Higher twisted K theory & applications

Occasion: Recent work of Dadarlat, Pennig, Evans on "selfabsorbing C^* algebras"

$\Rightarrow$ rigid model for spectrum of units in K-theory and promise of genuine equivariant versions

Can hope for applications going beyond what's accessible with equivariant cohomology (similar to what's happened w/ additive story)

Several new directions suggested by older work.

One theme:

dilogarithm as a Chern character for $B^2 GL_1(K)$

$$G \ni g \mapsto \Phi(g) = Tr_V\left(Li_2\left(m^{-1}g^{-1}\right)\right)$$

V rep of G

$m = e^{\mu} \in \mathbb{C}^{\times}$ complex mass parameter (near $\mu = +\infty$)

$$Li_2 x = \sum_{n>0} \frac{x^n}{n^2}$$

$$\exp d\Phi = \prod_{\substack{\text{weights} \\ \nu \text{ of } V}} \left(1 - m^{-1} g^{-\nu}\right)^{\nu} : \underset{\text{max torus}}{T} \longrightarrow \underset{\text{dual torus}}{T^{\vee}}$$

$(g \in T)$

Background

$K^0(X)$ = group completion of Iso Vect(X) under $\oplus$

$X \mapsto K^0(X)$ generalized cohomology thry

$\Rightarrow$ represented by a "spectrum"

0^{th} space $\mathbb{Z} \times BU$ $\left(BU(\infty) = Gr(\infty, 2\infty) \right)$

$\left(K^0(X) = [X; \mathbb{Z} \times BU] \right)$

- 2-periodic by Bott, $\beta: \Omega U \xrightarrow{\sim} BU$
- $\oplus, \otimes \Rightarrow$ "E_∞ ring spectrum"
- $GL_1(K) = \{\pm 1\} \times BU_\otimes$ is also a cohomology thry (Segal; also Atiyah; Madsen; Rezk)
- $GL_1^+(K) = \underset{\substack{\uparrow \\ \text{line bdles}}}{BU(1)} \times \underset{\substack{\uparrow \\ \text{vectr bdles with } \det = \mathbb{C}.}}{BSU_\otimes}$

more exotic

Recall $[X; BU(1)] = H^2(X; \mathbb{Z})$

Equivariant version: G acts on X; consider G-bundles

Have a version of BU with strict G-action, then $K^0_G(X) = \pi_0 \mathrm{Maps}(X; BU)^G$.

$K^0_G(pt) = R(G)$; few genuine units but many "formal" units (add a formal var. m, $R_G[[m]]$)

Applications of "lower" $GL(k)$

2D TQFT — index of line bundles over the moduli of G bundles on a Riemann surface

Paradigm — Path integral

Space of fields: $\mathrm{Map}(M; F) \ni \varphi$

Have eval: $M \times \mathrm{Map}(M, F) \longrightarrow F$

define $S(\varphi) = \int_M (\text{Some function of eval})$

then path integrate $\int_{\mathrm{Map}(M,F)} \exp(iS(\varphi))\, \mathcal{D}\varphi$

Note: 1st integral really happens in $\mathbb{C}^\times$ because only $\exp(iS(\varphi))$ is used

group law: multiplication in $\mathbb{C}^\times$

2nd integral: embed $\mathbb{C}^\times \hookrightarrow \mathbb{C}$ and integrate

Path integral: often well defined in topological setting

$$\overset{M}{\Sigma} \times \overset{\mathrm{Maps}(M,BG)}{\mathrm{Bun}_G(\Sigma)} \xrightarrow{\text{eval}} \overset{F}{BG} \quad \text{universal } G \text{ bundle}$$

$$\mathrm{eval}^*\tau \longleftarrow \tau \in H^4(BG; \mathbb{Z}) = B^2\mathrm{Pic}_G$$

$$\int_\Sigma \downarrow$$

$$H^2 = \mathrm{Pic}(\mathrm{Bun}_G(\Sigma))$$

$\mathcal{O}(\tau) \longrightarrow \mathrm{Bun}_G(\Sigma)$ can ask for Index of that

= additive K theory integration

Solution (Freed, Hopkins, –)

