The Jones Polynomial

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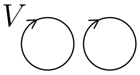
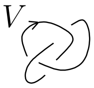
1 Introduction

A *link* is a finite family of disjoint, smooth, oriented or unoriented, closed curves in $\mathbb{R}^3$ or equivalently S^3. A *knot* is a link with one component. The *Jones polynomial* $V_L(t)$ is a Laurent polynomial in the variable $\sqrt{t}$ which is defined for every oriented link L but depends on that link only up to orientation preserving diffeomorphism, or equivalently isotopy, of $\mathbb{R}^3$. Links can be represented by diagrams in the plane and the Jones polynomials of the simplest links are given below.

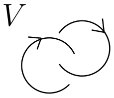
V  $= 1$

V  $= -(\frac{1}{\sqrt{t}} + \sqrt{t})$

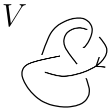
V  $= t + t^3 - t^4$

*Supported in part by NSF Grant DMS 0401734, the NZIMA and the Swiss National Science Foundation.

$$V \quad = \quad \sqrt{t}\ (1+t^2)$$



$$V \quad = \quad \frac{1}{t^2} - \frac{1}{t} + 1 - t + t^2$$

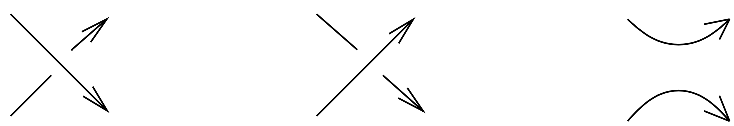


The Jones polynomial of a knot (and generally a link with an odd number of components) is a Laurent polynomial in t.

The most elementary ways to calculate $V_L(t)$ use the "linear skein theory" ideas of [7]. Indeed it is not hard to see by induction that $V_L(t)$ is defined by its invariance under isotopy, the normalisation V  $(t) = 1$ and the skein formula

$$\frac{1}{t}V_{L_+} - tV_{L_-} = (\sqrt{t} + \frac{1}{\sqrt{t}})V_{L_0}$$

which holds for any 3 oriented links having diagrams which are identical except near one crossing where they differ as below.



L_+ $\qquad$ L_- $\qquad$ L_0

As such the Jones polynomial resembles the Alexander polynomial $\Delta_L(t)$ of [1] which can be calculated in exactly the same manner as $V_L(t)$ except that the skein relation becomes

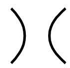
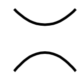
$$\Delta_{L_+} - \Delta_{L_-} = (\sqrt{t} + \frac{1}{\sqrt{t}})\ \Delta_{L_0}.$$

A two variable generalisation P_L of both Δ_L and V_L, sometimes called the HOMFLYPT polynomial, was found in [16] and [34]. It satisfies the most general skein relation

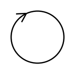
$$xP_{L_+} + yP_{L_-} + zP_{L_0} = 0$$

for homogeneous variables x, y and z.

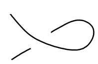
The other skein-like definition of V_L was found in [23]. Begin with *un-oriented* link diagrams up to *planar* istotopy. The Kauffman bracket $\langle L \rangle$ of such a diagram is calculated using

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

where the $\langle \cdot \rangle$ notation means that the relation may be applied to that part of the link diagrams inside the bracket, the rest of the diagrams being identical. If $\langle L \rangle$ were to be an invariant of three-dimensional isotopy it is easy to see that

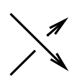
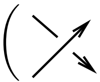
$\langle$  $\rangle = -A^2 - A^{-2}$

which further implies

$\langle$  $\rangle = A^{-3} \langle$  $\rangle$

Thus $\langle L \rangle$ cannot be a 3-dimensional isotopy invariant as such. However if L is given an orientation (then called $\vec{L}$), a simple renormalisation solves the problem and it is true that

$$(*) \qquad V_{\vec{L}}(A^4) = A^{-3 \ \mathrm{writhe}\,(\vec{L})} \langle L \rangle$$

where writhe$\,(\vec{L})$ is the sum over the crossings of $\vec{L}$ of $+1$ for a positive crossing () and -1 for a negative crossing ().

The formula (*) is readily proved by induction but a more structural proof will be discussed later on, connected with physics. If the crossings in a link alternate between over and under as one follows the string around, the highest and lowest degree terms in the Kauffman bracket can readily be located. This led to the proof of some old conjectures about alternating knots in [32],[23] and [38].

