Euclid's "Geometric Algebra"

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Euclid considers plane *figures* (σχῆματα) which are certain subsets of the plane. This includes all *rectilinear* figures. Below, I focus my attention only on the latter ones.

Let us consider the following three equivalence relations on set $\mathbf{\Sigma}$ of all rectilinear figures. Figures σ and σ' are said to:

- be *congruent* ("fitted" to each other, to use a literal translation of Euclid's terminology) if one can be "placed" over the other with perfect match. Such a placement can be always accomplished as a result of rigid motion of the whole plane (notation: $\sigma \sim_c \sigma'$, the congruence class of σ will be denoted $\bar{\sigma}$);

- be *scissors-equivalent* if both figures can be dissected into a finite number of mutually congruent figures (notation: $\sigma \sim_s \sigma'$, the equivalence class of σ will be denoted $[\sigma]$);

- be *equivalent in the sense of Euclid* if our figures are obtained by removing a finite number of congruent figures from two scissors-equivalent figures (notation: $\sigma \sim_e \sigma'$; the equivalence class of σ will be denoted $|\sigma|$). Subsequently, I will often refer to such figures as *"equal"*.

Each subsequent relation is a priori weaker than its predecessor:

$$\sigma \sim_c \sigma' \quad \Rightarrow \quad \sigma \sim_s \sigma' \quad \Rightarrow \quad \sigma \sim_e \sigma'. \tag{1}$$

Each satisfies the conditions spelled out in the *Common Notions* section of Book I. Only the last equivalence relation, however, satisfies an additional requirement of Euclidean Geometry, first *implicitly* referred to in the proof of fundamental Proposition i.35:[1]

> If one removes *congruent* pieces from two *"equal"* figures then one again obtains *"equal"* figures. (2)

[1](...) *the triangle* EAB *will be equal to triangle* ΔZΓ. *Let (triangle)* ΔΓE *have been taken away from both. Thus the remaining trapezium* ABHΔ *is* "equal" *to the remaining trapezium* .

Equivalence classes of *congruent* figures are naturally represented by "free" figures, i.e., figures literally "lifted into the air" from their actual location on the plane.

Addition of segments. Given two straight line segments ("segments", in short) AB and $\Gamma\Delta$, we can attach one to the other in a straight line. The resulting segments depend on whether we attach $\Gamma\Delta$ to AB, or vice-versa, and whether we place point Γ at B or at A. The resulting segments are however congruent to each other.

Moreover, if we replace segments AB and $\Gamma\Delta$, by segments AB and $\Gamma\Delta$ congruent to AB and $\Gamma\Delta$, respectively, then the resultsing segments are still congruent. We obtain thus a well defined operation of addition on the set E of congruence classes of segements which can be identified with the set of "free" segements:

$$+\colon \mathsf{E} \times \mathsf{E} \longrightarrow \mathsf{E}. \tag{3}$$

Equipped with binary operation (3) the set of free segments becomes a commutative semigroup.

One can define addition of angles exactly in the same manner.

Addition of figures. If we try to do the same with arbitrary rectilinear figures, the resulting figures will not be congruent to each other in most cases, they will be *scissors-equivalent* instead. This behavior is only seemingly different from the behavior of the operation of attachment for segments: for segments, the relations of being *congruent* and *scissors-equivalent* coincide!

If $\boldsymbol{\Sigma}$ denotes the set of all (finite) rectilinear figures, let

$$\boldsymbol{\Sigma}' := \boldsymbol{\Sigma}/\sim_s \tag{4}$$

denote the set of scissors-equivalence classes of rectilinear figures. Attachment of figures yields thus a well defined binary operation on set $\boldsymbol{\Sigma}'$:

$$+\colon \boldsymbol{\Sigma}' \times \boldsymbol{\Sigma}' \longrightarrow \boldsymbol{\Sigma}' \tag{5}$$

which makes it a commutative semigroup in complete similarity to the set of free segments E.

