Group Representations over Vect

Ruminations on

groups, vector spaces, categories and Chern-Simons theory

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Based on conversations with

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(Modest) Goal:

Topological understanding of 3 dim. Chern-Simons theory as a topological quantum field theory of manifolds with boundaries, edges, corners.

CS theory is (or should be) defined for any compact Lie group G and (regular) twisting $\tau \in H^4(BG; \mathbb{Z})$.

Relates closely to Conformal Field Theory in 2D; Representations of the loop group LG of G; Moduli of holomorphic $G_{\mathbb{C}}$-bundles on Riemann surfaces; Representations of the quantum group.

The latter provide the only construction of CS theory. (Reshetikhin-Turaev)

(but no codim 3 corners)

Some Tech background

Loop groups have an interesting class of projective representations, with projective cocycles classified by levels in $H^4(BG)$. CFT gives them a richer structure than first expected.

Thus, every Riemann surface with boundary components labelled by reps of LG defines a vector space of conformal blocks inside the product of the boundary reps. These spaces have structural importance in both CS and CFT (Segal's modular functor).

Their construction used to require either the theory of holomorphic bundles on Riem. surf. or quantum groups.

This was partly changed by the

Theorem. (Freed, Hopkins, –; 2000)
The twisted K-theory group ${}^{\tau}K_G^d(G)$ is canonically isomorphic to the $\mathbb{Z}$-span of the irreps of LG at level (related to) $\tau \in H^4(BG)$. The convolution product on K corresponds to the fusion product on representations, defined from the conformal block for the 3-holed sphere.

Note: The dimensions of the conformal blocks give the structure constants for multiplication, but the higher structure on conformal blocks, required for CS theory, is lost. The maximum one recovers from K-theory is the 2dim. reduction of Chern-Simons.

Still, one could hope for a topological construction of the missing structure.

Recent ideas subsumed by the term categorification are now pointing the way.

Note: we claim no credit for the individual ideas, many of which have appeared in the literature.

Categorification:

natural numbers	$\longrightarrow$	vector spaces
vector spaces	$\longrightarrow$	linear objects, eg linear categories (?)
linear categories	$\longrightarrow$	linear 2-categories

This is approaching science fiction, but for instance

algebras (examples of linear cats)	$\longrightarrow$	tensor categories (have associated 2-category of modules)

Plan for Chern-Simons

a point (corner in a 3-manifold)	$\longrightarrow$	tensor category ${}^{\tau}\mathrm{Vect}[G]$ of (τ-twisted) vector bundles over G, with convolution.
circle (edge)	$\longrightarrow$	Drinfeld centre of the above; a version of ${}^{\tau}\mathrm{Vect}_G[G]$, vector bundles that are equivariant for the conjugation action

One must arrange that this Drinfeld centre with its braided structure, is equivalent to the category of reps of the Loop group LG.

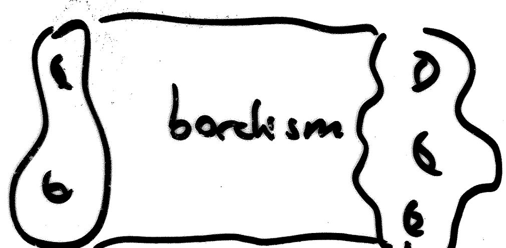
It is known that the remaining part of Chern-Simons theory can be reconstructed from that.

Topological Quantum Field Theories

= (strict) symmetric monoidal functors from the (oriented) bordism category in dims. n, $n-1$ to Vect, the $\overset{\text{tensor}}{\text{cat}}$ of vector spaces (with $\otimes$ as monoidal structure)

Means closed $(n-1)$-manifold $\rightarrow$ vector space

n-manifold w. boundary $\rightarrow$ vector in boundary space



$n-1$: object $\xrightarrow{\text{morphism}}$ object

$$V_{\text{left}} \longrightarrow V_{\text{right}}$$

$$\Updownarrow$$

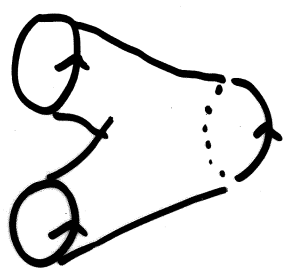
vector in

$$V_{\text{left}}^{*} \otimes V_{\text{right}}$$

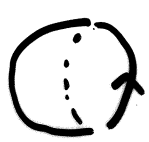
Current focus: allowing corners of various codimensions. Leads to higher categorical structures.

