Gauge Theory, Mirror Symmetry, and Langlands Duality

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Gromov-Witten theory

Among *2D topological quantum field theories* studied in the past decade, *Gromov-Witten theory* has enjoyed enduring interest.

GW assigns to a compact symplectic manifold X a *space of states* $H^*(X)$. Surfaces with points labeled by states give *correlators* (numbers) counting pseudo-holomorphic maps to X with incidence conditions.

Varying the surfaces refines these numbers to cohomology classes on the Deligne-Mumford spaces $\overline{M}_g^n$.
(*Topological: cohomology* of the parameter space replaces the *functions*.)

These invariants could contain enormous information; a structural classification is still missing in general.

Mirror symmetry (Lerche, Vafa, Warner and refined by many others) reduces GW theory to more standard computations in the complex geometry of a conjectural mirror manifold $X^\vee$.

Homological mirror symmetry

was introduced by Kontsevich to spell out the structure of the invariants.

Key idea: include *open* as well as *closed* strings, surfaces with corners, and *boundary conditions* (*branes*) forming a linear category with structure. This category should determine all invariants.

On the symplectic side (X, ω): Fukaya's A_∞ category $\mathcal{F}(X)$;
on the complex side: $D^b Coh(X^\vee)$, with its Yoneda structure, (plus ?).

HMS, the conjectural match of the two categories, is known in many cases: elliptic curves (Polishchuk-Zaslow); $K3$ (Seidel); del Pezzos, weighted projective spaces (Auroux, Katzarkov, Orlov); toric Fanos (FO^3+Abouzaid +others); Calabi-Yau hypersufaces (Sheridan)

The mirror of a toric variety X with torus $T_\mathbb{C}$ is the dual torus $T_\mathbb{C}^\vee$,
plus a *super-potential* Ψ, a Laurent polynomial.
The associated category of Ψ-*Matrix factorizations* is $\mathbb{Z}/2$-graded.

Group actions and Hamiltonian quotients

Many GW computations involve *Hamiltonian quotients* of simpler varieties. Thus, projective toric varieties are quotients of vector spaces by tori.

Example (Givental-Hori-Vafa mirror; simplified)

The best-known case is $\mathbb{P}^{n-1} = \mathbb{C}^n//\mathrm{U}(1)$, with mirror

$$(\mathbb{C}^*)^{n-1} = \{(z_1, \ldots, z_n) | z_1 z_2 \cdots z_n = q\}, \Psi = z_1 + \cdots + z_n$$

For $Y = \mathbb{C}^n$, with standard $(\mathbb{C}^*)^n$ action, declare the mirror to be

$$Y^\vee = (\mathbb{C}^*)^n, \quad \Psi = z_1 + \cdots + z_n.$$

For $K_\mathbb{C} \subset (\mathbb{C}^*)^n$ and $X = \mathbb{C}^n//K$:

$X_\mathbf{q}^\vee =$ fiber over $\mathbf{q} \in K_\mathbb{C}^\vee$ of the dual surjection $(\mathbb{C}^*)^n \twoheadrightarrow K_\mathbb{C}^\vee$,
the super-potential is the restricted Ψ.

The *Novikov variables* $\mathbf{q}$ track degrees of holomorphic curves.

Mirror of a Lie group action. Langlands dual group

Addressing HMS in relation to Hamiltonian quotients raises the following

Basic Questions

1. Find the mirror structure on $X^\vee$ for a Hamiltonian group action on X.
2. In terms of this structure, describe the mirror to the quotient.

Basic Answers (Torus case; 0th order approximation)

1. The mirror to a T-action on X is a holomorphic map $X^\vee \to T^\vee_{\mathbb{C}}$.
2. The mirror of $X//T$ is the fiber of $X^\vee$ over 1.

Basic Answers (Compact, connected G; (-1)st order approximation)

1. The mirror to a G-action on X is a holomorphic map from $X^\vee$ to the space of conjugacy classes in the *Langlands dual group* $G^\vee_{\mathbb{C}}$.
2. Cannot state just yet ... (the fiber over 1 is wrong).

Problem with the answers

They are wrong: pursuing them within HMS leads to paradoxes.

Thus, in the GHV mirror, the original Ψ has no critical points on Y, so its matrix factorization category is zero.

We can't get the Fukaya category of a toric variety from the zero category; and indeed, GHV tell us to *first* restrict Ψ to the fiber of $(\mathbb{C}^*)^n \to K_{\mathbb{C}}^{\vee}$ *and then* compute MF.

