Gauge Theory & Mirror Symmetry in 3D

I. Mirror Symmetry in 2D: Conjectural equivalence between A/B models in a pair of "mirror manifolds"

In 3D: ???, but seems specific to gauge theory
2D mirror symmetry "takes place on the 2D boundary"

Today: Update on 3D Mirror Symmetry program in TQFT
[Refs: • Perspectives in Geometry Lectures, Austin 2011
• ICM Lecture, 2014]

Antecedents:

- Full Locality in TQFT ("Cobordism Hypothesis", Lurie)
- Topological Verlinde ring (${}^{\tau}K_G(G)$, Freed, Hopkins, –)
- Proposal for ∂ conditions (Kapustin, Rozansky, Saulina)
- Deformation theory for E_n algebras (Francis, Lurie)
- Much input from physics. [Seiberg, Witten '93]

Full locality in TQFT
(Moore – Segal; Kontsevich; Costello; Hopkins-Lurie; Lurie)

A fully local* TQFT Z is determined by the object $Z(+)$ attached to a point, in a symmetric monoidal D-category.
Fully local boundary conditions are objects in $\mathrm{Hom}(\mathbb{1}, Z(+))$.
[Must satisfy some strong finiteness conditions.]

* These are theories for framed manifolds.
Tangential structures (SO, Spin, etc) require extra data.

II. Example (Moore–Segal; Finite 2D gauge theory)

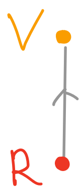
$Z(+) = \mathbb{C}\langle G\rangle$ group ring, in (Alg, Bimods, Intertwiners)

Boundary conditions $= \mathbb{C}\langle G\rangle$-mods $=$ G-reps

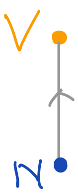
Have: two distinguished boundary conditions:

Dirichlet, R, regular representation

Neumann N, trivial representation

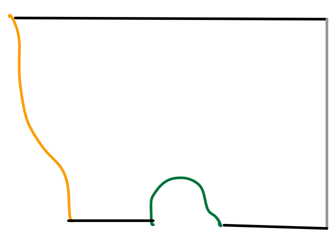


$= \mathrm{Hom}^G(R, V) =$ underlying vector space



$= \mathrm{Hom}^G(\mathbb{1}, V) =$ $=$ invariants V^G

Calculus of representations — captured by TQFT pictures

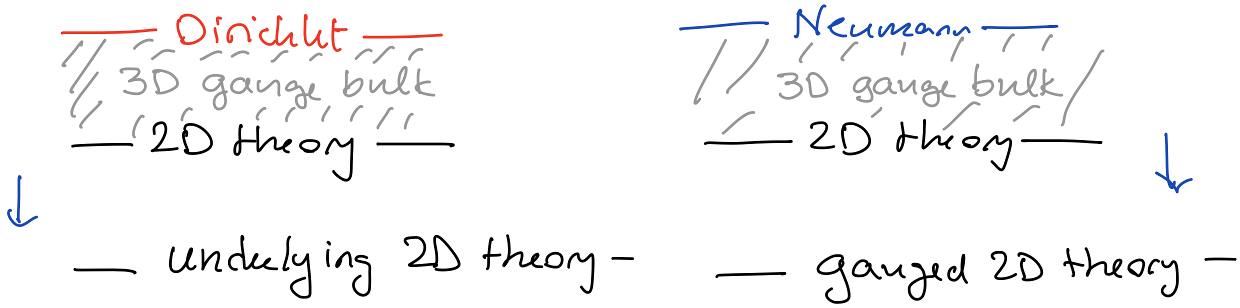


$$\begin{array}{c} V^* \\ \uparrow \\ \mathrm{Hom}(V,W) \otimes W^* \end{array}$$

III. 3D gauge theory: Representations of G on categories

= Boundary conditions for pure 3D gauge theory

These are the 2D theories with G-gauge symmetry



Several types of representations for Lie groups!

<u>In 2D</u>: B-type = Linear representations

A-type = "topological representations"

[Need a derived context (complexes) to see topology of G]

<u>In 3D</u>

A-type	A/B type	B-type
G acts on a topol. space X $\Downarrow$ acts on $\mathrm{DLoc}(X)$ = derived local systs. gauged "string topology" of X (Chas-Sullivan)	Chern-Simons theory conformal ∂ conds, but usually, no topological ones ("no fin. dim reps") [Kapustin-Saulina] chiral factorization of WZW model	G acts on an algebr. variety X $\Downarrow$ action on $(\mathrm{D})\mathrm{Coh}(X)$ gauged 2D B-model

$\to$ Focus: Hamiltonian G-actions on symplectic manifolds.

