

Gauge Theory, Toda spaces & Coulomb branches

Physics and recent mathematical understanding:

3D (topological) gauge theory
is controlled by
Hyperkähler spaces closely related to Toda system

Pattern: low energy behavior of a QFT should be
 \Leftrightarrow sigma-model in moduli space of vacua \mathcal{M}

Some landmarks for gauge theory w/ linear matter:

- Seiberg and Witten on 3D pure gauge theory for $SU(2)$
- Argyres - Fanaghi, Wanner - generalization to $SU(n)$
- Seiberg - Intriligator on 3D mirror symmetry
- Witten - Hanany on Poincaré series for Coulomb branches
- * Bezrukavnikov, Finkelberg, Mirkovic:

Topological description of Toda space from affine Grassmannian

- - Gromov-Witten boundary conditions \Leftrightarrow holomorphic Lagrangians
- Bullimore - Dimofte - Gaiotto - abelian Coulomb branches
- Braverman - Finkelberg - Nakajima: Chiral rings for polarized reps
- Braverman et al: proposal for quaternionic reps
- - construction of chiral ring

In SUSY gauge theories 3D & higher: have

Coulomb and Higgs branches $\mathcal{M}_c, \mathcal{M}_h$ of \mathcal{M}

For 3D X/G , X hyperkähler

Higgs: $X//G$; Coulomb: Toda + quantum corrections

2. The Toda spaces $\mathcal{C}_{3,4}(G;0)$

↑ \mathbb{C} -K-theory
homology

- Hyperkähler manifolds;
- in one complex structure, completely integrable abelian gps over:

$$\mathcal{C}_3 \rightarrow \mathcal{J}\mathbb{C}/G_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}}/W; \quad \mathcal{C}_4 \rightarrow G_{\mathbb{C}}/G_{\mathbb{C}} = T_{\mathbb{C}}/W$$

- Abelian cases: $\mathcal{C}_3 = \frac{T^*T_{\mathbb{C}}^{\vee}}{W}$, $\mathcal{C}_4 = \frac{T_{\mathbb{C}} \times T_{\mathbb{C}}^{\vee}}{W}$ monodromy
- General cases: affine blow-ups of Weyl quotients

- BFM: $\mathbb{C}[\mathcal{C}_3(G;0)] = H_*^G(\Omega G)$ Pontryagin product
 $\mathbb{C}[\mathcal{C}_4(G;0)] = K_*^G(\Omega G)$ & homology co-product
 \Rightarrow Hopf algebras over H_*^G, K_*^G

- Thm (-) Some boundary conditions for 3D topological gauge theory correspond to bundles of categories w/ Lagrangian support on $\mathcal{C}_{3,4}$ (Kapustin - Rozanov - Saulina 2-category)

Eg from symplectic mfolds with Hamiltonian G action:

- Symplectic cohomologies of certain open mfolds
- Quantum cohomologies of compact mfolds

- Examples: - a point (Verlinde formulas) ←
- a α -representation (Generalized & Coulomb br.)
 - Compact Fano (tomorrow)

3. Gauged point with a bulk deformation

$$W = \frac{h}{2} \cdot \Sigma^2, \quad \Sigma \in \mathcal{O}_G \text{ (invariant quadratic form)} \\ h \in H^4(BG)$$

The exponentiated graph $\Gamma(dW)$ meets the unit section of the Toda groups at lattice points in $t_c/w, T_c/w$.

The Hessian determinants are the structure constants for a Frobenius algebra. This is the 2D TQFT " \ast/G " with bulk deformation W .

4. Complex representation V

Noncompact \Rightarrow use \mathbb{C}^\times scaling to render things finite
Equivariant parameter μ (complex mass in physics)
 $\in H^2(B\mathbb{C}^\times)$

The associated Lagrangian is again $\Gamma(\exp(dW))$ for the GLSM superpotential in $\underline{H_\ast}$ and $\underline{K_\ast}$

$$t_c \ni \Sigma \mapsto \prod_{\text{wts. } \nu} (\mu + \langle \nu | \Sigma \rangle)^\nu \in T_c^V \quad \text{Toda sections}$$

$$T_c \ni x \mapsto \prod_\nu (1 - m^\nu x^\nu)^\nu$$

open \mathcal{Q} :
extend to moduli of curves

Thm The associated TQFT computed by intersecting with the unit section is the Gromov-Witten gauged theory V/G (w/ Chris Woodward, generalizing Witten)

5. Main Theorem on Chiral Rings $\mathcal{C}_{3,4}(G; E)$

G = compact connected Lie gp; E = quaternionic rep;
 "polarized" means $E = V \oplus V^*$

Nakajima; Bullimore-Dimofte-Gaiotto; yours truly;
 Braverman - Finkelberg - Nakajima;

