Group actions on Categories & Langlands duality

I. Index formula on moduli of G-bundles and TQFT.

- Revolves around the Verlinde formula
- Will recall its formulation in twisted K theory
- [Generalization from line bundles to "arbitrary" vector bundles]
- For Line bundles, relation between Loop group reps and twisted K theory
- Will recall the construction of (twisted) K theory classes from families of Dirac operators.

The same K classes naturally form a basis of simple objects in a linear category (the fixed-point category for a loop group action on the category of vector spaces)

We'll discuss some puzzling appearance of Langlands dual groups in relation to Lie group actions on categories (and tensor categories)

$G =$ compact Lie group [connected, simply conn.]

$T \subset G$ max torus, W Weyl group $T = L$

$T^\vee =$ dual torus

$k \in \mathbb{Z} \cong H^4(BG; \mathbb{Z})$ "level"

$M_G(\Sigma) =$ moduli space of flat G-bundles on a closed surface Σ

[inherits a complex structure from Σ]

$\mathcal{O}(k) \longrightarrow M_G(\Sigma)$ "level k line bundle"

[always exists on the stack, not always on space]

$= \det^{-k} H^*(\Sigma; \text{standard vector bdle})$ for $G = SU(n)$

Intense activity was generated ~20 yrs ago by a formula proposed by E. Verlinde for

$d = \dim H^0(M_G(\Sigma); \mathcal{O}(k)) \quad (k \geq 0)$

using ideas from Conformal Field Theory

$d = Z_{G,k}(\Sigma)$, "partition function" for Σ in a 2d top. field theory, the Verlinde ring

Digression Recall that 2d tqfts $\leftrightarrow$ commutative Frobenius algebras, algebras A with a trace $\theta : A \to \mathbb{C}$ giving a nondeg. pairing $a \times b \mapsto \theta(ab)$.

If A is semi-simple $\cong \bigoplus \mathbb{C} \cdot P_i$, then $\theta(P_i) := \theta_i \in \mathbb{C}^\times$ and $Z(\Sigma) = \sum \theta_i^{1-g(\Sigma)}$.

In our case, the Verlinde ring is a Frobenius algebra $/\mathbb{Z}$. It is a quotient of the ring of representations R_G by the ideal of characters vanishing at $\left[\begin{array}{l} F = \ker\left(T \xrightarrow{(k+c)} T^\vee\right) \\ \text{the reg. pts of} \end{array}\right.$ the isogeny (k) being defined by the "level"

$$k \in H^4(BG;\mathbb{Z}) \to H^4(BT;\mathbb{Z}) = \mathrm{Sym}^2 \pi_1(T)^\vee.$$

The projectors P_f range over $f \in F^{reg}/W$

The traces $\theta(P_f) = \dfrac{vol(C_f)}{|F|} = \dfrac{\Delta(f)^2}{|F|}$.

Substantial work went into proving various cases of the formula (Tsuchiya et al; Beauville et al; Faltings; ..)

A distinction arose between the space and the moduli stack of all algebraic $G_{\mathbb{C}}$-bundles

Turned out to be immaterial for holomorphic sections (and higher cohomology of line bundles, which vanishes in both cases)

But for more general vector bundles it became clear that Verlinde-style formulae apply to the moduli stack and not the space.

Recall that Narasimhan-Seshadri identify the moduli space of semi-stable algebraic $G_{\mathbb{C}}$ bundles (modulo grade equivalence) with that of flat G-bundles.

So we are missing unstable bundles (and squashing some things).

The stack of all $G_{\mathbb{C}}$-bundles

Has an attractive complex-analytic presentation due to G. Segal.

Choose a disk $\Delta \subset \Sigma$ with smooth parametrized boundary $\partial\Delta$. Then, the stack $\mathcal{M}_G(\Sigma)$ is the double coset $_{\mathrm{Hol}(\Delta; G_{\mathbb{C}})}\backslash LG_{\mathbb{C}} / \mathrm{Hol}(\Sigma \setminus \Delta; G_{\mathbb{C}})$

where the holomorphic maps have smooth bdry values.

The variety $X_{\Sigma\setminus\Delta} := LG_{\mathbb{C}} / \mathrm{Hol}(\Sigma\setminus\Delta; G_{\mathbb{C}})$ is a complex Kähler homogeneous space for $LG_{\mathbb{C}}$, analogous in many ways to flag varieties of cx semisimple Lie groups.