Euclid's equivalence relation $\sim_e$ is compatible with this operation of addition: given two *"equal"* figures, $\sigma \sim_e \sigma'$, attaching either of them to a figure τ produces *"equal"* figures. In particular, oparation (5) induces an operation of addition also on the set of *Euclid's equivalence* classes:

$$\mathbf{\Sigma}'' := \mathbf{\Sigma} / \sim_e \,, \tag{6}$$

so that we have the following commutative diagram

$$\begin{tikzcd} \mathbf{\Sigma}' \times \mathbf{\Sigma}' \arrow[r, "+"] \arrow[d, two heads] & \mathbf{\Sigma}' \arrow[d, two heads] \\ \mathbf{\Sigma}'' \times \mathbf{\Sigma}'' \arrow[r, "+"] & \mathbf{\Sigma}'' \end{tikzcd} \tag{7}$$

where the vertical maps correspond to the canonical map $\mathbf{\Sigma}' \longrightarrow \mathbf{\Sigma}''$ which sends any equivalence class, $[\sigma]$, of relation $\sim_s$ to the corresponding equivalence class, $|\sigma|$, of relation $\sim_e$.

Multiplication of segments. Given two segments λ and μ, one can construct a rectangle having λ and μ as its sides. Its congruence class depends only on the congruence classes of λ and μ. We will denote it $\lambda \times \mu$.

We thus obtain a pairing:

$$\times : \mathsf{E} \times \mathsf{E} \longrightarrow \mathbf{\Sigma}. \tag{8}$$

If we *fix* one of the segments, say λ, then the induced map

$$\mathsf{E} \longrightarrow \mathbf{\Sigma}, \qquad \mu \mapsto \lambda \times \mu, \tag{9}$$

is injective: *rectangles having pairwise equal sides are congruent*.

Euclid's crowning achievement in Book I of the *Elements* is the following theorem.

Proposition i.45 *Given a segment λ and a figure σ, there exists a segment μ such that $|\sigma| = |\lambda \times \mu|$. Segment μ is unique up to congruence.*

The same in modern mathematical idiom:

for any segment λ, the map

$$E \longrightarrow \Sigma'', \qquad \mu \mapsto |\lambda \times \mu|, \tag{10}$$

defines an isomorphism between the semigroup, E, of free segments and the semigroup, Σ'', of ***Euclid's equivalence*** *classes of figures.*

Let us *fix* some segment $\eta \in \mathsf{E}$. The inverse map to (10)

$$m_\eta \colon \mathbf{\Sigma}'' \longrightarrow \mathsf{E}, \quad |\sigma| \mapsto \text{the unique } \mu \in \mathsf{E} \text{ such that } |\sigma| = |\eta \times \mu|, \tag{11}$$

whose existence is guaranteed by the above theorem of Euclid, can be used to define multiplication on E:

$$\mathsf{E} \times \mathsf{E} \longrightarrow \mathsf{E}, \qquad \mu\mu' := m_\eta(\mu \times \mu'). \tag{12}$$

It is essential to remember that this multiplication depends on the choice of segment η even though our notation does not reflect it.

Exercises.
(*a*) What $\eta\mu$ is equal to?
(*b*) Show that multiplication (12) is distributive with respect to addition of segments.
(*c*) What is the *meaning* of $\mu\mu'$?

Associativity and commutativity of multiplication (12) is a question involving a priori theory of volume for 3-dimensional figures. There is however an approach that allows one to stay within the framework of plane geometry. It is based on the theory of *similar figures* which is developed in Book VI, and on the celebrated *Theorem of Pappus*, another great mathematician from Alexandria, who lived there some 6 centuries later. He was the last significant mathematician of Antiquity.

With all this additional nontrivial work it is possible to demonstrate that the set of free segments, E, equipped with operations of addition, (5), and multiplication, (12) becomes a *semifield*.[2]

Having chosen the *unit* segment η, one can identify Pythagorean Arithmetic as the arithmetic of segments of special kind, namely of positive integral multiples of η (Euclid would have said: *of segments that are* ***measured*** *by* η).

To me this seems to be the key to understanding the methods of Book VII of the *Elements*. On the other hand, theory of proportions is developed in Book V along different lines. Modern commentators regard the theory of proportions of Book V to be a creation of Eudoxus. Book VI that was mentioned above studies the equivalence relation of *similarity* of figures;

[2]If E had zero and $-\lambda$ for any $\lambda \in \mathsf{E}$ then E would be a *field*.

it can be considered as an application of the theory of proportions to plane geometry.