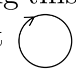
The Kauffman 2-variable polynomial $F_L(a, x)$ is defined in [24] by considering the linear skein relation involving all four possibilities at a crossing:

L_+ $\qquad$ L_- $\qquad$ L_0 $\qquad$ L_∞.

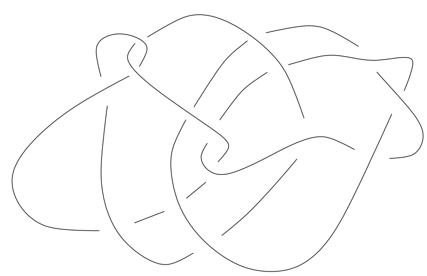
This polynomial contains $V_L(T)$ as a specialisation but not the Alexander polynomial.

The above polynomials are quite powerful at distinguishing links one from another, including links from their mirror images, which corresponds for the Jones polynomial to replacing t by t^{-1}. More power can be added to the polynomials if simple geometric operations are allowed. "Cabling" entails replacing a single strand with several parallel copies and the polynomials of cables of a link are also isotopy invariants if attention is paid to the writhe of a diagram.

The following problem, however, is open at the time of writing this article:

"Does there exist a knot in $\mathbb{R}^3$, different from the unknot , whose Jones polynomial is equal to 1?"

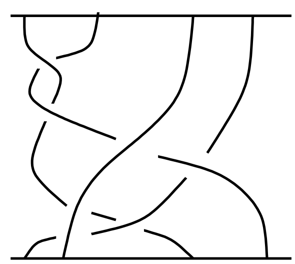
For links with more than one component it is known ([39], [10]) that the answer to the corresponding question is yes, the simplest example being:



One of the reasons that the question above has not been answered is presumably that, unlike with the Alexander polynomial, we have little intuitive understanding of the meaning ot the "t" in $V_L(t)$. Perhaps the most promising theory in this context is in [25] where a complex is constructed whose Euler characteristic, in an appropriately graded sense, is the Jones polynomial. The homology of the complex is a finer invariant of links known as "Khovanov homology".

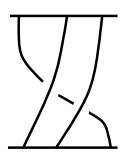
2 Braids

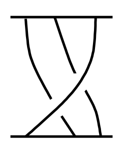
A braid (see [5]) on n strings is a collection of curves in $\mathbb{R}^3$ joining n points in a horizontal plane to the n points directly below them on another horizontal plane. If the end points of the braid are on a straight line the braid can be drawn as in the example below (where $n = 4$).

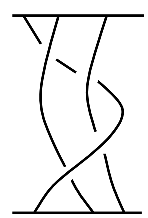


The crucial property of a braid is that the tangent vector to the curves can never be horizontal. Braids are considered up to isotopies which are supported between the top and bottom planes.

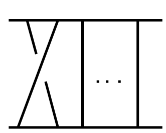
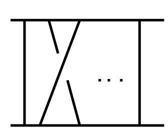
Braids on n strings form a group, called B_n, under concatenation (plus some isotopy) as below:

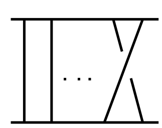
$\alpha =$ 

$\beta =$ 

$\rightarrow \alpha\beta =$ 

Let $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ be the braids below:

$\sigma_1 =$  , $\sigma_2 =$  ,

$\ldots,\ \sigma_{n-1} =$ 

Artin's presentation ([5]) of the braid group is on the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with the relations

$$\left\{ \begin{array}{rcll} \sigma_i\sigma_{i+1}\sigma_i & = & \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } 1 \leq i \leq n-2 \\ \sigma_i\sigma_j & = & \sigma_j\sigma_i & \text{if } |i-j| \geq 2 \end{array} \right. \qquad (3)$$

Thus to find linear representations of B_n it suffices to find matrices $\rho_1, \rho_2, \ldots, \rho_{n-1}$ satisfying (3) (with σ replaced by ρ). One such representation (of dimension n) called the (non-reduced) Burau representation is given by the row-stochastic matrices

$$\rho_1 = \begin{pmatrix} 1-t & t & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1-t & t & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}, \ldots$$

$$\rho_{n-1} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1-t & t \\ 0 & \ldots & 1 & 0 \end{pmatrix}$$

this representation is known not to be faithful for $n \geq 5$ but faithful for $n \leq 3$. The case $n = 4$ remains open. (See [30], [28], [4].)