1. There exist⁴ constructible, equivariant coefficient systems $\mathcal{H}_E, \mathcal{K}_E$ over the loop Grassmannian $G_{\mathbb{C}}(\mathbb{R}^1) \backslash G_{\mathbb{C}}(\mathbb{Z}^1) / G_{\mathbb{C}}(\mathbb{Z}^1)$
 $G \backslash \Omega G = G \backslash LG/G$
2. They are E_2 -multiplicative under Pontryagin products and their equivariant cohomologies $[\mathcal{C}_{3,4}(G; E)]$ are E_3 ("Poisson structures of degree -2")
3. They are multiplicative in E , $\mathcal{H}_E \otimes \mathcal{H}_F \rightarrow \mathcal{H}_{E \oplus F}$
 so $\mathcal{C}_{3,4}(G; E) \times \mathcal{C}_{3,4}(G; F) \rightarrow \mathcal{C}_{3,4}(G; E \oplus F)$
Toda
4. Non-polarized E require the removal of obstructions
5. $H_*^G(\Omega G; \mathcal{H}_E)$ and $K_*^G(\Omega G; \mathcal{K}_E)$ are birational to $\mathcal{C}_3, \mathcal{C}_4$ and are expected to be the chiral rings for E/G
6. (Abelianization) $\mathcal{C}_{3,4}(G; E) \cong \mathcal{C}_{3,4}(\mathbb{T}; E - \sigma_M) / W$
 if E contains the roots of σ . [-]
7. Polarized case: construction from GLSM boundary cond.
 [-]

6. Construction in the Polarized case

(Physics; Nakajima; B-F-N; BDG)

Morally Choose a polar half V of E

Get an index bundle " $H^0 - H^1$ " $(P^1; p_G^* V \otimes \sqrt{K})$ along P^1
over $\text{Bun}_G(P^1) \sim_G \Omega G = G \backslash LG/G$

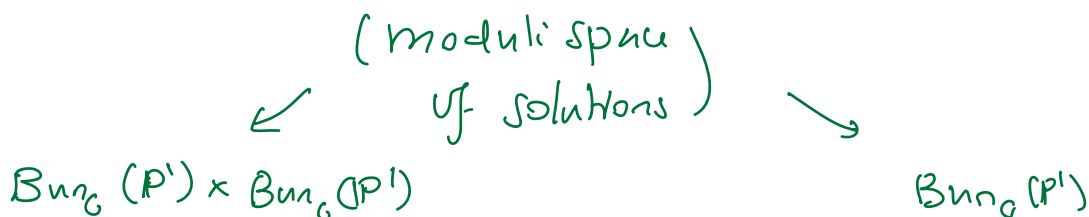
Build the associated linear space Spec Sym (dual sheaf)

Coefficient systems $\mathcal{H}_E, \mathcal{K}_E$ are cohomologies with compact vertical supports

Morally $\mathcal{O}_{3,4}(G; E) = \text{Spec } H_G^* K_G^*(\Omega G; \mathcal{H}_E, \mathcal{K}_E)$
with Pontryagin products.

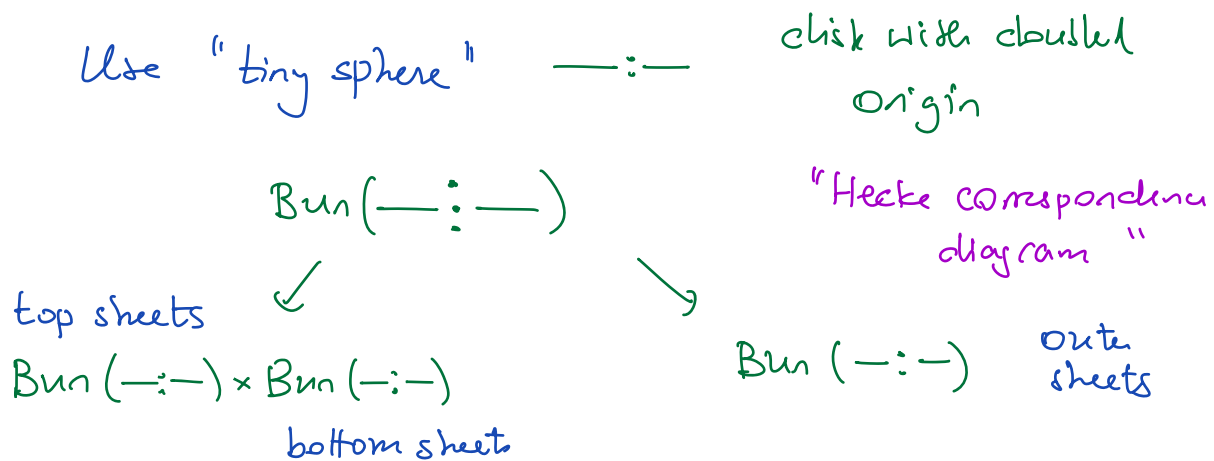
unit = volume form \Rightarrow difficult to make precise

Product structure should come from 3D pair of pants
by solving a gauged Dirac equation w/ prescribed
boundary conditions



7. Algebraic Geometry Rewording (B F N)

The splitting $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ reduces the 3D Dirac equation to the $\bar{\partial}$ equation (and TQFT \Rightarrow constant in t)
 \Rightarrow complex geometry can be used:



The correspondence diagram is now well-defined and gives an E_3 multiplication on $H_*^G(\Omega_G; \mathcal{H}_E, \mathcal{K}_E)$.