However, this operation is not defined in terms of categories.
So we have just destroyed the *raison d'être* of HMS.

The right answers can be found using arguments from 4D QFT; the tour covers some beautiful geometry. (I learnt these ideas from Ed Witten.)

This beautiful story is a fairy-tale for two reasons:

1. it is not rigorous, 2. history did not happen this way.

(But it is more entertaining than the actual answer.)

SU(2) magnetic monopoles

are solutions of the **Bogomolny equation** $\boxed{F = *D\phi}$ on $\mathbb{R}^3$.
$F = dA + A \wedge A$ is the curvature of an $\mathfrak{su}(2)$-connection $D = d + A$, and the *Higgs field* ϕ is valued in the ad-bundle.

They correspond to time-invariant ASD connections $D + \phi\, dt$ on $\mathbb{R}^4$.
Finiteness of the energy breaks the symmetry to U(1) on the sphere at ∞, leading to a discrete invariant, the monopole charge $n \geq 0$.

The moduli spaces are hyper-Kähler manifolds; studied by Atiyah, Donaldson, Hitchin, Hurtubise, Manton, Nahm, Taubes ... .
The charge n moduli space was described in many ways; among them,

1. A specific Zariski-open subset of the nth Hilbert scheme of $\mathbb{C}^* \times \mathbb{C}$.
   This is a resolution of singularities of $T^*(\mathbb{C}^*)^n/S_n$,
   and is naturally associated to the group $\mathrm{U}(n)$.
2. The space of solutions of *Nahm's equations* $\frac{dT_i}{ds} + \varepsilon_i^{jk} T_j T_k = 0$,
   $T_i(s) \in \mathfrak{u}(n)$, simple poles with $\mathrm{Res}_{s=0,2} T_i$ giving the irrep of SU(2).

3D reduction of Yang-Mills theory

Seiberg and Witten studied the $3D$ reduction of $4D$ quantum Yang-Mills theory for SU(2) (a cousin of the Donaldson invariants).

They described the low-energy limit as a Σ-*model* in the space of vacua, which they identified with the Atiyah-Hitchin moduli space of charge 2 monopoles.

They identified the resulting 3-dimensional SU(2) gauge theory with the *Rozansky-Witten* theory of the hyper-Kähler Atiyah-Hitchin manifold.

Later, Argyres-Farragi described SU(n) gauge theory in terms of the charge n monopole space.
Martinec-Warner related a general G to the *periodic Toda system*, revealing a first connection to the Langlands dual Lie algebra $\mathfrak{g}^\vee$.

But what does this have to do with gauged GW theory?

Equivalent field theories admit the same branes

Branes for the $3D$ pure gauge theory are general 2D gauged TQFTs.
Mathematical description: categories with locally trivial G-action.

Branes for RW theory were recently described by Kapustin and Rozansky.

Their 2-category is contains smooth holomorphic Lagrangians L.
Locally, the category $End(L)$ is the tensor category $D^b Coh(L)$.
An L' near L is the graph of a $d\Psi$; $Hom(L, L') = \mathrm{MF}(L, \Psi)$, lives on $L \cap L'$.
(The global description is deformed by the ambient symplectic manifold.)

Better: localized branes at L are $\mathcal{O}$-linear categories on L.
Kapustin and Rozansky assert that these local descriptions patch together.

Example (Cotangent bundle T^*L)

The matrix factorizations for $\Psi \in \mathcal{O}(L)$ constitute $Hom(L; \Gamma(d\Psi))$.
This *micro-localization* of the MF category circumvents the paradox of a zero category having a non-zero restriction.

Key features of an A-model group action

Theorem (Connected groups)

1. *A Hamiltonian G-action on (X, ω) induces a locally trivial action of G on the Fukaya category $\mathcal{F}(X)$.*
2. *This is described (up to homotopy) by a morphism of E_2-algebras*
$$C_*(\Omega G) \to HCH^*(\mathcal{F}(X));$$
or, a module category structure of $\mathcal{F}(X)$ over $(C_*(\Omega G)$-modules$, \otimes)$.
3. *The invariant part $\mathcal{F}(X)^G$ is the fiber over $0 \in Spec\, C_*(\Omega G)$ of $\mathcal{F}(X)$.*
4. *The latter "gauged category" should be equivalent to $\mathcal{F}(X//G)$.*

Remark

$H_*(\Omega G; \mathbb{C})$ is a (Laurent) polynomial ring, and is truly commutative (E_∞).
The same is true of $HH^*(\mathcal{F}(X))$ when $\cong H^*(X)$.
But an E_2 morphism between commutative algebras has *more information* than the underlying morphism of algebras.