Braids can be viewed in several ways, which lead to several generalisations. For instance, identifying the vertical axis for a braid with time and taking the intersection of horizontal planes with the braids shows that elements of B_n can be thought of as *motions* of n distinct points in the plane. Thus it is natural that

$$B_n \cong \pi_1(\{\mathbb{C}^n \setminus \Delta\}/S^n)$$

when Δ is the set $\{(z_1, \ldots, z_n) | z_i = z_j \text{ for some } i \neq j\}$ and the symmetric group S_n acts freely on $\mathbb{C}^n \setminus \Delta$ by permuting coordinates. But Δ is the zero-set of the frequently encountered function

$$\prod_{i<j} (z_i - z_j)$$

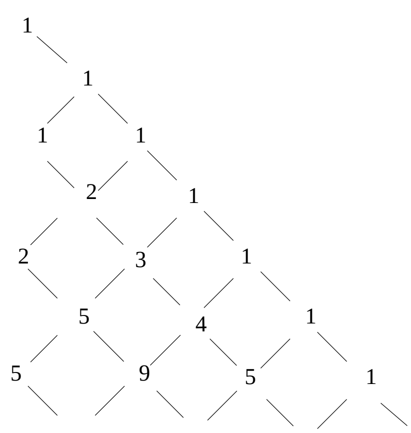
so the braid group may naturally be generalised as the fundamental group of $\mathbb{C}^n$ minus the singular set of some algebraic function ([5]). Or, motions of points can be extended to motions of the whole plane and a braid defines a diffeomorphism of the plane minus n points. Thus the braid group may be generalised as the *mapping class group* of a surface with marked points ([5]).

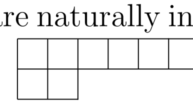
3 The Temperley–Lieb algebra

If $\tau \in \mathbb{C}$ one may define the algebra $TL(n,\tau)$ with identity 1 and generators $e_1, e_2, \ldots, e_{n-1}$ subject to the following relations:

$$\begin{aligned} &e_i^2 = e_i \\ &e_i e_{i\pm1} e_i = \tau e_i \\ &e_i e_j = e_j e_i \quad \text{if } |i-j| \geq 2. \end{aligned}$$

Counting reduced words on the e_i's shows that $\dim\{TL(n,\tau)\} \leq \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}$ and in [19] it is shown that these numbers, the Catalan numbers, are indeed the dimensions of the Temperley–Lieb algebras. In the obvious way, $TL(n,\tau) \subseteq TL(n+1,\tau)$. If τ^{-1} is not in the set $\{4\cos^2 q\pi; q \in \mathbb{Q}\}$, $TL(n,\tau)$ is semisimple and its structure is given by the following Bratelli diagram:



where the integers on each row are the dimensions of the irreducible representations of $TL(n,\tau)$ and the diagonal lines give the restriction of representations of $TL(n,\tau)$ to $TL(n-1,\tau)$. These representations are naturally indexed by Young diagrams with n boxes and at most two rows:  with the diagonal lines in the Bratelli diagram corresponding to removal/addition of a box. The dimension of the representation corresponding to the diagram whose second row has r boxes ($r \leq n$), is $\begin{pmatrix} n \\ r \end{pmatrix} - \begin{pmatrix} n \\ r-1 \end{pmatrix}$.

One may attempt to make $TL(n,\tau)$ into a C*-algebra and look for Hilbert space representations (with $e_i \neq 0$), by imposing $e_i^* = e_i$. From [41] this is only possible (for all n) when

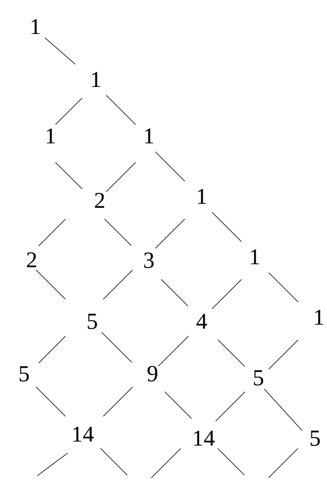
$$\begin{array}{rl} \text{(i)} & \tau \in \mathbb{R},\ 0 < \tau \leq 1/4, \quad \text{or} \\ \text{(ii)} & \tau^{-1} \in \{4\cos^2 \pi/m,\ m = 3,4,5,\ldots\}. \end{array}$$

The proof uses the fact that f_n, inductively defined by

$$f_{n+1} = f_n - \frac{[2]_q[n+1]_q}{[n+2]_q}\, f_n e_{n+1} f_n,$$

must be an orthogonal projection with $e_i f_n = f_n e_i = 0$ for $i \leq n$. These f_n are sometimes called Jones–Wenzl idempotents. (Here $\tau^{-1} = 2+q^2+q^{-2}$ and for this and later formulae we define the quantum integer $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$.)

