Go and No-Go results in Chern-Simons theory

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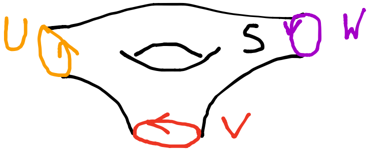
and work in progress with Dan Freed, some with Claudia Scheimbauer.

It concerns TQFT, an algebraic structure whose complexity increases with dimension: operations ⇔ gluing of compact manifolds with corners.

Surprisingly, Lurie's *Cobordism Hypothesis* showed that *fully extended* TQFTs, with corners of all codimensions, have a simpler structure, in any dimension, than the partially extended ones considered earlier.

Our theme is exploiting the notion of **boundary conditions** in TQFT.

Boundary conditions are properly called **boundary theories**, as they include *data*. They are controlled by manifolds with corners and with a codimension-one face colored. One can also have multiple boundary conditions.



Examples

Topological boundary conditions for a TQFT $\leftrightarrow$ modules for an algebra.

In $2D$ this is exactly right. One can show that TQFTs (over the ground field $\mathbb{C}$) correspond precisely to finite semi-simple algebras, and boundary conditions to their finite modules. (Disclaimer: some assumptions needed)

One point of view is that to the point one associates the *category of boundary conditions*, the modules over the algebra. But we will see that this is not quite right.

In $3D$, a large class of theories can be generated by *fusion categories* — finite semi-simple linear categories with multiplication (tensor product) and internal duals. These are the *Turaev-Viro* theories.

For example, the "Pontryagin category" of vector bundles over a finite group, with convolution product, constructs the finite gauge theory in $3D$. The same TQFT arises from the category of representations of the group.

Non-Example

Not all interesting $3D$ theories arise in this way. An early example of Witten's, the *Chern-Simons* theory for a compact group, has resisted such a construction.

In the early 90's, Reshetikhin and Turaev described a more complex combinatorial construction of this theory from a *modular tensor category*.

Briefly, this object is what a Turaev-Viro theory would attach to a *circle*. If a fusion category F is attached to a point, the general formalism of TQFT says that the circle computes the *Drinfeld center* $Z(F)$, which has additional structure. (Braiding and ribbon.)

Reshetikhin and Turaev constructed the TQFT by working up from the circle; point corners are (usually) not allowed in that theory.

A related problem was the suspected *absence of topological boundary conditions* for Chern-Simons theory. (Kapustin et al).

Non-Example, continued

However, a *conformal* boundary theory for Chern-Simons was known from the start, the $2D$ Wess-Zumino-Witten model. In fact, CS can be constructed from that, following Witten's original ideas.

In a crude simplification, quantum field theories can be *gapped* or *gapless*, referring to the spectrum of the Hamiltonian in the macroscopic limit. Gapped theories (may) have a lowest-energy topological sector, gapless ones a *conformal* sector.
Gapless theories correspond to conduction, gapped ones to insulation.

This also applies to boundary theories: if the bulk theory is gapped, the boundary theory may or may not be so. Examples are known where insulators are forced to admit conduction on the boundary.
In our language, assuming this simplified translation is correct, these are topological theories which admit no topological boundary theories.

Reconstruction failure

We see here in $3D$ a new pheonomenon, the (possible) failure to reconstruct certain TQFTS from their (2-)categories of boundary conditions.

Recall that in $2D$, we can define the TQFT from a semi simple algebra A, but what we really see is the category of modules, the boundary conditions.

In fact, the algebra A does not appear unless we single out an object in the category, namely the regular module. This is a *generating* boundary condition; the endomorphism algebra $\mathrm{End}_A(A)$ is A itself.
But what really matters is *Morita equivalence* not isomorphism: if we chose instead the module $A^{\oplus n}$, we would get a matrix algebra $M_n(A)$, which generates an equivalent TQFT with an equivalent category of boundary conditions.

However the Chern-Simons example suggests that the $3D$ analogue fails, and one of our results (the No-Go theorem) confirms that this is the case.

Extending Reshetikhin-Turaev theorie to points

Much attention was given to the problem of ‘extending Chern-Simons theory down to a point’ (vid. Douglas, Henriques et al, Conformal Nets).