The monopole and Rozansky-Witten connection

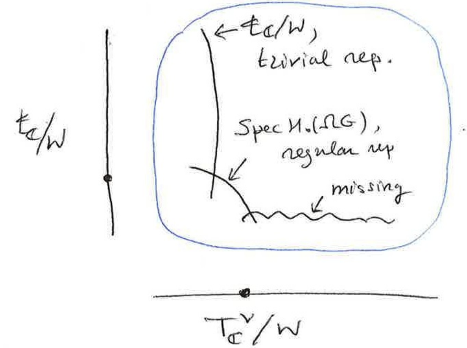
Theorem (Bezrukavnikov-Finkelberg-Mirkovic)

1. *Spec* $H_*^G(\Omega G)$ *is an affine resolution of singularities of* $(T^*T_{\mathbb{C}}^\vee)/W$.
2. *Spec* $H(\Omega G) \subset$ *Spec* $H_*^G(\Omega G)$ *is the fiber over* $Z(G^\vee) \subset (T^*T_{\mathbb{C}}^\vee)/W$.
3. *Spec* $H_*^G(\Omega G)$ *is algebraic symplectic, and Spec* $H_*(\Omega G)$ *Lagrangian. Completed there,* $H^G(\Omega G) =$ *E_2 Hochschild cohomology of* $H_*(\Omega G)$ *(a.k.a. the cotangent bundle.)*

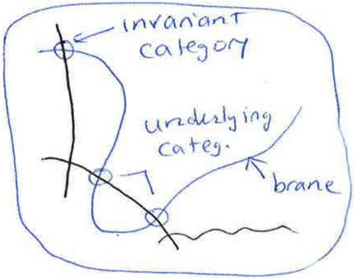
Remark

1. This $E_2 HH^*$ controls the formal E_2 deformations of $H_*(\Omega G)$. Algebras with E_2-action of $H_*(\Omega G)$ micro-localize to Spec$(E_2 HH^*)$, defining germs of branes for the Rozansky-Witten theory. The above BFM space provides a natural *uncompletion*.
2. The BFM space for SU(n) is the SU(2)−monopole space of charge n. General: solns. to *Nahm's equations* in $\mathfrak{g}^\vee$ with principal $\mathfrak{sl}(2)$ poles.

Pictures instead of thousands of words



The BFM space with the trivial and the regular representations



Invariant category and underlying category

Decategorification: the space of states

A general construction (Costello, Kontsevich-Soibelman, Hopkins-Lurie) yields a 2D partial TQFT from a category $\mathcal{F}$ with a perfect cyclic trace.
The input states are in $HH_*(\mathcal{F})$, outputs in $HH^*(\mathcal{F}) \cong HH_*(\mathcal{F})^\vee$.
If lucky: $HH_* = HH^*$ and we get a full TQFT.

Conjecture

For compact symplectic X, $HH_*(\mathcal{F}) \cong HH^*(\mathcal{F}) \cong H^*(X)$.

This may well fail, but Fukaya has indicated a way around it: the TQFT maps factor through $H^*(X)$ anyway.

A G-Hamiltonian X gives GW invariants on $H^*_G(X)$, but the $GW(X/\!/G)$ maps factor through a smaller space. Woodward defined an A_∞ *quantum Kirwan map* $H^*_G(X) \to QH^*(X/\!/G)$, expected onto for projective X and orbifold $X/\!/G$. This fails in general, e.g.

1. $X = *$, space is $H^*_G(G)$, we get string topology of BG;
2. $X = T^*(G/F)$, F finite: the space is $\mathbb{C}[F]^F$ (twisted sectors).

Several TQFT's are read off the brane $\mathcal{B}$ associated to a G-Hamiltonian X. Call $\mathcal{E} = Spec\, H_*\Omega G$ and $\mathcal{T}$ the cotangent fiber in BFM.