Theorem (folklore)

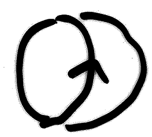
A TFT in dimensions $(2,1)$ is equivalent to the datum of a commutative Frobenius algebra structure on the vector space A associated to the circle



mult: $A \otimes A \longrightarrow A$

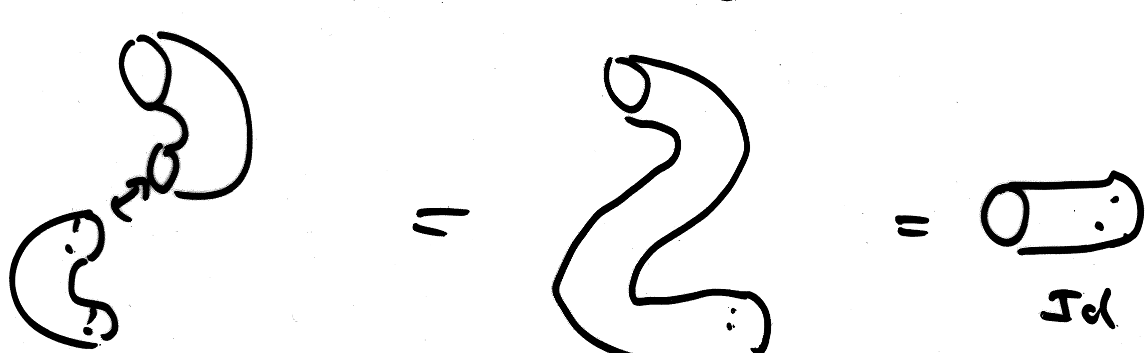


unit
$\mathbb{C} \longrightarrow A$



co-unit
$\theta : A \longrightarrow \mathbb{C}$

Frobenius condition: $a \times b \mapsto \theta(ab)$ is non-degenerate

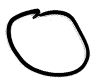


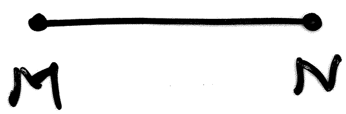
Moore - Segal took a key step towards extending this to surfaces with corners

$\Rightarrow$ notion of Open - Closed field theory involves a pair A, B of Frobenius algebras, $A \longrightarrow Z(B)$. Their version was suited to semi-simple algebras but was vastly generalised by Kontsevich, Costello (and then Hopkins - Lurie)

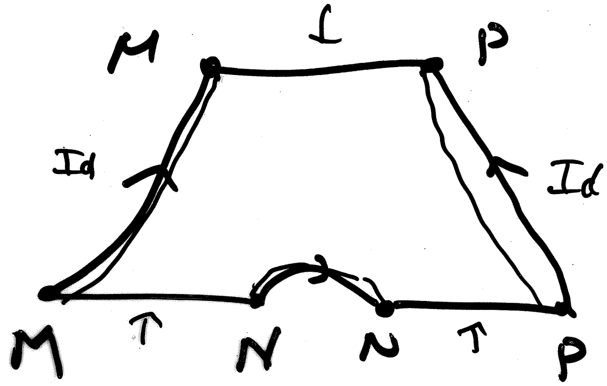
- algebra $B \longrightarrow$ linear A_∞ Frobenius category (eg: A_∞ algebra with nondeg. cyclic trace)
- surfaces can move in families, get cohomology classes on the base instead of numbers (linear maps)
- Kontsevich can describe the deformation space of such theories.

Boundary states : B-modules

 $\longrightarrow$ A



$\longrightarrow$ $\mathrm{Hom}_B(M,N)$



$$\mathrm{Hom}(M,N) \otimes \mathrm{Hom}(N,P) \longrightarrow \mathrm{Hom}(M,P)$$

composition

The pictures have meanings as follows:

Labelling boundaries & corners makes a surface "behave like" a closed surface, and produces a number. "Sewing" boundaries = summing over intermediate states.

Refresher on finite groups

$G \longrightarrow$ group ring $\mathbb{C}[G]$, $\sum a_g \cdot g$

This is a semi-simple algebra, $\bigoplus_V \mathrm{End}(V)$;

The map $\theta : \mathbb{C}[G] \to \mathbb{C}$, $\sum a_g \cdot g \mapsto \frac{a_1}{|G|}$,

makes it into a non-comm Frobenius algebra.