Coordinatisation of Euclidean geometry. A model for Euclidean geometry of the plane is provided by F^2 whenever F is a *field* equipped with the following two properties:

(a) F is *ordered;*[3]

(b) for any $a \in F$, the equation $x^2 = 1 + a^2$ has a solution in F (we might say that $\sqrt{1+a^2} \in F$).

Points of the plane are described by pairs (a, b) of elements of F (their "coordinates").

Thus, the *real plane* $\mathbb{R}^2$ provides a familiar model of Euclidean plane geometry while the *complex plane* $\mathbb{C}$ does not. A natural question arises whether *any* model of Euclidean geometry of the plane must be necessarily of this kind. The answer is *yes*: and the *coordinate field F* is obtained from the semifield of segments, E, by formal addition of the *zero* segment and the symbol $-\lambda$ for each segment $\lambda \in \mathsf{E}$. Note that E then becomes the set of all *positive* elements in F.

This beautiful result is referred to as the Coordinatization Theorem of Euclidean geometry, and Euclid himself played a significant part in proving it..

Archimedean planes. We shall say that a Euclidean plane satisfies the so called *Archimedes Axiom* if for any segment $\lambda, \mu \in \mathsf{E}$, there exists a positive integer n such that

$$\mu < n\lambda := \underbrace{\lambda + \cdots + \lambda}_{n \text{ times}}. \tag{13}$$

Consider the smallest field K of functions which contains all real *rational* functions of a single variable, say t,

$$f(t) = \frac{p(t)}{q(t)} \tag{14}$$

and which, for any function $g \in K$, contains also the function $\sqrt{1+g^2}$. The coordinatized plane. K^2, over this field provides an example of a *non* Archimedean plane.

[3]The ordering is of course assumed to be compatible with the operations of addition and multiplication.

It is an elementary fact, though it seems it was first recorded only in XIX-th Century, that

$$\text{\textit{The two relations: of \textbf{scissors-equivalence}, $\sim_s$, and of \textbf{equivalence in the sense of Euclid}, $\sim_e$, coincide \textbf{if and only if} the plane satisfies the Archimedes Axiom.}} \tag{15}$$

The sufficiency of the Axiom of Archimedes for these two relations to be the same relation is usually referred as the Theorem of Bolyai and Gerwein. The necessity can be established as follows.

Let a segment $\lambda \in \mathsf{E}$ be given. We will say that a figure σ *has diameter **greater** than* λ, if *there exist* points A and B belonging to σ such that the free segment $|\mathrm{AB}|$ is bigger than λ:

$$|\mathrm{AB}| > \lambda. \tag{16}$$

Conversely, we will say that figure σ *has diameter **less** than* λ, if *for every* pair of distinct points A and B belonging to σ

$$|\mathrm{AB}| < \lambda. \tag{17}$$

The above notions depend only on the congruence classes of figures, so they make sense for free figures.

Before we proceed, let us make an observation that if the diameters of two figures σ and τ are less than λ and μ, respectively, then the diameter of any figure obtained by attaching τ to σ has the diameter less than $\lambda + \mu$.

Suppose that our plane is non Archimedean. This means that there exist segments λ and λ' such that

$$n\lambda < \lambda' \tag{18}$$

for any positive integers n.

Now consider two figures σ and σ' such that

$$\sigma \; \text{ has diameter less than } \lambda \tag{19}$$

and

$$\sigma' \; \text{ has diameter greater than } \lambda'. \tag{20}$$

Such figures *cannot* be scissors-equivalent. Indeed, when we cut σ into finitely many pieces, say

$$\sigma_1, \ldots, \sigma_n \tag{21}$$

the diameter of each piece is likewise less than λ. But then *any* figure obtained by reattaching pieces congruent to figures (21) will, by the observation made above, have the diameter less than

$$\underbrace{\lambda + \cdots + \lambda}_{n \text{ times}} = n\lambda < \lambda'. \tag{22}$$

In particular, one cannot obtain figure σ' by reattaching parts of σ *no matter how, and into how many pieces, we cut it*!

It follows that two triangles with the same base and the same height, *are not* scissors-equivalent if the diameter of one of them is "infinitely" bigger than the diameter of the other one. At the same time they are equivalent in the sense of Euclid.