Requires intermediate step of path integral on circle

$\mathrm{Bun}_G(S^1) \sim G/_{Ad}G$

$\mathrm{Bun}_G(S^1) = \mathrm{Maps}(S^1; BG) = LBG \simeq G/_{Ad}G$

$S^1 \times \mathrm{Bun}_G(S^1) \xrightarrow{\mathrm{eval}} BG$

$\mathrm{eval}^*\tau \in B\mathrm{Pic}(G/G) \;\; (= H^3_G(G))$
$= $ coeff. systems for K theory

addition integral = K theory groups with coefficients

$= {}^{\tau}K_G(G)$

1. Thm = free $\mathbb{Z}$-module on representations of $LG = \mathrm{Map}(S^1; G)$ projective, of + energy..

Consider now

$$\begin{array}{ccc} \mathrm{Bun}_G^{flat}(\Sigma\text{-disk}) & & {}^{\tau}K_G(G^{\times 2g}) \\ \downarrow & & \pi \downarrow \;\; \Pi \text{ commutators} \\ \mathrm{Bun}_G^{flat}(S^1) & & {}^{\tau}K_G(G) \end{array}$$

2. Thm Multiplicity of "vacuum rep" in $\pi_! 1$
$= $ index of $\mathcal{O}(\tau)$ on $\mathrm{Bun}_G(\Sigma)$.

("topological path integral works").

Application 2

Categorification of Thm 1.

C^∞ or C^ω.

$\downarrow$

$\int_{S^1} \tau \rightsquigarrow$ class in $H^3_G(G;\mathbb{Z}) = H^2_G(G;\mathcal{O}^\times)$

This defines a family of Azumaya algebras over G/G.

Now the "stack G/G has a canonical central section"

$\rho: g \in G(\text{space}) \longrightarrow g \in Z(g)$ the stabilizer group.

(This is a "semiclassical ribbon")

An algebra with a central unit has a matrix factorization category of "modules"

Abstractly: unit $\to$ automorphism of Id of category of modules

$\Longleftrightarrow$ topological action of S^1 ("$B\mathbb{Z}$-action")

MF = (Tate) fixed point category

Thm 3 $\quad {}^{\tau}MF_G(G;\rho) \equiv {}^{\tau-c}\mathrm{Rep}(LG)$

(Freed, –) $\quad$ (shift c = dual Coxeter # – from Dirac operator: really $LG \ltimes \mathrm{Cliff}(Lg)$ – modules).

Application 3 – (related)

Thm: this is a categorification of Kirillov's correspondence

representations $\longleftrightarrow$ integral coadjoint orbits

* The character formula for reps as an orbital integral can be deduced by taking Chern characters

– proved by Kiran Luecke for reps of compact group G ($\Leftrightarrow$ ∞ level limit of LG) and discrete series reps of real semisimple G

(gets Kirillov / Harish-Chandra character formulas)

True but not written out for LG.

Igor Frenkel: formula should pop out

Relation to CFT allows one to show

The spaces $\mathcal{D}$-index $(Bun_G; \mathcal{O}(\tau))$ form vector bundle on M_g (conformal blocks). CFT $\Longrightarrow$ projectively flat with known Chern slope (central charge)

Proof by direct topological computation difficult (Not written down; student Catherine Lee working on generalization – below. Combinatorics difficult!)

Higher units in K-theory:

Lots of formal ones

In $K^0_G(X)[[m]]$: $\sum_{k\geq 0} m^k V_k$, $V_k \in K^0_G(X)$, $\mathrm{rk}\, V_0 = 1$

For instance $\mathrm{Sym}_m(V) = \sum_{k\geq 0} m^k \mathrm{Sym}^k(V)$

Comes with an exponential functor

$$K^0_G(X)_{\oplus} \xrightarrow{\mathrm{Sym}} K^0_G(X)[[m]]_{\otimes}$$

transformation of cohomology theories

$\longrightarrow$ may define complex orientation (integration) on

Repeat Path integral story with a class in

$B^2 \mathrm{GL}_1(K_G[[m]])$

eg of the form $\tau \in H^4(BG) \times$ "$B^2\mathrm{Sym}\, V$" $V \in \mathrm{Rep}(G)$

integrates to line bdle higher unit

$$\begin{array}{ccc} \Sigma \times \mathrm{Bun}_G \Sigma & \xrightarrow{\mathrm{eval}} & BG \\ \int_\Sigma \downarrow & & \\ \mathrm{Bun}_G \Sigma & & \end{array}$$