The original hope was that the simplicity of the Cobordism Hypothesis would clarify the constructions. But this has been thwarted; and in fact the question was not well-posed.

Whether a TQFT defined in dimensions $1, 2, 3$ can be reconstructed from an object assigned to a point depends on the choice of the target (symmetric monoidal) 3-category. It usually has a negative answer in $\mathcal{FC}$, the fusion categories (Müger et al).

The good question is whether there exist a target in which this can be accomplished, and perhaps a universal one.
The Go part of our result affirms this. Moreover, the universal answer is formally determined from the world of braided fusion categories.
“Nothing is learned by extending RT theories down to points.”

Key Results I: No-Go

1. A modular tensor category $\mathcal{A}$ is usually not the center of a fusion category; the obstruction is detected by its class in the *Witt group* (defined by Ostrik, Müger et al.). Examples from finite abelian groups with quadratic forms show that this contains the Witt group of $\mathbb{Q}$.
2. Nonetheless, Reshetikhin-Turaev associates to them a TQFT $T_{\mathcal{A}}$ in dimensions $1, 2, 3$. This may/not extend down to dimension 0 in $\mathcal{FC}$.
3. No-Go theorem: unless $\mathcal{A}$ is a Drinfeld center to begin with, so that $T_{\mathcal{A}}$ is secretly a TV theory, no extension of the theory down to points (in a target that contains fusion categories, and with some other qualifications) admits topological boundary conditions.
4. More precisely: any non-zero topological boundary condition must be *generating*: the endomorphism category of the boundary condition is a fusion category whose Drinfeld center is $\mathcal{A}$, and constructs $T_{\mathcal{A}}$.

Key Results II: Go

1. Given the modular tensor category $\mathcal{A}$, we can define a *symmetric monoidal category with duals* $\mathcal{FC}[X, X^\vee]$ with relation $X \otimes X^\vee \equiv \mathcal{A}$.
2. "The square norm of a RT theory is the TV theory on its center"
3. This satisfies $Z(X) = \mathcal{A}$. $(Z(X) = \mathrm{End}(\mathrm{Id}_X).)$
4. Setting $\mathcal{T}(pt) = X$ defines a fully extended TQFT which, in dimensions $1, 2, 3$ recovers the theory $\mathcal{T}$ defined from $\mathcal{A}$.
5. There is a *universal* symmetric monoidal 3-category $\mathcal{UFC}$ with full duals, containing $\mathcal{FC}$ as a full subcategory, in which every RT theory becomes a TV theory as above.
6. For the 'roots' X, Y of $\mathcal{A}, \mathcal{B}$ in $\mathcal{UFC}$, $\mathrm{Hom}(X, Y) = 0$ unless $\mathcal{A}^{rev} \otimes \mathcal{B}$ is the Drinfeld center of a fusion category $\mathcal{F}$.
   In the latter case, $\mathrm{Hom}(X, Y)$ is the 2-category of $\mathcal{F}$-modules.
7. An analogue holds for super categories. The group of units in super-$\mathcal{UFC}$ is $\mathbb{Z}/24$. (if we get our signs to line up ...)

Ingredients of the proof

Basics on Fusion categories (Etingof, Gelaki, Nykshik, Ostrik, Müger)

1. Complete reducibility: every fusion category is a direct sum of *simple* ones, each of which is Morita equivalent to one with simple unit
2. The center of such a simple fusion category F is *modular*.
3. Double centralizer theorem: a non-zero semi-simple module category over F gives a Morita equivalence with its F-endomorphism category

Basic results from TQFT

1. Fusion categories form a symmetric monoidal 3-category with duals (Douglas, Schommer-Pries, Snyder)
2. A modular tensor category $\mathcal{A}$ defines an invertible oriented $4D$ TQFT.
3. Fusion categories with a central action $\mathcal{A}$ define boundary conditions for that (Brochier, Jordan, Snyder)
4. The boundary theory defined by $\mathcal{A}$ as a module over itself "is" the RT theory $T_{\mathcal{A}}$ (K. Walker).

