The Quantum GIT Conjecture

Application of the mirror picture of 3D gauge theory to GW theory of GIT quotients of compact Fanos

X compact symplectic, w. Hamiltonian G-action,
$\mu : X \longrightarrow \mathfrak{g}^*$ moment map, $X/\!\!/G := \mu^{-1}(0)/G$

Moral statement: gauged GW theory of X = GW $(X/\!\!/G)$
in Fano, smooth quotient case

History: Proved by Batyrev in toric case
(reinterpreted in Givental-Hori-Vafa description)
Conjectured by physicists long ago

Recent proof joint with Dan Pomerleano (for QH^*)
based on earlier ('17) incomplete proposal by (-)

Noteworthy: Proof is Floer-theoretic,
no obvious path to alg geom. proof!

Complements: Loosening assumptions ⇒ progressive failure
* orbifold quotients: Iritani
additive statement OK, corrections to multiplication
* non-Fano: $QH^*(X/\!\!/G)$ additive summand in gauged
Known from toric examples; in physics, complement
is called "Landau-Ginzburg sector"

1. Mirror picture of G-gauge theory

Involves the Toda space $\mathcal{J}_G := \mathcal{C}_3(G;0)$ for G.

Thm Equivariant space of states for an oriented local TQFT with topological G-symmetry gives a sheaf of algebras with Lagrangian support in $\mathcal{J}_G$.

[Assumptions: Finiteness of $HH^*_G(\mathcal{C})$ over $H^*(BG)$
Orientability $\Leftrightarrow$ CY condition $\Rightarrow$ unobstructedness
$\Rightarrow$ quantizes to a module over NC def of $\mathcal{J}_G$]

Even though assumptions don't quite apply to $GW(X)$ the conclusion does

Thm $QH^*_G(X)$ is an E_2 algebra over $\mathbb{C}[\mathcal{J}_G]$ (is E_3)
Finite of rank $QH^*(X)$ over the Toda base.

Examples:

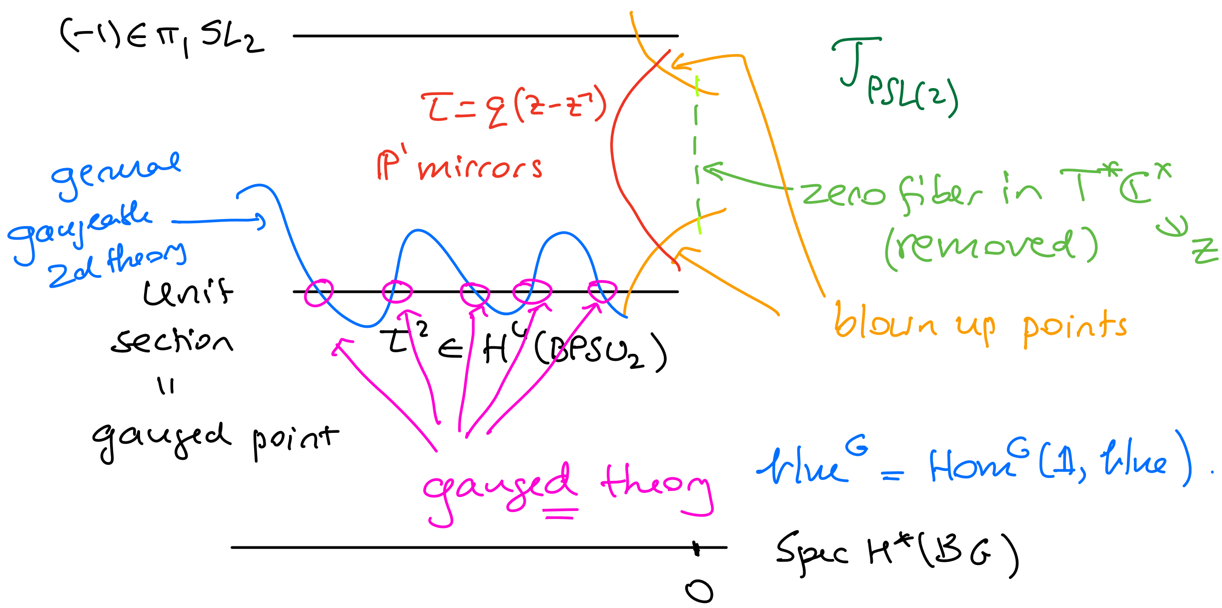
* Flag varieties G/L are leaves of the Whittaker foliation:

$$\mathcal{J}_G \simeq N_\chi \backslash\!\backslash T^*G^L_{\mathbb{C}} /\!/_\chi N \quad \text{and these are the cotangent fibers}$$

* $z = \tau^n$ in $\mathbb{C}[z^\pm, \tau] \leftrightarrow T^*\mathbb{C}^\times$ goes with $\mathbb{C}_1^{\oplus n}$

Functions on it are the equiv. symplectic cohomology

2. Interpretation: Character calculus for gauge theory



Theorem (-, Pomerleano) If X is compact Fano with smooth quotient $X/\!/G$,

$$QH^*(X/\!/G) \cong QH^*_G(X) \otimes_{H^G_*(\Omega G)} H^G_*(\text{point}).$$

Special case (G = torus)

$$QH^*(X/\!/T) = QH^*_T(X) / \langle \text{Seidel ops} = 1 \rangle$$
or fixed value.

Similarly can perturb unit section to very nearby leaf of Toda foliation (staying in the Fano case)

Proof by Floer Theory. (Algebraic Geometry ??)

3. Rewriting the tensor product

Over G, we have the family of Floer theories

$$g \in G \longmapsto HF(X; g)$$

This is:

A (derived) local system (Floer continuation maps)

G-equivariant for conjugation

Multiplicative and E_2 multiplicative equivariantly
(pair of pants product)

Fiber at 1 carries the monodromy representation of ΩG

Equiv. fiber $HF^*_G(X)$ —"— of $H^G_*(\Omega G)$
(Remark: read chains wherever needed)

This is the action of $\mathbb{C}[J_G]$ on $QH^*_G(X)$.

Proposition $H_*^G(G; \mathcal{HF}^*(X)) = QH^*_G(X) \otimes_{H^G_*(\Omega G)} H^*(BG)$

[This is the space of states for the gauged theory]

We're now set up for a Floer calculation.

4. Additive structure

A. Flow the cohomology into $\mu^{-1}(0)$ by turning on $K \cdot |\mu|^2$ as a Floer Hamiltonian, $K \to \infty$.

Theorem In Fano case, Every equivariant class flows eventually into a periodic orbit in a Guillemin-Sternberg normal nbhd which is a (disk in T^*G)-bundle over $\mu^{-1}(0)/G$.

Remark X is stratified by the Morse function $|\mu|^2$ (Kirwan stratification) which breaks up the cohomology. Local calculation shows that in the Fano case all classes flow out of the strata and into the bulk (see $\mathbb{P}^1$ example below).

However, the Fano case allows the definition of an invariant for each Floer class, monotone index, which controls the rate of flow globally.

Remark The non-Fano case has trapped cohomology at some fixed-point sets. This accounts for the physics Landau-Ginzburg summand. Combinatorial description — work in progress ()

Remark One difficulty with local calculation: $|\mu|^2$ need not be Morse-Bott. Circle case is easier.

B. The flow defines a lattice in $QH^*_G(X)$ over $\mathbb{Z}[q, q^{-1}]$. Namely, declare every geometric orbit in the G-S normal neighborhood to have q-degree 0.

Theorem

1. The Floer differential between degree 0 orbits in GS neighborhood is topological + $O(q)$.

2. The q-power filtration is bounded below in each degree.

Rmk (1) from a priori energy estimates
(2) from degree bounds & Fano condition

Meaning of topological

Spaces of Orbits in a fiber of projection to $X/\!/G$ are $G \times$ flag varieties of G. The topological differential is the one computing $H^*_G(G \times G/T) = H^*(G/T)$.

Corollary The q-filtration on the Floer complex gives a convergent spectral sequence with topologically computed E_1 term.