Get: (line bundle) $\times$ Sym ($\not{D}$-index bundle of V-bundle on Σ

$\mathcal{O}(\tau)$

Have	Should also have
higher hoisting for $K_G(G)$	higher central extension of loop groups (by $U_\otimes$)
${}^{h\tau}K_G(G)$ = free $\mathbb{Z}$-module	∴ on iso classes of irreducibles of LG
Higher Azumaya algebra over stack G/G	MF category for semi-classical ribbon elt $\equiv$ rep category of higher projective reps of LG
Virtual bundle over $Bun_G(\Sigma)$ of unit rank (line bdle) $\times$ $\mathrm{Sym}(\mathrm{Index}(\Sigma; V))$	✓
Index formula over Bun_G In terms of a Frobenius algebra (m-deformed Verlinde ring) (–, Woodward)	? Higher projective flatness of these bundles over M_g? (central charge in $U_\otimes$)

The (K-theoretic) mirror of the GLSM and "Coulomb branches"

The index of $\mathcal{O}(\tau) \otimes \mathrm{Sym}\left(H^0(\Sigma; \underline{V}) \ominus H^1(\Sigma; \underline{V})\right)^\vee$ interpreted as K-theoretic integration of $\mathcal{O}(\tau)$ over the space of holomorphic maps $\Sigma \to V/G_{\mathbb{C}}$.

("Path integral in the GLSM").

"Mass" m = equivariant scaling of V to "control" this

The controlling Frobenius algebra is (closely related to) Jacobian ring on G/G of

$$\psi(g) = \frac{1}{2}\tau \cdot (\log g)^2 + \mathrm{Tr}_V\left(\mathrm{Li}_2\left(m^{-1}g^{\tau}\right)\right)$$

cohomological: $\mathrm{Tr}_V\left((\xi+\mu)\left(\log(\xi+\mu)-1\right)\right)$

("K-theoretic mirror")

The exponentiated differential of ψ is a single-valued map $T \to T^\vee$

$$\exp d\Psi = \prod_{\substack{\text{weights}\\ \nu \text{ of } V}} \left(1-m^{-1}g^{-\nu}\right)^\nu : \underset{\substack{\text{max}\\ \text{torus}}}{T} \longrightarrow \underset{\substack{\text{dual}\\ \text{torus}}}{T^\vee}$$

Lagrangian section in $T \times T^\vee \to T$, Weyl invar.

Nonabelian version of $T \times T^\vee$ is the

<u>Toda integrable system</u>

(Bezrukavnikov, Finkelberg, Mirković)

$\mathrm{Spec}\, K_0^G(\Omega G)$ with Pontryagin product

group (scheme) over $\mathrm{Spec}\, K_G^0(pt) = \mathrm{Rep}(G)$

$$\begin{tikzcd} \mathrm{Toda} \arrow[d] & \arrow[l, hook'] \text{generic fiber: } T_{\mathbb{C}}^\vee \\ T_{\mathbb{C}}/W = \text{conj classes in } G_{\mathbb{C}} \arrow[u, bend left, "\text{unit section}"] & \end{tikzcd}$$

<u>Fact</u> class in $B^2 GL_1(K_G)$ gives a Lagrangian section of the Toda system

$$\begin{tikzcd} S^2 \times \mathrm{Bun}_G(S^2) \arrow[r, "\mathrm{eval}"] \arrow[d, "\int_{S^2}"'] & BG \\ \mathrm{Bun}_G(S^2) \arrow[r, squiggly, "K\text{ thy}"] & K_G^0(G) \end{tikzcd}$$

[Can think of Lagrangian sections of the Toda system as Chern characters of classes in $B^2\, GL_1(K_G)$.]

From this section one can construct a new space over T_c/W by glueing two copies of Toda after vertical shift

Singularities in the section $\Rightarrow$ get collapsing and blow-ups of the original space

Thm this is the "Coulomb branch" of 3D gauge theory with matter in $\underbrace{V \oplus V^\vee}_{\text{Hyperkähler}}$
[N=4 SUSY]

Rmk. Has tentative def in physics as space of monopoles w/ singularities
Has precise alternate def in math (Nakajima; Braverman-Finkelberg-Nakajima)

- this 3D theory admits GLSM as boundary theory

Observation: something interesting only happens if m^{-1} is a genuine variable (not formal near $m=\infty$) else Φ is regular and Coulomb branch is same as for $V=0$

Q Relation with higher central extensions of LG?

Recent paper (BDRJF): $LG \ltimes \text{Heisenberg}(LV \oplus LV^\vee)$.

Happy Birthday !!