Flavor of the argument

Our $\mathcal{FC}_{\mathcal{A}} := \mathcal{FC}[X, X^\vee]/(XX^\vee = \mathcal{A})$ is a kind of *amalgamated pushout* from the 3D bordism category of framed manifolds and $\mathcal{FC}$. The universal $\mathcal{UFC}$ uses the bordism category *with singularities* ("domain walls").
We define $\mathcal{FC}_{\mathcal{A}}$ first as a linear 3-category, then add the monoidal structure, associator and symmetry. First,

$$\mathcal{FC}_{\mathcal{A}} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}^{\otimes n} - \mathbf{Mod}$$

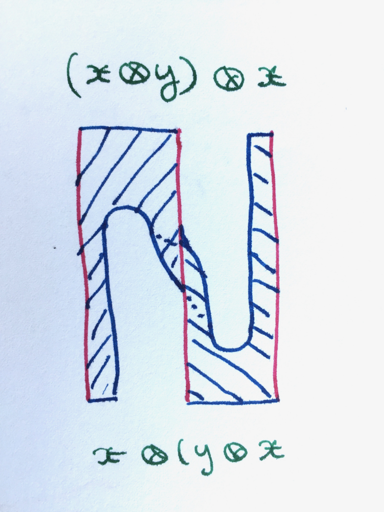
the sum of 3-categories of fusion categories with central $\mathcal{A}^{\otimes n}$-action. The objects $X, X^\vee$ correspond to the summands $n = \pm 1$.

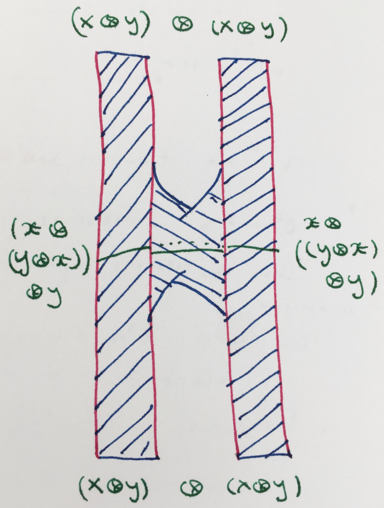
The monoidal structure is obvious on the parts $n \geq 0$ and $n \leq 0$ separately. To complete it we add an isomorphism $XX^\vee = \mathcal{A}$ by tensoring over $\mathcal{A}$.

In Walker's construction of $T_{\mathcal{A}}$ from $\mathcal{A}$ and its regular module, $T_{\mathcal{A}}$ becomes a boundary theory, so all our pictures must be faces of a bulking manifold. For instance, the point X is represented by an interval with one end marked by the regular boundary condition.

Associativity pictures

In the standalone theory, multiplication is disjoint union; associativity is strict, the associator on $X \otimes X^\vee \otimes X$ is a triple of straight identity lines. But now we need a bulking 2-manifold; an option for an associator is shown here (read $X^\vee$ for y):





Working out the pentagon identity, we see that fails on the bulking surface, as nicely as it holds on the red boundary. Yet we wanted an associative structure!

The solution to this quandary is that the bulk theory is *invertible*. So we can erase the bulking manifold, up to a *phase*, a central unit in the $3D$ theory.

Working out the phase is a topology calculation; it is related to the central charge. On general grounds, it is coupled to a tangential characteristic class which must be trivialized. This can be done on oriented 3-manifolds with p_1-structure, in 6 different ways it turns out.

Concluding Speculation

The $\mathbb{Z}/24$ that we (expect to) find is the dual of π_3^s.
This is the next step in a bold conjecture of Michael Hopkins about a universal linear higher categorical target for TQFTs.

An n-dimensional TQFT takes values in a symmetric monoidal n-category.
Known sequence: $\mathbb{C}$ (or the cyclotomic subfield) for $n = 0$, then vector space for $n = 1$, semi-simple algebras for $n = 2$, fusion categories $n = 3$.
We could continue categorizing by forming the category of modules over the previously defined structures and seek finite objects inside.
But, known examples show that we must add super-vector spaces (with their Koszul symmetric tensor structure) and therefore super-algebras . . .

Looking at the groups of units, we obtain the sequence $\mu_\infty, \mathbb{Z}/2, \mathbb{Z}/2, ..?$
Hopkins recognized here the duals of the stable stems, and conjectured that the universal answer should have as spectrum of units the Pontryagin dual of the sphere spectrum. (Anderson dual.) Hence $\mathbb{Z}/24$ next.

THANK YOU FOR LISTENING