When $\tau^{-1} = 4\cos^2(\pi/m)$, the Hilbert space representations decompose according to Bratteli diagrams obtained by truncating—eliminating the 1 on the m-th row, and all representations below and to the right of it, so that for $m = 7$ we would obtain



In terms of Young diagrams this corresponds to only taking those those diagrams whose row lengths differ by at most $m-2$. The existence of these Hilbert space representations is from [19].

The Temperley-Lieb algebras arose in [19] as orthogonal projections onto subfactors of II_1 factors. As such the Hilbert space structure was manifest. The trace on a II_1 factor also yielded a trace on the $TL(n, \tau)$.

To be precise, there is for each m a unique linear map $\mathrm{tr} : TL(n, \tau) \to \mathbb{C}$ with:

(i) $\mathrm{tr}(1) = 1$

(ii) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$

(iii) $\mathrm{tr}(xe_{n+1}) = \tau\, \mathrm{tr}(x) \qquad$ for $x \in TL(n+1, \tau)$.

This trace may be calculated either from (i), (ii) and (iii), or using the representations, as a weighted sum of ordinary matrix traces. The weight for the representation of $TL(n, \tau)$, the second row of whose Young diagram has r boxes, is

$$\frac{[n-r+1]_q}{([2]_q)^n}$$

. Thus if $x \in TL(n, \tau)$ and π_r is the $\binom{n}{r} - \binom{n}{r-1}$ dimensional irreducible representation then

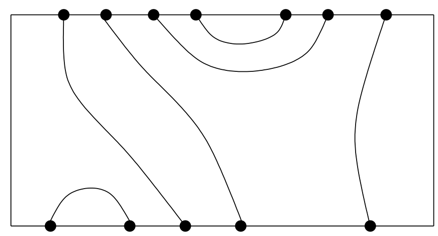
$$\mathrm{tr}(x) = \frac{1}{(q+q^{-1})^n} \sum_{r=0}^{[\frac{n}{2}]} [n-r+1]_q \text{ trace } (\pi_r(x)).$$

One also has $\mathrm{tr}(f_n) = \dfrac{[n+2]_q}{([2]_q)^{n+1}}$ so that the disappearance of the "1" from the Bratteli diagram is mirrored by the vanishing of the trace of the corresponding projection.

Positivity of tr, $\mathrm{tr}(a^*a) \geq 0$, is responsible for all the Hilbert space structure. To explicitly construct the Hilbert space representations one may use the GNS construction: take the quotient of the $*$-algebra by the kernel of the form $\langle a, b\rangle = \mathrm{tr}(b^*a)$ which makes this quotient a Hilbert space on which $TL(n, \tau)$ will act with the e_i's as orthogonal projections. Explicit bases can be obtained easily if desired, using paths on the Bratteli diagram, or Young tableaux.

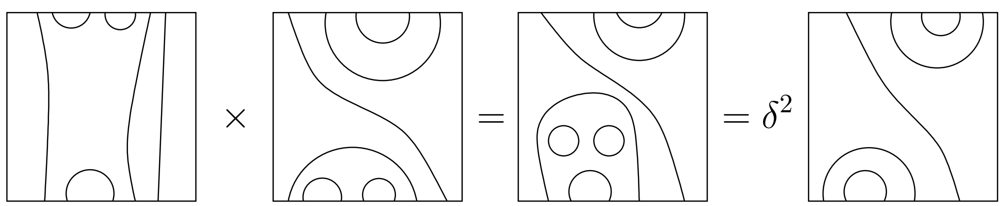
A useful diagrammatic presentation of $TL(n, \tau)$ was discovered in [23]. A (Kauffman) TL diagram (for non-negative integers m and n) is a rectangle with n marked points on the top and m on the bottom with non-intersecting smooth curves inside the rectangle connecting the boundary points as illustrated below.

A (5,7)-diagram

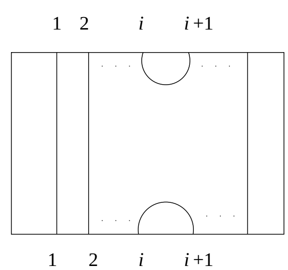


Two Kauffman TL diagrams are considered the same if they connect the same pairs of boundary points.

