WHAT IS...

a Grope?

Peter Teichner

The mathematical term *grope* first appeared in print in Jim Cannon's 1978 *Bulletin* exposition [1]. He credited the term to his Madison colleague Russ McMillan, a geometric topologist. Cannon explained that the object in question was called a grope "because of its multitudinous fingers." He went on to warn that "this terminology suggests any number of bad puns," some of which he failed to resist. Over the years research articles have even been rejected because they used the term *grope*.

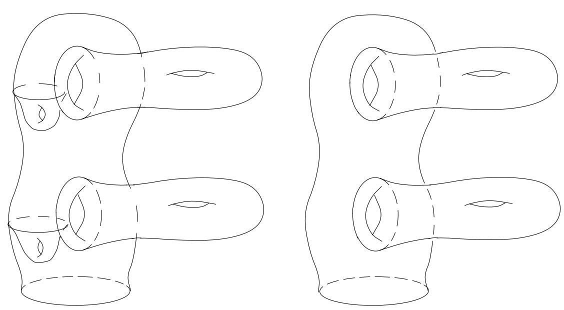

Figure 1. Gropes of height 2 (L) and class 3 (R).

Mathematically, *gropes* are certain 2-dimensional complexes (with one boundary circle) which are unions of *surfaces* (here taken to be compact, connected, oriented 2-manifolds with a single boundary circle). To organize the gluing of these surfaces, we introduce a complexity, the *height* of the grope. For height $h = 1$, a grope is just a surface Σ. For $i = 1, \dots, 2 \cdot \text{genus}(\Sigma)$, let α_i be a full symplectic basis of circles in Σ. Then a grope of height $(h+1)$ is formed by attaching gropes of height h to each α_i along the boundary circles.

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Gropes, therefore, are not quite manifolds, but the singularities that arise are of a very simple type, so that these 2-complexes are in some sense the next easiest thing after surfaces. To motivate the definition of gropes and their complexity, let us next explain a relation to group theory.

Group Commutators and Gropes. A continuous map $S^1 \to X$ (from the circle to any space X) represents an element in the fundamental group $\pi_1 X$. The map extends to a map of a surface (a grope of height 1) to X if and only if it represents a commutator in $\pi_1 X$. This is most easily seen by thinking of a surface Σ of genus g as a (punctured) $4g$-gon with sides identified in pairs. The pattern of these identifications is given by reading the following word along the boundary of the $4g$-gon:

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}.$$

Since the boundary circle of Σ is in the middle of the $4g$-gon, it must equal this commutator. Iterated commutators can similarly be expressed by continuous maps of gropes to X: A map $S^1 \to X$ represents an element in the h-th term of the *derived series* of $\pi_1 X$ if and only if it extends to a continuous map of a grope of height h. Recall that the derived series of a group $G = G^{(0)}$ is defined by iterated commutators $G^{(h+1)} := [G^{(h)}, G^{(h)}]$. A group is *solvable* if this series terminates at 1. There is a close cousin of the derived series, namely the *lower central series* of a group $G = G_1$, defined by $G_{c+1} = [G, G_c]$. A group is *nilpotent* if this series terminates at 1. The reader might imagine how to define certain 2-complexes (with one boundary circle), called *gropes of class c*, such that a map $S^1 \to X$ represents an element in the c-th term of the lower central series of $\pi_1 X$ if and only if it extends to a continuous map of a grope of class c. In fact, such gropes are more general than the ones previously defined, and the terminology has shifted over the years as follows: gropes that have a height h are now also called *symmetric gropes*. Group theory tells us that they have class $c = 2^h$. Not every grope is symmetric, as shown in Figure 1.

Geometric Group Commutators. Once one can describe iterated commutators in $\pi_1 X$ by maps of gropes, one might as well look at *embedded* gropes in order to study more geometric questions. The most direct applications seem to be the most recent ones, namely to knot theory. This is the theory of *embedded* circles in 3-space (rather than continuous maps of a circle as in the case of the fundamental group). Recall that every knot bounds a Seifert surface in 3-space but that only the trivial knot bounds an embedded disk. Thus all of knot theory is created by the difference between a surface and a disk. As we saw, this is just like the difference between a commutator in $\pi_1 X$ and the trivial group element. Gropes give us a way to filter this difference in analogy to iterated commutators in group theory.

Thinking 4-dimensionally, one is led to studying knots in $S^3 = \partial D^4$ which extend to *embeddings* of symmetric gropes of height h into D^4. This gives a filtration of the *knot concordance group,* which was introduced by Cochran, Orr, and the author in 1998. We showed that all the previously known concordance invariants can be recovered for small h. For example, if a knot bounds a symmetric grope of height 4 in D^4, then all its Casson-Gordon invariants vanish. Using von Neumann signatures of solvable covers of the knot complement, it was shown that each of the successive quotients of the terms of this filtration are nontrivial.

Schneiderman showed that all knots with trivial Arf invariant bound (nonsymmetric) gropes of arbitrarily large *class* in D^4. However, if one asks for such a grope to be embedded in 3-space, then a rich obstruction theory arises. It was developed by Conant and the author and is closely related to Vassiliev's knot invariants, with the *class* of the grope corresponding exactly to the *finite type* of the invariant.

See [3] for a survey and references for these 3- and 3.5-dimensional applications of gropes.

A Brief History of Gropes. Grope-like objects first appeared in a 1971 article by Stanko who proved that certain wild embeddings in codimension 3 are limits of tame embeddings. In 1975 Cannon and Ancel extended Stanko's technique to codimension 1. In 1977 Cannon introduced gropes and the *disjoint disks property* to prove several manifold recognition theorems. Among them was the famous *Double Suspension Theorem,* which says that for any homology n-sphere, the double suspension is homeomorphic to the standard $(n+2)$-sphere. (The suspension of a space X is the union, along X, of two cones on X.) The result was extremely surprising, since a single suspension of a manifold X can be a manifold only if X is the standard sphere. Without using gropes, Bob Edwards had proven the Double Suspension Theorem before Cannon in many cases (as well as the Triple Suspension Theorem). Inspired by their success in these problems, Edwards suggested using gropes in 4-dimensional topology. Michael Freedman introduced them in his paper that appeared in the proceedings of the 1983 International Congress of Mathematicians in Warsaw. In that paper, he extended his *Disk Embedding Theorem* from the simply connected case to 4-manifolds with *good* fundamental group. This included finite and cyclic groups (it is still an open question which groups are good, groups of subexponential growth being the most general class known). In [2] the topological theory of 4-manifolds is formulated entirely in terms of symmetric gropes.

It is amusing that the applications of gropes have moved down in dimensions over the years. However, the slogan has always remained the same: If you are looking for a disk, try to find a grope first.

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References

[1] J. Cannon, The recognition problem: What is a topological manifold? *Bull. Amer. Math. Soc.* **84** (1978), 832–866.

[2] M. Freedman and F. Quinn, *The Topology of 4-Manifolds*, Princeton Math. Series, vol. 39, Princeton, NJ, 1990.

[3] P. Teichner, Knots, von Neumann Signatures, and Grope Cobordism, *Proceedings of the International Congress of Mathematicians, Vol. II: Invited Lectures*, (Beijing, 2002), pp. 437–446.

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