"gauged A-models"

IV The MetaTheorem

* 3D $N=4$, topologically twisted SUSY gauge theory for G
  $\Longleftrightarrow$ Rozansky-Witten theory of the Toda space for $G^\vee$.

* The character calculus is captured by holomorphic Lagrangian geometry of the Toda space $\mathcal{J}(G^\vee)$
  Eg: Characters of 2D TQFTs with G action
  = coherent sheaves with Lagrangian support

* Underlying TQFTs and Gauged TQFTs are computed as intersections with a Dirichlet/Neumann Lagrangian.

* Toric case: Givental-Hori-Vafa recipe

Toda space examples

* $\mathcal{J}(T^\vee) = T^* T_{\mathbb{C}}^\vee \longrightarrow \mathfrak{t}_{\mathbb{C}} = T_1^* T_{\mathbb{C}}^\vee$ Holomorphic Integrable System

* $\mathcal{J}(G^\vee)$ = affine resolution of singularities of $T^* T_{\mathbb{C}}^\vee / W$

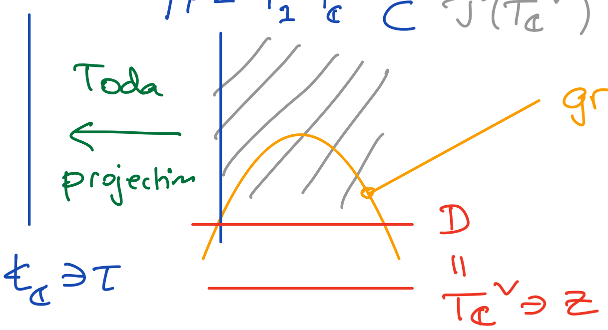
  $\cong H_*^G(\Omega G)$ (Bezrukavnikov, Finkelberg, Mirković)

  Projects to $\mathfrak{g}_{\mathbb{C}} /_{Ad} G_{\mathbb{C}} \cong \mathfrak{t}_{\mathbb{C}}/W = \operatorname{Spec} H^*(BG)$

  Unit section = Neumann condition
  Fiber over $0 \in \mathfrak{t}_{\mathbb{C}}$ = Dirichlet condition

$G = U(1)$

$N = T_1^* T_{\mathbb{C}}^\vee \subset \mathcal{J}(T_{\mathbb{C}}^\vee) = T^* T_{\mathbb{C}}^\vee$



$\tau = q(z - z^{-1}) = $ "dW"

$W = q(z + z^{-1})$, superpotential for mirror of $\mathbb{P}^1$

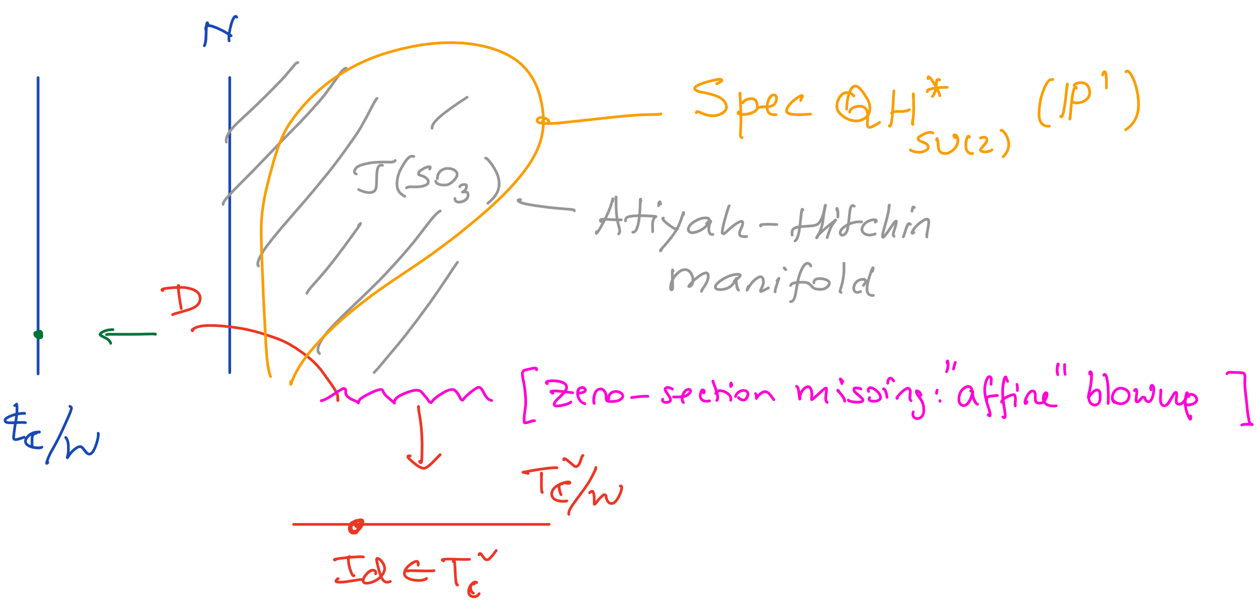
$\mathbb{C}[\text{graph}] = QH^*_{U(1)}(\mathbb{P}^1) = \mathbb{C}[\omega_0, \omega_\infty]/(\omega_0 \omega_\infty = q^2)$