As a symplectic manifold it can be realized as (Flat G-bundles on $\Sigma\setminus\Delta$) / gauge equivalence trivial on $\partial\Delta$.

The LG-action is projective-Hamiltonian (a central extension lifts to the prequantum line bundle)

So holomorphic and cohomological questions on the stack of all $G_{\mathbb{C}}$-bundles on Σ

$\Longleftrightarrow$ $Hol(\Delta; G_{\mathbb{C}})$-equivariant holomorphic and cohomological q.s on $X_{\Sigma \setminus \Delta}$.

Moreover, it seems reasonable to ask $LG_{\mathbb{C}}$-equivariant questions, and indeed those contain the answers we need.

Example: What is $H^0(X_{\Sigma\setminus\Delta}; \mathcal{O}(k))$ as an $LG_{\mathbb{C}}$-representation? Turns out the multiplicity of a certain "vacuum" representation inside is equal to $\dim H^0(M_G(\Sigma); \mathcal{O}(k))$.

* While complex analysis on such manifolds is still out of reach there exists an algebraic model for $X_{\Sigma\setminus\Delta}$ for which we can ask and answer analogous questions.

Example: $H^{>0}(X^{alg}_{\Sigma\setminus\Delta}; \mathcal{O}(k)) = 0$.

("Kodaira vanishing").

Thanks to the work on the Verlinde formula we were in the position to compute $H^0(X^{\mathrm{alg}}_{\Sigma\setminus\Delta}; \mathcal{O}(k))$ and show the vanishing of higher cohomology.

So we had an "analytic index theorem" for these varieties but without a topological side.

[This was paradoxical, usually the topological side is easier]

But this required finding a receptacle for the topological index (Riemann-Roch).

* When G acts on X, RR takes values in $\mathrm{Rep}(G) = K_G(\text{point})$: $K_G(X) \xrightarrow{p_*} K_G(\text{point})$
* Here, $K_{LG}(\text{point})$ is not "topological" and the map p is not proper.

Turns out these problems have a simultaneous solution.

- Instead of $K_{LG}(\text{point})$, consider $K_{LG}(\mathcal{A}_{S^1})$ for the gauge action of LG on the space $\mathcal{A}_{S^1}$ of flat G-connections of the circle
- Instead of projecting $X_{\Sigma\setminus\Delta}$ to a point, use the flat connection model and restrict to the boundary $\partial\Delta$: $X_{\Sigma\setminus\Delta} \longrightarrow \mathcal{A}_{\partial\Delta}$.

This is a proper map!

In fact, the based loop group ΩG (at some point on S^1) acts freely and we can divide out to get

$$\frac{X_{\Sigma\setminus\Delta}}{\Omega G} = G^{2g} \xrightarrow[\text{commutators}]{\text{product of}} G = \frac{\mathcal{A}_{S^1}}{\Omega G}$$

conjugation G-action   conjugation G-action

leading to a well-defined map

$$K_G(G^{2g}) \xrightarrow{p_*} K_G(G)$$

which is our topological Riemann-Roch.

Some amendments

We wanted projective representations of LG with cocycle $k \in H^2_{LG}(\mathcal{O}^\times)$ $\left(\xrightarrow{\sim} H^3_G(G;\mathbb{Z})\right)$

So we should map to the twisted K-group ${}^{k}K_G(G)$. Actually there is an extra shift by c ($=$ dual Coxeter number) ($= n$ for $SU(n)$) coming from the spinors ($\sqrt{K}$) on $X_{\Sigma\setminus\Delta}$, and the key theorem is

Theorem [FHT] ${}^{k+c}K^{\dim G}_G(G)$ = free abelian gp generated by the positive energy irreps of LG at level k.

Theorem [FHT; generalized by Woodward-T]
Analytical index = topological index for $X^{alg}_{\Sigma\setminus\Delta}$ for "essentially all" K-theory classes, at least after inverting $k+c$. [Proof by fixed-point reduction to max torus, no "conceptual" proof].

Rmk H. Posthuma in his thesis has an account of the line bundle case

II. From Loop group representations to K theory

1. Preparation: Compact groups.

Recall the Kirillov correspondence between

irreducible representations

$\updownarrow$

Co-adjoint orbits (+ line bundles)

In the connected case, it can be summarized by saying that both of those correspond to the set of dominant integral regular weights.