Computation of $H^G_*(G; \mathcal{HF})$: As we move over G, the Floer orbits are all geodesics in the group. These form spaces $G \times \mathfrak{g}$. G-equivariant cohomology learns the base $X/\!/G$.

C. The answer

We described gr of the map

$$(\text{gauge theory space}) \longrightarrow QH^*(X/\!/G).$$

[In principle there could be extra differentials but the answer already has the right size]

The actual map has quantum corrections.

We know how to define it using the Lagrangian correspondence

$$\mu^{-1}(0) \hookrightarrow X \times X/\!/G$$

(monotone case, Wehrheim-Woodward; generalized by Fukaya et al).

Building the Floer family over G for X, and the constant family for $X/\!/G$, we get a correspondence bimodule between Floer cohomologies that is additively isomorphic to both sides

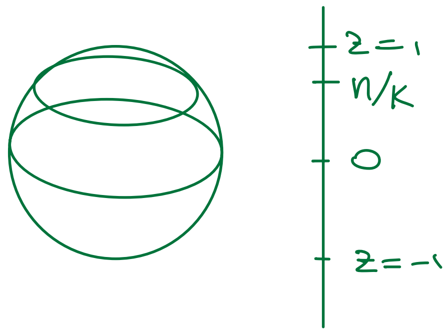
$\Longrightarrow$ isomorphism of full QH^* structures.

Future plans: • describe the map in closed form (expected to be Woodward's Quantum Kirwan map)

• Generalize to complete GW structure?

D. $\mathbb{P}^1$ example

$$H = \frac{1}{2} K z^2$$



$0 \le n \le K$

Natural presentation of Floer complex
* upper half orbits originate from top
* lower half orbits originate from bottom

		Floer degree
$P_n :=$ Poincaré dual class of point	on n^{th} orbit	$2n+1$
$S_n :=$ —"— circle		$2n$ for $n \ge 0$

Floer differential:

topological $\qquad \tau \in H^2(BS^1)$

$$\begin{cases} \delta P_n = S_{n+1} + q^2 S_{n-1} + \boxed{\tau \cdot S_n} & n>0 \\ \delta P_0 = S_1 + S_{-1} & \\ \delta P_n = q^2 S_{n+1} + S_{n-1} + \tau S_n & n<0 \end{cases}$$

Correct normalization: $\pi_n := q^{-|n|} P_n, \quad \sigma_n := q^{-|n|} S_n$

Floer degree: $\quad 1 \qquad\qquad 0$

$$\delta \pi_n = \underbrace{q(\sigma_{n+1} + \sigma_{n-1})}_{\text{pure Floer}} + \boxed{\tau \cdot \sigma_n} \longrightarrow \text{topological}$$

Lattice: $q^{\ge 0} \langle \pi_n, \sigma_n \rangle$.

The Seidel operator z acts as $\pi_n \longrightarrow \pi_{n+1}, \sigma_n \longrightarrow \sigma_{n+1}$

Computing the classical differential and fixing z gives the $\mathbb{Z}[\tau]$-module $\mathbb{Z}[\tau]/(\tau)\,[z, z^{-1}]$.

This is indeed the classical cohomology of $\mathbb{P}^1/\!/S^1$ as module over $H^*(BS^1)$.

With the full quantum differential instead, computes

with $\sigma_n = \pm z^n$ depending on sign of n

$\leftarrow QH^*_{S^1}(\mathbb{P}^1)$

$\boxed{\mathbb{Z}[\tau, z^{\pm}, q^{\pm}] \big/ \left(\tau = q(z - z^{-1})\right)}$, z = fixed value)

Note that $\mathbb{C}[\tau, z^{\pm}, q^{\pm}] \big/ \tau = q(z - z^{-1})$

is $QH^*_{S^1}(\mathbb{P}^1) \subset \mathcal{T}_{S^1} = \operatorname{Spec} \mathbb{C}[\tau, z^{\pm}]$

and we find the intersection with a fiber z = fixed

This is the Batyrev construction.