Errata to *Five Lectures on Topological Field Theory*

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Abstract

I attach a list of known errors or omissions to my lectures from Barcelona (2012). A couple of mathematical errors and omissions have substance; there is also a complete salad of left versus right adjunction which I will attempt to dress up. Two-number references pertain to the arXiv version, three-numbered ones to the Birkhäuser book.

Proposition 3.12/3.3.7. The first assertion is careless: it holds in the simply connected case, but, as the second assertion indicates, we *can* have k-invariants relating π_1 with the higher homotopy groups. The correct version of the first statement would be *... if and only if all k-invariants are lifted from* $B\pi_1$.

Definition 4.7/3.4.4. Right and left are exactly backwards: $Z(+)$ is the left dual and $Z(-)$ the right dual subject to (4.5-4.6); (3.5-3.6) in the book. The Hom definition in Remark 4.8.iv that follows is equivalent to (4.5-4.6). Mercifully, this does not matter in the current section because we are in a *symmetric* monoidal setting so nothing else needs changing here. Definition 4.13/3.4.7 is correct as stated.

Theorem 4.22/3.4.4 (Important!) The theorem is correct as stated when the target of the TQFT is a $(2,2)$-category. In the $(\infty,2)$ case, invariance imposes an extra condition on the trivialization of the Serre functor S. Namely, the space $\mathrm{Hom}(I,S)$ can be identified with the dual of the Hochschild homology of the object $Z(+)$. As such, it carries a cyclic structure, translating into a topological circle action. The isomorphism $I \simeq S$ is required to be invariant under this action. This is automatic in the $(2,2)$-case, when there is no 'room' for a topological circle action.

Theorem 4.34/3.4.20. The Serre functor includes a dimension shift in addition to the canonical bundle, and the statement does not explain how that is trivialized. When $\dim X$ is even, we have the easy option of changing the target 2-category of DG-categories to that of $\dim X$-periodic DG-categories, where the shift by $\dim X$ comes identified with the identity functor (in a way that commutes with all structure). When $\dim X$ is odd, this runs into possible trouble with the symmetric monoidal structure there, because the Koszul sign rule is not compatible with an odd shift.

The clean way to impose this periodicity in any case is by adapting the physics notion of a *topological twist*: a coupling of the fields to the tangent bundle of the space-time manifold, the argument of the TQFT functor. Namely, we change the natural action of $\mathrm{SO}(2)$ on the target 2-category by incorporating the dimensional shift. (In other words, $\pi_1\mathrm{SO}(2) = \mathbb{Z}$ acts by the automorphism of the identity which is the shift by $(-\dim X)$ on each DG-category.) The framed TQFT functor is now equivariant for this action; however, it takes values in a target that is twisted over space-time (by this dimension shift, coupled to the Euler characteristic).

(Thanks are due to Domenico Fiorenza for pointing out this omission.)

§5.21/3.5.8. The use of left/right duals is mangled up: the correct reading of that argument perversely has $x^\vee$ for left dual and ${}^\vee x$ for right dual, resulting in a confused statement. Here is the corrected version with natural notation, $x^\vee$ for right dual and ${}^\vee x$ for left dual:

Let us spell out its identification of Serre with the double right dual functor, in the rigid case. This does not rely on semi-simplicity of $\mathscr{T}$. Return to the identification $\mathscr{T}^\circ = \mathscr{T}^\vee$ of §5.11/3.5.5, sending $a^\circ \in \mathscr{T}^\circ$ to the functional $X \mapsto \mathrm{Hom}_{\mathscr{T}}(a, X)$; I claim that we have

$$x \otimes a^\circ \otimes y = \left({}^\vee y \otimes a \otimes x^\vee\right)^\circ$$

because on a functional $X \mapsto F(X)$, we have $(x \otimes F)(X) = F(X \otimes x)$; following which, we can use the Hom-duality properties of $x^\vee, {}^\vee y$,

$$\mathrm{Hom}_{\mathscr{T}}(a, y \otimes X \otimes x) = \mathrm{Hom}_{\mathscr{T}}({}^\vee y \otimes a \otimes x^\vee, X).$$

Now let us use the left dual to identify $\mathscr{T}$ with $\mathscr{T}^\circ$: $a \mapsto ({}^\vee a)^\circ$. The relation becomes

$$x \otimes ({}^\vee a)^\circ \otimes y = \left({}^\vee y \otimes {}^\vee a \otimes x^\vee\right)^\circ = \left[{}^\vee \left(x^{\vee\vee} \otimes a \otimes y\right)\right]^\circ$$

The result is identifying the Serre bi-module $\mathscr{T}^\vee$ with $\mathscr{T}$, but with the left tensor action twisted by double right dualization. This is the bimodule implementation of the double right dual functor.