Example n-dimensional rep of $SU(2)$

$\longleftrightarrow$ sphere of radius n in $su(2) \cong \mathbb{R}^3$

For $SO(3)$, odd radii

But one can describe a canonical correspondence w/o direct reference to classification of irreps.

Each coadjoint orbit has a symplectic form,

$$\omega_\lambda \left(ad^*_\xi(\lambda), ad^*_\eta(\lambda) \right) = \langle \lambda | [\xi, \eta] \rangle$$

(for which the $\mathfrak{g}$-action is Hamiltonian)

For a vector bundle V on the orbit $\mathcal{O}_\lambda$ of
which is a sum of line bundles w/ curvature ω
and which carries a lifted G-action,*
the Dirac index $\not{D}\text{-Ind}(\mathcal{O}_\lambda; V)$ is a
representation of G, and this establishes
the Kirillov correspondence.

Remarks for connected G, V carries no
information beyond its rank
When $\pi_1 G \neq 0$, the G-action on V should
be projective and cancel the spinor projective cocycle
(hence odd radii for $SO(3)$, not even).

The Dirac family construction (Freed, Hopkins, -)
provides an inverse to this, assigning an
orbit & line bundle w/ G-action to any irrep.
[It also "categorifies" Kirillov's character formula]

Input: • (irreducible) representation V_λ of G

w highest weight λ

• invariant metric on $\mathfrak{g}$

• Spinor space $S(\mathfrak{g})$ based on $\mathfrak{g}$
$\mathbb{C}\times$

[if you prefer: $\mathrm{Cliff}_{\mathbb{C}}(\mathfrak{g})$ as a right $\mathrm{Cliff}_{\mathbb{C}}(\mathfrak{g})$-module]

Recall: $\mathrm{Cliff}_{\mathbb{C}}(\mathfrak{g})$ generated by $\mathfrak{g}$ with relations

$$\xi\cdot\eta + \eta\cdot\xi = 2\langle\xi|\eta\rangle$$

• $\mathbb{Z}/2$ graded, has a filtration with

$$gr \cong \Lambda^{\bullet}\mathfrak{g}$$

• If $\dim\mathfrak{g} = 0 \mod 2$,

$$\mathrm{Cliff}(\mathfrak{g}) \cong \mathrm{End}(\underbrace{S^+ \oplus S^-}_{\text{spinors}})$$

If $\dim\mathfrak{g} = 1 \mod 2$,

$$\mathrm{Cliff}(\mathfrak{g}) \cong \mathrm{End}(S) \otimes \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R})$$

$$\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}) \cong \mathbb{C}_0 \oplus \mathbb{C}_1, \quad e \ \deg 1, \quad e^2 = 1.$$

• Kostant's cubic Dirac operator on G:

$$\not{D} : L^2(G; S^+) \rightleftarrows L^2(G; S^-)$$

$$= R_a \otimes \psi^a - \frac{1}{12} f_{abc}\, \psi^a \psi^b \psi^c$$

R_a = right transl; $\|\psi^a\|^2 = 1$; f_{abc} = structure consts of $\mathfrak{g}$

Remark. We have trivialized the tangent bundle (hence the spinor bundle) of G by left translations.

The right translation action of $\xi_a \in \mathfrak{g}$ on spinor fields would be

$$T_a = R_a + \underbrace{\sigma_a}_{\text{adjoint action on } S(\mathfrak{g})}$$

The Levi-Civita (differential geometer's) Dirac operator would be

$$R_a \otimes \psi^a - \frac{1}{8} f_{abc} \psi^a \psi^b \psi^c$$

So Kostant's operator favours a bit right transl

Can decompose $L^2(G; S^\pm)$ into

$$\bigoplus_\lambda \underset{\text{Left action}}{V_\lambda^*} \otimes \underbrace{V_\lambda \otimes S^\pm}_{\text{Right action of } G}$$

and correspondingly, $\not{D}$ decomposes into operators $\not{D}^\lambda : V_\lambda \otimes S^\pm \longrightarrow V_\lambda \otimes S^\mp$

on finite-dimensional spaces.

Two key properties of $\not{D}$:

- $[\not{D}, \psi(\xi)] \; (= \not{D}\psi(\xi) + \psi(\xi)\not{D}) = 2T(\xi)$

Total right action on spinors

(this is a quantization of Cartan's relation $[d, \iota(\xi)] = \mathcal{L}_\xi$).