The vector space $TL(m, n, \delta)$ with basis the set of (m, n) diagrams, and $\delta \in \mathbb{C}$, becomes a category with this concatenation together with the rule that closed curves may be removed, each one counting a (multiplicative) factor of δ. We illustrate their product in $TL(m, n, \delta)$ below:

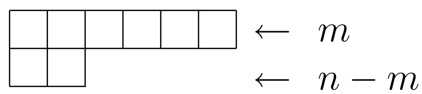


Of special interest is the algebra $TL(n, n, \delta)$. If we define E_i to be the diagram below:



then $E_i^2 = \delta E_i$, $E_i E_{i\pm1} E_i = E_i$ and $E_i E_j = E_j E_i$ for $|i - j| \geq 2$. Thus provided $\delta \neq 0$ we have an isomorphism between $TL(n, \delta^{-2})$ and $TL(n, n, \delta)$ by mapping e_i to $\frac{1}{\delta} E_i$.

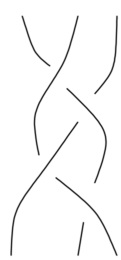
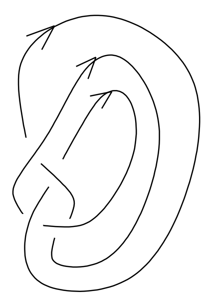
One of the nicest features of the Kauffman diagrams is that they yield simple explicit bases for the irreducible representations. To see this, call a curve in a diagram a "through-string" if it connects the top of the rectangle to the bottom. Then all (m, n) diagrams are filtered by the number of through-strings and if we let $TL(m, n, k, \delta)$ be the span of (m, n) diagrams with at most k through-strings, we have $TL(k,n,\delta)TL(n,m,k,\delta) \subseteq TL(k,m,k,\delta)$. Thus $V_{n,m} = TL(n,m,m,\delta)/TL(n,m,m-1,\delta)$ is a $TL(n,\delta^{-2})$-module, a basis of which is given by (m,n)-diagrams with m through-strings ($m \leq n$). The number of such diagrams is $\binom{n}{m} - \binom{n}{m-1}$ and it follows from [19] that all these representations are irreducible for "generic" δ (i.e. $\delta \notin \{2\cos \mathbb{Q}\pi\}$) and that they may be identified with those indexed by Young diagrams as below:

$V_{n,m} \longleftrightarrow$ 

The invariant inner product on $V_{n,m}$ is defined by $\langle v, w\rangle = w^* v$ for the natural identification of $V_{m,m}$ with $\mathbb{C}$ (* is the obvious involution from (m,n) diagrams to (n,m) diagrams.)

4 The original definition of $V_L(t)$

Given a braid $\beta \in B_n$ one may form an oriented link $\hat{\beta}$ called the closure of β by tying the top of the braid to the bottom as illustrated below:

$\beta =$  $\longrightarrow$ $\hat{\beta} =$ 

All oriented links occur in this way ([5]) but if $\alpha \in B_n$, $\alpha\beta\alpha^{-1}$ and $\beta\sigma_n^{\pm 1}$ (in B_{n+1}) have the same closure.

Theorem 1 (Markov, [5]) *Let $\sim$ be the equivalence relation on $\coprod_{n=1}^{\infty} B_n$ (all braids on any number of strings) generated by the two "moves" $\beta \sim \beta\sigma_n^{\pm 1}$ and $\beta \sim \alpha\beta\alpha^{-1}$. Then $\beta_1 \sim \beta_2$ if and only if the links $\hat{\beta}_1$ and $\hat{\beta}_2$ are the same.*

It is easily checked that, if $1, e_1, e_2, e_3, \ldots$ satisfy the TL relations of §3 then sending σ_i to $(t+1)e_i - 1$ (with $\tau^{-1} = 2 + t + t^{-1}$) defines a representation

ρ_n of B_n inside $TL(n,\tau)$ for each n. The representation is unitary for the C*-algebra structure when $\tau^{-1} = 4\cos^2 \pi/n,\ n = 3,4,5,\ldots$ (and $t = e^{\pm 2\pi i/n}$). It is an open question whether ρ_n is faithful for all n. It contains the Burau representation as a direct summand.

Combining the properties of the trace tr defined on TL with Markov's theorem one obtains immediately that, for $\alpha \in B_n$, the following function of t depends only on $\hat{\alpha}$:

$$\left(-\sqrt{t} - \frac{1}{\sqrt{t}}\right)^{n-1} \sqrt{t}^{\ -e}\, \mathrm{tr}(\rho_n(\alpha))$$

(here $e \in \mathbb{Z}$ is the "exponent sum" of α as a word on $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$).