$(\tau = \omega_0 - \omega_\infty \qquad z = q^{-1}\omega_0 = -q\omega_\infty^{-1})$

Intersection with D : $QH^*(\mathbb{P}^1)$, $\mathbb{Q}[\omega_0]/\omega_0^2 = q$

Intersection with N : $\mathbb{Q} = QH^*(\mathbb{P}^1 /\!\!/ U(1))$

$G = SU(2)$



V. applications (old)

(1) $J(G^\vee)$ has a holomorphic Lagrangian foliation by the mirrors of the Flag Varieties of G, (with their small quantum parameters)

Givental–Kim: projections to $\mathfrak{t}/W$ and QH^*

Rietsch: construction of B-model mirrors

(—) : (Knörrer) Equivalence of $\mathcal{J}$ with Toda foliation

(2) 2D TQFTs with G gauge symmetry $\rightsquigarrow$ objects in $DCoh\, J(G^\vee)$ w/ Lagrangian support

- Intersection with D: state space of original TQFT
- Intersection with N: —"—"— gauged TQFT
- Intersection with leaves: "reduction at flag varieties"

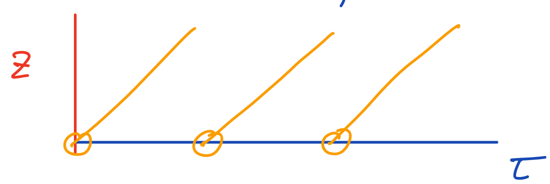
Geometric interpretation needs caution; see "Quantum GIT conjecture" later.

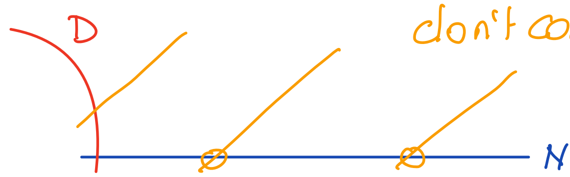
(3) 2D Topological YM theory with parameter $h \in H^4(BG)$
(Witten integrals over moduli of flat G-bundles)

$\longleftrightarrow$ graph of $\exp(dW)$, $W(\tau) = \frac{1}{2} h \tau^2$

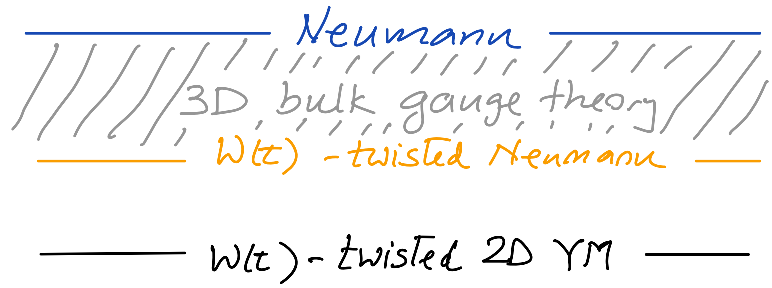
abelian case, $z = e^{\tau h}$

Nonabelian : singular weights don't contribute





Dimensional reduction picture $\downarrow$



(4) Variant : Can add higher terms to $W(\tau)$ to get all Witten integrals

(5) Variant : (w. Woodward) K-theory integrals uses the finite difference Toda space ($t_{\mathbb{C}} \rightsquigarrow T_{\mathbb{C}}$)

(6) Variant : (w. Freed, 2015)
Category of loop group representations
as a "matrix factorization" category
(also on the finite difference Toda space)

VI Applications (newer and ongoing)

(1) 3D Coulomb branches from the GLSM

Background:

3D SUSY G-gauge theory with matter $E = V \oplus V^*$

$\longrightarrow$ "Coulomb branch", a hyperkähler space $\mathcal{C}(G;E)$ birational to $J(G^\vee)$.