- $\not{D}^2 = -(\lambda+\rho)^2$ on $V_\lambda \otimes S^\pm$

The Dirac family

is the $\mathbb{Z}/2$ graded vector bundle with fiber $V \otimes S^\pm$ over $\mathfrak{g}$, and odd operator

$$\not{D}^V_\xi : \not{D}^V + \psi(\xi) \quad \text{at } \xi \in \mathfrak{g}.$$

Theorem [FHT] The kernel of $\not{D}^{V_\lambda}_\xi$ is supported on the orbit of $(\lambda+\rho)$ and equal (Kirillov line bundle) $\otimes$ (Spinors to normal bdle)

In fact, $\not{D}^{V_\lambda}_\xi$ is a model for the Atiyah-Bott-Schapiro K-theory Thom class of (Kirillov orbit, pre-quantum line bundle)

Corollary The Dirac Index (K-integral) of $\not{D}^V_\xi$ over $\mathfrak{g}$ is V again.

Theorem (Kirillov)
The Fourier transform of the $\exp^*$-pullback of the character of V (as a half-density on G) is the associated coadjoint orbit.

The Dirac family is a "categorification" of this

$$\mathrm{Rep}(G) \longrightarrow \underbrace{\begin{array}{l}\mathbb{Z}/2 \text{ graded vectr bundles} \\ \text{on } \mathfrak{g} \text{ with } G\text{-action} \\ \& \text{ endomorphism } \not{\partial}\end{array}}$$

we'll improve this to get an <u>equivalence</u> of cats

The "kernel" for this Fourier transform is $T^*G \cong G \times \mathfrak{g}^*$ projecting to $\mathfrak{g}^*$, with Kostant's cubic Dirac operator along the fibers, coupled to the standard connection "pdq". This gives

$$\bigoplus_V V^* \otimes \left(V \otimes S^\pm, \not{D}^V + \mu(\xi) \right)$$

2. Dirac family for Loop Groups

This implements the isomorphism of abelian gps

$$(*)\quad {}^{k}\mathrm{Rep}(LG) \xrightarrow{\sim} {}^{k+c}K_G(G) \cong {}^{k+c}K_{LG}(\mathcal{A}_{S^1})$$

(and can be strengthened to an equivalence of cats)

The Lie algebra $\widehat{L\mathfrak{g}} := \widetilde{L\mathfrak{g}} \,\tilde{\oplus}\, \mathbb{R}\frac{d}{dt}$

$\uparrow$ ctral extension

has an invariant bilinear form, and we can identify $\mathcal{A}_{S^1}$ with the gauge LG action with the slice at level $(k+c)$ in $(\widehat{L\mathfrak{g}})^*$. There is again a Kirillov correspondence between irreps and integral coadjoint orbits.

Kostant's cubic Dirac operator and the Dirac family over $\mathcal{A}_{S^1}$ can be defined as before and

Theorem [FHT] The assignment

$$H \longmapsto \left(H \otimes S^{\pm}_{L\mathfrak{g}},\ \not{D}^{H} + \psi(\xi)\right) \quad (\xi \in \mathcal{A})$$

$\Omega^1(S^1;\mathfrak{g}) \cong L\mathfrak{g}^* \cong L\mathfrak{g}$

induces the isomorphism $(*)$ of abelian groups

The Dirac family for H is a model of the ABS Thom class for a conjugacy class $C_H \subset G$.

Specifically, with a basis $\xi^a(n)$ of Lg (ranging over Fourier modes and a basis of $\mathfrak{g}$)

$$\not{D}^H = \sum_{n\in\mathbb{Z}} R_a(n) \otimes \psi^a(-n)$$

$$- \frac{1}{12} \sum_{\substack{m,n \\ \in \mathbb{Z}^2}} f_{abc} :\psi^a(m)\psi^b(n)\psi^c(-m-n):$$

normal ordering places negative modes in last position

acts on $H \otimes S^{\pm}$ and satisfies

- $[\not{D}^H, \psi^a(n)] = 2T_a(n)$ ← Lg action on $H \otimes S^{\pm}$

- $(\not{D}^H)^2 = -2(k+c)\frac{d}{dt} - (\lambda+\rho)^2$ (λ: highest weight of H)

- The kernel of $\not{D}^H + \psi(\xi)$ is supported on a single gauge orbit of LG in $\mathcal{A}$, and equals the Kirillov line bundle $\otimes$ normal spinor bundle

- (Moral) The K-integral over $\mathcal{A}$ recovers H.