The centre of $\mathbb{C}[G]$ is the convolution ring of class functions, and is spanned by projectors

$P_V := \dfrac{\dim_V \cdot \mathrm{Tr}_V(g)}{|G|}$ for irreps V. The associated open-closed field theory is topological Yang-Mills on surfaces with gauge group G, and the number associated to a surface counts the flat G-bundles. By computing we get

$\mathrm{inv}\, Z(S^2) = \sum_V \dfrac{\dim^2_V}{|G|^2}$, $\mathrm{inv}\, Z(T^2) = \sum_V 1$

giving $\sum \dim^2 V = |G|$ and $\#$ irreps $= \#$ conj classes

Remarkable theorem (Brauer)

The representation ring is spanned (even integrally) by representations induced from 1-dimensional reps of subgroups (of special form).

In particular we can find all reps we want over the cyclotomic integers. (But usually not over $\mathbb{Z}$).

Twisted version. Any central extension of G by $\mathbb{C}^\times$ gives a twisted version of the group ring. The extension defines a complex line bundle $\mathcal{L}$ over G equivariant for the conjugation action. Coefficients a_g take values in the fibres of $\mathcal{L}$, as do the characters of those "projective" reps of G on which the central $\mathbb{C}^\times$ acts naturally.

Same story goes through $\longrightarrow$ twisted Yang-Mills.

Categorification

In this case involves replacing the ground field $\mathbb{C}$ with the tensor category of dg vector spaces (almost a ring in the world of categories with operations $\oplus$ and $\otimes$)

$\mathbb{C}[G] \rightsquigarrow \mathrm{Vect}[G]$, the category of vector bundles over G with convolution as $\otimes$

(Caution: Meaning is clear for a finite group but Lie groups require some redefinition!)

$$\left\{ \begin{array}{c} \text{Reps of } G \text{ on} \\ \text{vector spaces} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \text{Linear actions of } G \\ \text{on linear categories} \\ \text{(with some constraints?)} \end{array} \right\}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{1-category} & \text{2-category} \\ \parallel & \parallel \\ \text{modules over } \mathbb{C}[G] & \text{modules over } \mathrm{Vect}[G] \end{array}$$

The gadgets on the right are often called linear 2-representations of G.

Example of a 2-representation

A linear category with a single object is an algebra A. Assume it's unital.

Let Z = centre of A

$\mathrm{Inn}(A)$, $\mathrm{Aut}(A)$, $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$

the automorphisms groups of A

Have a 2-extension

$$1 \to Z^\times \to A^\times \to \mathrm{Aut}(A) \to \mathrm{Out}(A) \to 1$$

of $\mathrm{Out}(A)$ by $Z^\times$.

A homomorphism $G \to \mathrm{Out}(A)$ is analogous to a projective representation of G. The analogy is exact if $Z^\times = \mathbb{C}^\times$, or if the 2-extension comes with a reduction to $\mathbb{C}^\times$.

(Otherwise, it's similar to a group extension

$$1 \to Z^\times \to \tilde{G} \to G \to 1 \;)$$

The analogy means this "model" suffers from the same problem as modelling a representation of a $\mathbb{C}^\times$-central extension of G on V by a homomorphism

$$\mathbb{P}\rho : G \to \mathbb{P}GL(V):$$

* have lost track of the extension cocycle
* cannot distinguish representations related by tensoring with a 1-dim. rep.

Instead, lift $\mathbb{P}\rho$ to a map $\rho : G \to GL(V)$.

The failure of ρ to be a homomorphism defines a cocycle $g_1, g_2 \mapsto \rho(g_1 g_2)\rho(g_1^{-1})\rho(g_2^{-1})$ for the central extension. A genuine rep. of G is one where the cocycle is $\equiv 1$.

2-cocycle w values in $\mathbb{C}^\times$

Similarly for a genuine 2-representation we ask for a lifting of $G \xrightarrow{\omega} \mathrm{Out}$ to a map $G \xrightarrow{\alpha} \mathrm{Aut}$, and a reconciliation of the failure of the group law by elements of $A^\times$:

$$\alpha(g \cdot h) = \mathrm{Ad}(a_{g,h}) \circ \alpha(g) \circ \alpha(h).$$

This "kills" the 2-extension of G.

Two choices of such $a_{g,h}$ differ by a 2-cocycle of G with values in $Z^\times$, which is the information about the 2-representation α that was lost to ω.

In particular, the $\mathbb{C}^\times$-central extensions of G define "1-dimensional" 2-reps of G, on the category with one object and $\mathrm{End} = \mathbb{C}$. They all have $\mathrm{Out} = \{1\}$, $\omega = 1$.