Fixing a complex structure: $\mathrm{Spec}\{Z(S^2)\}$ (space associated to S^2)

First constructed by Braverman - Finkelberg - Nakajima

$H_*^G(\Omega G;$ coefficients built from $V)$.

Described in terms of the GLSM for V (–, 2020); obtained by adding to Toda space the Lagrangian section

$\exp[dW(\tau)]$, $W(\tau) = \mathrm{Tr}_V(\tau(\log \tau - 1))$

[GLSM superpotential]

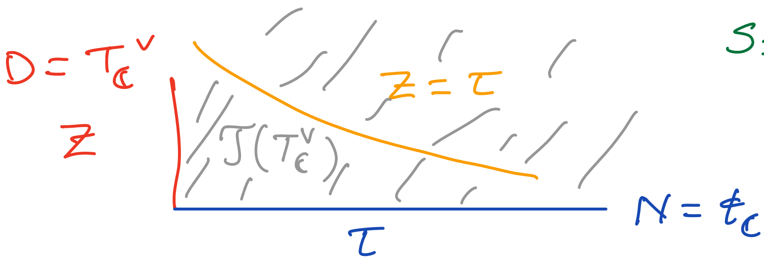
"The modification of $J(G^\vee)$ which makes V into a finite boundary theory (finite, rk 1 over the base)"

also predicted to be the equivariant symplectic cohomology of V

Recently proved (Gonzalez - Mak - Pomerleano):

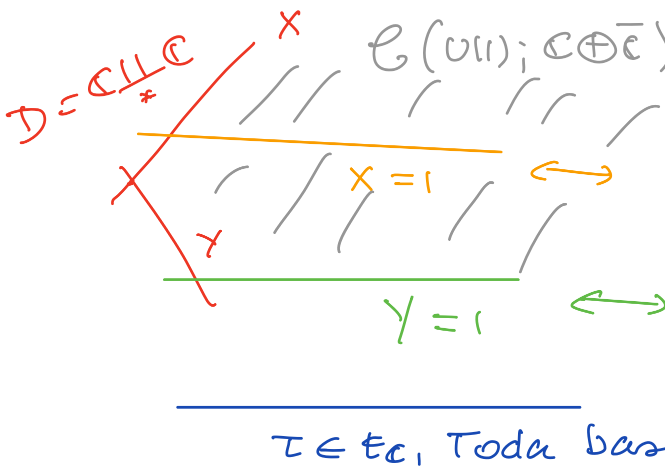
"$C(G;E)$ is the subring of $\mathbb{C}[J(G^\vee)]$ which preserves the $QH_G^*(V)$ lattice in $SH_G^*(V)$"

Example $U(1)$ with $E = \mathbb{C} \oplus \overline{\mathbb{C}}$



$$SH^*_{U(1)}(\mathbb{C}) = \mathbb{C}[z, z^{-1}] = \mathbb{C}[\tau, \tau^{-1}]$$

$$\mathcal{E}(U(1); \mathbb{C} \oplus \overline{\mathbb{C}}) = \mathrm{Spec}\left[\mathbb{C}[x, y, \tau] / (xy = \tau)\right]$$



$$QH^*_{U(1)}(\mathbb{C}) \cong \mathbb{C}[\tau]$$

$$QH^*_{U(1)}(\overline{\mathbb{C}}) \cong \mathbb{C}[\tau]$$

(1)' Categorical developments:

work of Gammage, Hilburn

Key Applications (Ongoing)

(2) $\mathcal{C}(G;E)$ when E is quaternionic, non-polarized.

Boundary condition V does not exist

Nonetheless, the "chiral ring" $\mathbb{C}[\mathcal{C}(G;E)]$ can be constructed by judicious use of KSp-theory. (–, '22)

It comes with a Lagrangian multi-section with $\#W$ sheets over Toda base.

(depends on a direction in the Kähler cone of G/T)

Conjecture: Construction of $\mathcal{C}(G;E)$ from $SH^*_G(G \times_T V)$, for a T-equivariant polar half V of E.

(3) "Quantum GIT Conjecture"
(Partially proven, joint w/ Pomerleano)

For compact Fano manifolds X,

$$QH^*(X/\!/G) \cong \text{gauged } QH^*(X)$$

computed from the "Lagrangian calculus in $\mathcal{T}(G^\vee)$" as

$$QH^*_G(X) \underset{H^G_*(\Omega G)}{\otimes} H^*(BG).$$