics in Egypt and Mesopotamia," in: Michel Serres (ed.), *A History of Scientific Thought. Elements of a History of Science*: 44–72, Oxford: Blackwell 1995).
28. Annette Imhausen, *Ägyptische Algorithmen. Eine Untersuchung zu den mittelägyptischen mathematischen Aufgabentexten* (Ägyptologische Abhandlungen 65). Wiesbaden: Otto Harrassowitz 2003.
29. For a detailed discussion of these problems see Annette Imhausen, "Egyptian Mathematical Texts and their Contexts," *Science in Context* 16, 2003: 367–389.
30. Michel Guillemot, "Les notations et les pratiques opératoires permettent-elles de parler de 'fractions égyptiennes'?", in: Paul Benoit, Karine Chemla, Jim Ritter (eds.), *Histoire de fractions, fractions d'histoire*, Basel, Boston, Berlin: Birkhäuser 1992: 53–69.
31. Annette Imhausen, "Calculating the Daily Bread: Rations in Theory and Practice," *Historia Mathematica* 30 (2003): 3–16.
32. A first attempt to analyze the mathematical content of some parts of the Reisner Papyri has been made by Richard J. Gillings, *Mathematics in the Time of the Pharaohs*, Cambridge, Mass. MIT Press 1972, pp. 218–231.
33. A variety of architectural sources (with mathematical implications) can be found in Corinna Rossi, *Architecture and Mathematics in Ancient Egypt*, Cambridge: Cambridge University Press 2004.

AUTHOR

ANNETTE IMHAUSEN
Trinity Hall
Cambridge University
Trinity Lane, Cambridge CB2 1TJ
UK
e-mail: ai226@com.ac.uk

Annette Imhausen studied mathematics, history of mathematics, and Egyptology and received her PhD from Mainz University (Germany) under David E. Rowe. She has held fellowships at the Dibner Institute for the History of Science and Technology (Cambridge, Mass.) and at Trinity Hall (Cambridge University, England). She is currently completing a book that outlines the historical development of ancient Egyptian mathematics, situating it within the larger social, economic, and cultural background. In her spare time she enjoys running 1/2 and 2/3 marathons.