A simple check using the (oriented) skein theoretic definition of the Jones polynomial shows that this function of t is precisely $V_{\hat{\alpha}}(t)$. This is how $V_L(t)$ was first discovered in [20].

Although less elementary, this approach to $V_L(t)$ does have some advantages. Let us mention a few.

(I) One may use representation theory to do calculations. For instance using the weighted sum of ordinary traces to calculate tr as in §3 one obtains readily the Jones polynomial of a torus knot (i.e. $\hat{\alpha}$ where $\alpha = (\sigma_1\sigma_2\cdots\sigma_{p-1})^q \in B_p$ if p and q are relatively prime). It is

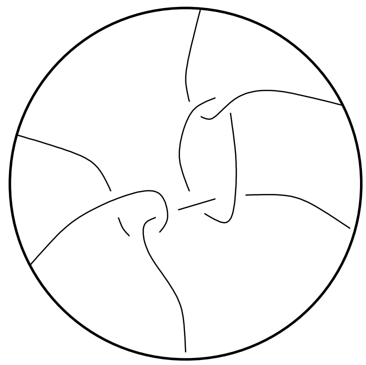
$$\frac{t^{\frac{(p-1)(q-1)}{2}}}{1-t^2}\,(1 - t^{p+1} - t^{q+1} - t^{p+q}).$$

(II) If one restricts attention to links realisable as $\hat{\alpha}$ for $\alpha \in B_n$ *for fixed* n, the computation of $V_{\hat{\alpha}}(t)$ can be performed in polynomial time as a function of the number of crossings in $\hat{\alpha}$. Thus one has computational access to rather complicated families of links.

(III) Unitarity of the representation when $t = e^{\pm\frac{2\pi i}{n}}$ can be used to bound the size of $|V_L(t)|$. For instance if $\alpha \in B_k$ and $V_{\hat{\alpha}}(t) = (-\sqrt{t} - \frac{1}{\sqrt{t}})^{k-1}$ then α is in the kernel of ρ_n, and $|V_{\hat{\beta}}(e^{\pm\frac{2\pi i}{n}})| \leq (2\cos\pi/n)^{k-1}$ for any other $\beta \in B_k$.

The representation of the braid group inside the TL algebra should be thought of as an extension of the Jones polynomial to "special knots with boundary". The coefficients of the words in the e_i's (or equivalently the Kauffman TL diagrams) are all invariants of the braid. We can further remove the braid restriction and consider arbitrary knots and links with boundary, known as "tangles" ([7]).

A 3-tangle



Tangles may be oriented or not and their invariants may be evaluated either by reduction to a system of elementary tangles using skein relations or by organising the tangle and representing it in an algebra. See [42].

A similar algebraic approach is available for the HOMFLYPT and Kauffman two-variable polynomials. The algebra playing the role of the TL-algebra is the Hecke algebra for HOMFLYPT ([16], [21]) and the BMW algebra ([6], [31]) for the Kauffman polynomial. The BMW algebra was discovered after the Kauffman polynomial in order to provide an analogue of the TL and Hecke algebras. For detailed analysis of the Hilbert space and other structures for both Hecke and BMW algebras see [45] and [46].

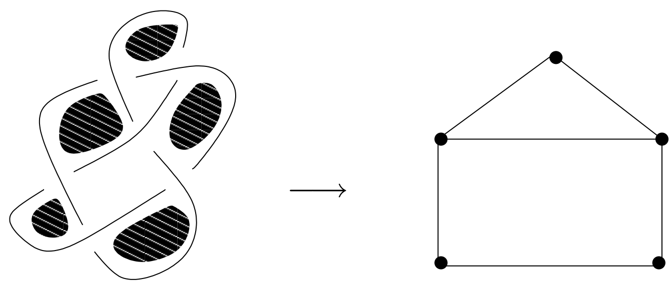
5 Connections with statistical mechanics

One might say that turning a knot into a braid organizes the knot by "putting it on a lattice", thereby creating a physical model with the crossings of the knot as interactions. Taking the trace of the braid is evaluating the partition function with periodic (vertical) boundary conditions.