8. Global construction from GLSM

$\mathcal{E}_{3,4}(G; E)$ arises by gluing two copies of the Toda space along the vertical shear by $\exp(dW)$ from GLSM.

Equivalently: The chiral ring for E is the subring of functions on the Toda space which survive $\exp(dW)$ translation

Reformulation (Pomerleano): This is the subring of functions that preserve the lattice $\text{QH}_G^*(V) \subset \text{St}_G^*(V)$ (including its bulk deformations).

9. Non-polarized case: $E \neq V \oplus V^*$

- I don't have a good interpretation in terms of Gromov-Witten boundary conditions.

Guess: in terms of $G \times_T V$ (E is a double over T)

the formula I have is not 'clean' though

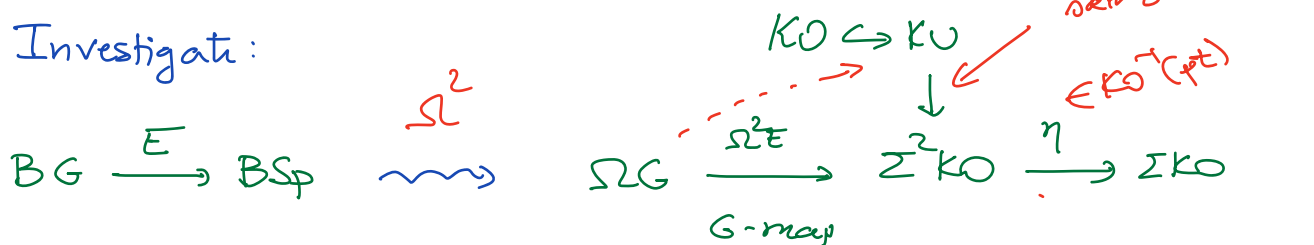
Caution: check paper linked from my website;
the arXiv version has many calculational mistakes

Problem: Invoking the construction for E instead of V
leads to $\mathcal{L}_{3,4}(G; E \oplus E)$.

Need to extract "square roots" of the $\mathcal{H}_1 \mathcal{K}$

Method: check real structures.

Investigate:



Polarization of E would lift $\Omega^2 E$ to KU

Obstructed by $\eta \circ \Omega^2 E \in KO^1$

In any case: want an E_2 lift so obstruction really is

$$BG \xrightarrow{E} BSp \xrightarrow{\eta} Z^3 KO$$

Seems unhelpful until we recall that

we don't need a complete left!
Just enough to build the coefficient systems.

So the obstruction is the image, via $\Sigma^4 J$, into
 $\Sigma^4 GL_1(H\mathbb{Z})$ or $\Sigma^4 GL_1(KU)$ (or $\Sigma^4 GL_1(ko)$)

For cohomology: obstruction class in $H^4(BG; \mathbb{Z}/2)$
(w_1) and is $c_2(E) \bmod 2 = w_4(E)$

For KO -theory: a secondary obstruction $\sigma \in H^5(BG; \mathbb{Z}/2)$
(w_2) is defined if $w_4(E) = 0$

For KU -theory: the 2nd obstruction is $B\sigma \in H^6(BG; \mathbb{Z})$
(w_3) (Essentially $\frac{1}{2} c_3(E)$)

Theorem (nasty calculation)

If G is connected and $w_4(E) = 0$, then $B\sigma = 0$.

(Fails for disconnected groups)

Improvements · One can weaken the obstruction to
 w_4 is the square of a class in $H^2(BG; \mathbb{Z})$

Witten: obstruction is in $\pi_4 G \xrightarrow{E} \pi_4 Sp$

- One can even reduce to the obstruction predicted by Ed Witten $\Leftrightarrow W_4$ has a square root $\in H^2(BG; \mathbb{Z}/2)$

at the price of collapsing
the cohomology grading mod 2:

$$\begin{array}{ccc}
 BG & \xrightarrow{E} & BSp \\
 \text{homology grading} & & \\
 & & \begin{array}{ccc}
 & \eta & \\
 \longrightarrow & \mathbb{Z}/2 & 5 \\
 & \mathbb{Z}/2 & 4 \\
 \longrightarrow & \mathbb{Z} & 3 \\
 & 0 &
 \end{array}
 \end{array}
 \left. \vphantom{\begin{array}{ccc}
 & \eta & \\
 \longrightarrow & \mathbb{Z}/2 & 5 \\
 & \mathbb{Z}/2 & 4 \\
 \longrightarrow & \mathbb{Z} & 3 \\
 & 0 &
 \end{array}} \right\} \mathbb{Z}^3 KO$$