On a general category, a representation of G is the datum of

* an endofunctor α_g for each $g \in G$
* a natural isomorphism of functors

$$\alpha_{gh} \xrightarrow[\sim]{\gamma_{g,h}} \alpha_g \circ \alpha_h$$

for each pair (g, h)

Subject to the obvious coherence condition

$$\begin{tikzcd} \alpha_g \circ \alpha_h \circ \alpha_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{commutes.}"] & \alpha_{gh} \circ \alpha_k \arrow[d] \\ \alpha_g \circ \alpha_{hk} \arrow[r] & \alpha_{ghk} \end{tikzcd}$$

A theorem of Ostrik's classifies all reps of a finite group on semi-simple linear categories: they are (sums of) induced representations from 1-dim. reps of subgroups.

Unfortunately, this means that there are "not enough" such representations!

Character of a 2-representation

[see Ganter - Kapranov for a treatment and relation to elliptic cohomology]

An endomorphism of a vector space has a trace which is a complex number

An endofunctor F of a linear category $\mathcal{C}$ also has a "trace" which is a vector space, the zeroth Hochschild homology $HH_0^{\mathcal{C}}(F)$.

When $\mathcal{C}$ is an algebra A,

$$HH_0^{A}(F) = A / \{a \cdot b - b \cdot F(a)\}_{a,b \in A}$$

This is the cokernel of an obvious map

$$\cdots \longrightarrow A \otimes A_F \longrightarrow A_F \dashrightarrow HH_0(F)$$

which continues to the left to give the Hochschild chain complex.

Proposition (Exercise)

If G commutes with F, (up to nat. isom)

it induces a linear map $G_* : HH_0^{\mathcal{L}}(F) \circlearrowleft$.

<u>Def</u> The character of a 2-rep of G on $\mathcal{L}$ is the conjugation-equivariant vector bundle over G, with fibre $HH_*^{\mathcal{L}}(g)$ at $g \in G$.

<u>Rmk</u> This is in $\mathrm{Vect}_G[G]$ if all is finite-dim, in ${}^{\tau}\mathrm{Vect}_G[G]$ if $\mathcal{L}$ is a projective? representation with 2-extension classified by $\tau \in H^4(BG;\mathbb{Z})$.

Remarkably the same $\mathrm{Vect}_G[G]$ appears in a different guise, as the Drinfeld centre of $\mathrm{Vect}[G]$. (for finite G). As such, it inherits a rich structure (Modular tensor category) which lets us reconstruct all of Chern-Simons theory for finite groups. (Implicit in Dijkgraaf-Witten, Freed-Quinn)

This was generalised by Muеger who showed that the Drinfeld double of a certain class of (spherical) tensor categories leads to a 3d TFT.

The old theorem about ${}^{\tau}K_G(G)$ suggests this should also work for compact Lie groups.

Status: There is hope.

The naive formulation is not correct.

For a torus, we can carry out the proposal literally and produce a 3dim topological field theory, but this turns out to be the L^2 version of CS theory.

The vector space associated to a surface is the L^2 space of the Θ^{τ}-line bundle over the Jacobian, instead of the holomorphic section space.

However, this can be corrected in pure category language by using a more sophisticated version of the Drinfeld centre, which produces exactly the category of positive energy reps.

There are clear directions of attack for non-abelian compact Lie groups, but nothing to report yet.

We'd like to say that we can construct CS theory for manifolds with boundaries edges and corners.

But we're not entirely certain what this means..

"monoidal functor from the 3d bordism 3-category to the 3-category of linear 2-categories" seems unenlightening

Problem Not enough 2-representations $\Rightarrow$ difficult to give a Moore-Segal description.

Comparison: 2D (YM) vs 3D (CS)

	2D assigns	3D assigns
point	$\mathbb{C}[G]$, group alg. over $\mathbb{C}$ possibly twisted by $H^3(BG;\mathbb{Z})$	$\mathrm{Vect}[G]$, group ring over Vect, possibly twisted by $H^4(BG;\mathbb{Z})$
kind of object?	$\mathbb{C}$-algebra (N.com), Frobenius	Tensor category
circle	Centre of $\mathbb{C}[G]$	Drinfeld centre $\mathrm{Vect}_c[G]$
kind?	Comm. Frobenius $\mathbb{C}$-algebra	braided tensor cat. + more.
surface	Integral (or sum) of 1 (?) over the space of flat G-bundles	"Integral of lines" over the moduli of hol. G bundles = space of holo. sections
kind	number	vector space
3-fold	————	(Path) Integral of CS action functional over all connections

Remark: The CS side has an L^2 version for Lie groups. Not quite a TFT because ∞-dimensional.

Same issue appears in 2D Y-M. for Lie groups.