The previous paragraph is more than wishful thinking. The Temperley-Lieb algebra arose from transfer matrices in both the Potts and ice-type models in two dimensions ([37]) and each "e_i" implements the addition of one more interaction to the system. (The same e_i's as in the ice-type models were rediscovered in the subfactor context in [33].) Thus the Jones polynomial of a closed braid *is* the partition function for a statistical mechanical model on the braid. In [19] it is observed that knowledge of the Jones polynomial for a family of links called French sinnets would constitute a solution of the Potts model in 2 dimensions.

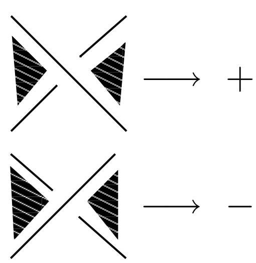
In [37] the TL relations are used to establish the mathematical equivalence of the Potts and ice-type (6-vertex) models. In [3] Chapter 12 this equivalence is shown for Potts models on an arbitrary planar graph. In view of this it is not surprising that statistical mechanical models can be defined directly on link diagrams to give explicit formulae for $V_L(t)$ (and other invariants) as partition functions. This works most easily for the Q-state Potts model.

Given an unoriented link diagram D, shade the regions of the plane black and white and form the planar graph Γ whose vertices are the black regions and whose edges are the crossings as below



D **Γ**

Assign $+$ and $-$ to each edge according to the following scheme:



Fix $Q \in \mathbb{N}$ and 2 symmetric matrices $w_{\pm}(a,b)$ for $1 \leq a,b \leq Q$. The partition function of the diagram is then

$$Z_D = \sum_{\text{states}} \prod_{\text{edges of } \Gamma} w_{\pm}(\sigma, \sigma')$$

where a "state" is a function from the vertices of Γ to $\{1, 2, ..., Q\}$ and, given an edge of Γ and a state, σ and σ' denote the values of the state at the ends of that edge (w_+ and w_- are used according to the sign of the edge).

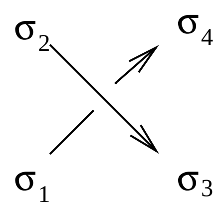
The Potts model is defined by the property that the "Boltzmann weights" $w_\pm(\sigma,\sigma')$ depend only on whether $\sigma=\sigma'$ or not. It is a miracle that the choice (with $Q=2+t+t^{-1}$)

$$w_\pm(\sigma,\sigma') = \begin{cases} t^{\pm 1} & \text{if } \sigma=\sigma' \\ -1 & \text{otherwise} \end{cases}$$

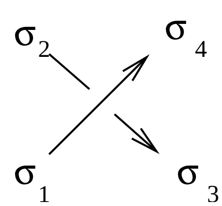
gives the Jones polynomial of the link defined by D as its partition function (up to a simple normalisation). See [22] for details.

It is natural to look for other choices of $w_\pm$ which give knot invariants. The Fateev-Zamolodchikov model ([15]) gives a classical knot invariant but besides that (and some variants on the Jones polynomial) there is only one other known choice of any interest, discovered in [17]. In this case $Q=100$ and the Boltzmann weights are symmetric under the action of the Higman-Sims group on the Higman-Sims graph with 100 vertices. The knot invariant is a special value of the Kauffman two-variable polynomial.

The other side of Temperley Lieb equivalence is the "ice-type" model which is a vertex model. That is to say the "spins" reside on the edges of a graph and the interactions occur at the vertices. To use vertex models in knot theory the knot projection D itself is the (four-valent) graph. The ice-type model has two spin states per edge so that a state of the system is a function from the edges of the graph to the set $\{\pm\}$. And the Bolzmann weights are given by two 4×4 matrices $w_\pm(\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ where the σ's are ± 1 and w_+ and w_- are the contributions of



and



to the partition function respectively. Furthermore we may think of a state as a locally constant function σ on D so for any $f:\{\pm 1\}\to\mathbb{R}$ we may form the term $\int_D f(\sigma)\,\mathrm{d}\theta$ corresponding to interaction with an external field ($\mathrm{d}\theta$ is the curvature or change of angle form on D). Then the partition function is

$$Z_D = \sum_{\text{states}} \left(\prod_{\text{crossings of } D} w_\pm(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \right) e^{\int_D f(\sigma)\,\mathrm{d}\theta}.$$

A (non-physical) specialization of the 6-vertex model yields values of f and $w_\pm$ for which Z_D is a link invariant equal to $V_L(t)$. See [22].