1. The GW theory of X, defined from $\mathcal{F}(X) = Hom_{BFM}(\mathcal{E}, \mathcal{B})$. The space of input states is $HH_*(\mathcal{F})$ (factors through $H^*(X)$).
2. Givental's equivariant GW theory, with ground ring $H^*(BG) = \mathbb{C}[\mathfrak{t}]^W$. Defining category: $\Gamma(\mathcal{B})$ as an $H^*(BG)$-linear category, space of states: $HH_*^{H^*(BG)}\Gamma(\mathcal{B}) \cong HH_*^G(\mathcal{F})$.
3. The gauged GW theory, generated by $\mathcal{F}(X)^G \cong Hom_{BFM}(\mathcal{T}, \mathcal{B})$, and conjectured to be $\mathcal{F}(X/\!/G)$ (if $X/\!/G$ orbifold).

There is a G-equivariant local system $\mathcal{HH}_*$ over G with fiber $HH_*(\mathcal{F}(X))$, whose $R\Gamma$ is $HH_*(\mathcal{F}(X)^G)$.
Heuristically, LG acts on $H^{\infty/2}(LX) \cong HH^*\mathcal{F}(X)$, and $BLG = G/G$.

Conjecture

If X is projective and $X/\!/G$ an orbifold, Woodward's quantum Kirwan map is $HH_*^G(\mathcal{F}(X)) \to R\Gamma^G(G; \mathcal{HH}_*\mathcal{F}(X))$, defined by $1 \in G$.

Theorem (Sort of; description of gauge theory via BFM space)

The BFM un-completion "governs A-models gauged by G".
Specifically, a G-action on a Fukaya category gives the germ of a brane in the RW theory of Spec $H_*^G(\Omega G)$, near the Lagrangian $H_*(\Omega G)$.
Gauging the theory requires extending this brane to the ambient space.

Remark

1. This theorem is partially a definition. We are refining the notion of a locally trivial G-action on a linear category enough to specify the gauged theory (the fixed-point category).
2. This un-completion strictifies a homotopy G-action to a "genuine" G-action, and is analogous to passing from $K(BG)$ to $K_G(*)$.
3. The 2-category of linear categories with locally trivial G-action has a forgetful *underlying category* functor.
   Unlike the case of G-action on vector spaces, this is not faithful.
   So the description as a (locally trivial, up to coherent homotopy) group action on a category was only a starting point.

Underlying category and invariant category

In the RW model, we must describe geometrically two functors from the (2-)category of linear categories with G-action to linear categories:

1. The forgetful functor, remembering the underlying category; this describes the original, pre-gauged TQFT.
2. The invariant category; this generates the gauged TQFT.

They are co-represented by the *regular*, resp. *trivial* representations of G, among categories with locally trivial action.

Theorem (Sort of)

1. *The regular representation is the Lagrangian* $\operatorname{Spec} H_*(\Omega G)$*.*
2. *The trivial representation is the Lagrangian* $\mathfrak{t}_{\mathbb{C}}/W = T_1^* T_{\mathbb{C}}^{\vee}/W$*.*

(I mean the categories of coherent sheaves over these Lagrangians.)

1 is clear: it describes $H_*(\Omega G)$-modules as a module category over itself.

2 is a key part to the BFM description of gauge theory.

Deformations by $H^*(BG)$ and the bulk of BFM space

The Fukaya category $\mathcal{F}(X)$ carries deformations parametrized by $H^*(X)$. The gauged category $\mathcal{F}(X)^G$ should carry deformations parametrized by $H^*_G(X)$, in particular, by $H^*(BG)$.

In fact, these deformations can be explained intrinsically:

1. As TQFT deformations: an $\alpha \in H^*(BG)$ transgresses to a $t(\alpha)$ on the moduli of G-bundles over a surface. The TQFT correlator deforms by twisting the path integrand by $\exp t(\alpha)$.
2. As deformations of the G-action on $\mathcal{F}(X)$: $H^*(BG)$ parametrizes $\mathbb{Z}/2$-graded deformations of the locally trivial G-action on **Vect**. The deformed $\mathcal{F}(X)^G$ is the invariant part of the twisted category.

This deformation has a clean geometric interpretation in the BFM space: projection to $\mathfrak{t}_{\mathbb{C}}/W$ turns α into a Hamiltonian, and we use its flow.

The twisted representation $\mathbf{Vect}_\alpha$ is the flow of the identity fiber $\mathfrak{t}_{\mathbb{C}}/W$. This allows one to access any part of brane in the BFM bulk.