As with the Potts model one may try to generalise to more general $w_\pm$ and f. This is much more successful for these "vertex" models than it was for models like the Potts model. The theory of quantum groups ([18],[9],[35]) allows one to obtain link invariants (as partition functions for vertex models) for each simple finite dimensional Lie algebra $\mathfrak{A}$ and each assignment of an irrreducible representation of $\mathfrak{A}$ to the components of the link. The images of the braid generators σ_i in the corresponding braid group representations are called "R-matrices". It is the Yang-Baxter equation that gives isotopy invariance of the partition function. In this way one obtains (by an infinite family of one-variable specialisations) the HOMFLYPT polynomial (sl_n) and the Kauffman polynomial (orthogonal and symplectic algebras) and more polynomials. The geometric operation of cabling corresponds to the tensor product of representations.

6 Connections with quantum field theory.

Conformal Field Theory. (CFT)

If φ is a (multicomponent) field in one chiral half of a 2-dimensional CFT, the correlation functions

$$\langle \varphi(z_1)\varphi(z_2)...\varphi(z_n)\rangle$$

(where $z_i \in \mathbb{C}$) are expected to be singular if $z_i = z_j$ for some $i \neq j$, holomorphic otherwise and satisfy a linear differential equation. Thus analytic continuation should determine a unitary monodromy representation of $\pi_1(\mathbb{C}^n \setminus \{(z_1, z_2, ..., z_n) | z_i = z_j \text{for some } i \neq j\})$ on the vector space of solutions to the differential equation near a point. In [40] these representations were calculated for the $SU(2)$ WZW model where the differential equation is know as the Khniznik-Zamolodchikov equaiton. The corresponding braid group representations were shown to be those obtained in section 4 and cablings thereof.

Topological quantum field theory. (TQFT)

In [47] the following formula appears:

$$V_L(e^{\frac{2\pi i}{k+2}}) = \int_A exp\{\frac{i}{\hbar}\int_{S^3} tr(A\wedge dA + 2/3 A\wedge A\wedge A)\} \prod_j Tr(Pexp \oint_j A) \quad [\mathfrak{D}A]$$

Where A ranges over all functions from S^3 to the Lie algebra $su(2)$, modulo the action of the gauge group $SU(2)$. Also $\hbar = \pi/k$ and j runs over the components of the link L, to each of which is assigned an irreducible representation of $SU(2)$. Parallel transport around a component j using A yields the linear map $Pexp\oint_i A$ whose trace is constant modulo gauge transformations. And $[\mathfrak{D}A]$ is a fictitious diffeomorphism invariant measure on all A's modulo gauge transformation.

There are at least two ways to interpret this formula.

1) As a solvable TQFT in $2+1$ dimensions, according to [48],[2]. One is then obliged to expand the context and conclude that $V_L(e^{\frac{2\pi i}{n}})$ is defined for (possibly empty) links in an arbitrary 3-manifold. The TQFT axioms then provide an *explicit formula* for the invariant if the 3-manifold is obtained from surgery on a link. in particular the invariant of a 3-manifold without a link is a statistical mechanics type sum over assignments of irreducible representations of $SU(2)$ to the components of the surgery link. The key condition making this sum finite is that only representations up to a certain dimension (determined by n) are allowed. This is the vanishing of the Jones-Wenzl idempotent of section 3. This explicit formula was rigourously shown to be a manifold invariant in [36]. For a more simple treatment see [27] and for the whole TQFT treatment see [14].

2) As a perturbative QFT. The stationary phase Feynmann diagram technique may be applied to obtain the coefficients of the expansion of Witten's formula in powers of $\hbar$ or equivalently $1/n$. These coefficients are known to be "finite type" or Vassiliev invariants and have expressions as integrals over configurations of points on the link-see [43],[26].

Algebraic Quantum Field Theory.

In the Haag-Kastler operator algebraic framework of quantum field theory ([13]), statistics of quantum systems were interpreted in [8] (DHR) in terms of certain representations of the symmetric group corresponding to permuting regions of space-time. To obtain the symmetric group the dimension of space-time needs to be sufficiently large. It was proposed in [11] that the DHR theory should also work in low dimensions with the braid group replacing the symmetric group, and that unitary braid group reprsentations defined above should be the ones occurring in quantum field theory. The "statistical dimension" of DHR theory turns up as the square root of the index of a subfactor (this connection was clearly established in [29]). The mathematical issue of the existence of quantum fields with braid statistics was established in [44] using the language of loop group reprsentations. Actual physical systems with non-abelian braid statistics have not yet been found but have been proposed in [12] as a mechanism for quantum computing.

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