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# Symmetric surgery and boundary link maps

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## 1. Introduction

In order to seperate 3-dimensional linking and knotting phenomena, John Milnor introduced the notion of a *link homotopy* [14]. He allowed self-intersections but did not allow different components to cross during a link homotopy. It is clear that any knot is link homotopically trivial but one of the most surprising and unintuitive (see the left hand side of Fig. 1) results of Milnor was that arbitrarily many parallel copies of a knot form a homotopically trivial link. In fact, one knows that any *boundary link* has vanishing $\mu$-invariants and thus it is homotopically trivial by the main result of Milnor. By definition, in a boundary link the components bound disjointly embedded Seifert surfaces. For example, any knot has a Seifert surface and is thus a boundary link. Similarly, parallel copies of a knot bound (disjoint) parallel copies of this Seifert surface.

The notion of a link homotopy makes clearly sense for links in any dimension. In fact, the correct category to work in seems to be the following: A *link map* is a continuous map such that connected components in the source are mapped *disjointly* into the target. A link homotopy between two link maps is then an ordinary homotopy *through link maps*. In this context, a *boundary link map* $L : \amalg_i M_i \to N$ is a link map which is the boundary of a link map $\amalg_i W_i \to N$ of oriented manifolds $W_i$, the "Seifert surfaces" for $L$.

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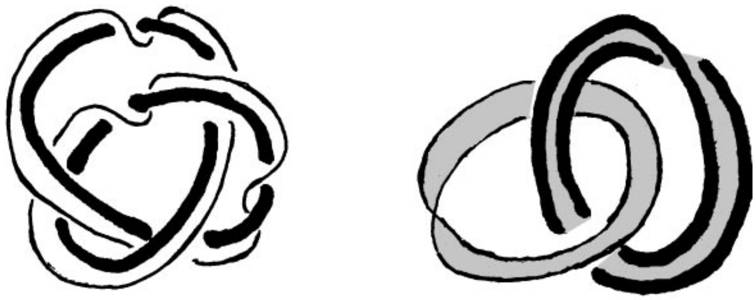

**Fig. 1.** Can you see the null-homotopies for these boundary links?

Answering an old question of Jerry Levine and Dale Rolfsen in the affirmative, our main result is as follows:

**Theorem 1.** *Any boundary link map $L : \amalg_i S^N \to S^{N+2}$ is link homotopic to the unlink.*

In the classical dimension $N = 1$ this can probably be proven along the lines of the embedded case discussed above. However, there is a more geometric argument, explained to the author by Mike Freedman, which uses a procedure known as *symmetric surgery* or *contraction/pushoff* from $4$-manifold theory. It seems to go back to an idea of Bob Edwards. It is this procedure which we generalize to all dimensions. We show that a boundary link map is link concordant to the trivial link and then we utilize our result from [15], namely that link concordance implies link homotopy in codimension $\geq 2$.

*Remark 1.* The attentive reader will have realized that the link on the right hand side of Fig. 1 has linking number $\pm 4$ and is thus *not* homotopically trivial. This is the reason why we used only *oriented* Seifert surfaces in the definition of a boundary link. (This also coincides with established notions, since Seifert used only orientable surfaces to bound knots in $S^3$.) Orientable but non-spin Seifert surfaces will be exploited in Example 1.

In higher dimensions, codimension 2 boundary links gained importance in the 1970's through the work of Cappell and Shaneson, see e.g. [1] or [2]. They observed that exactly for boundary links there is a degree one normal map to the unlink and thus their homology surgery approach works for that class of links. This lead to a complete algebraic computation of the boundary link concordance groups in high dimensions. The answer is in terms of $\Gamma$-groups of the free group (on the number of components of the links). As a consequence, many odd-dimensional boundary links are not concordant to split links, see [2]. In this light Theorem 1. is very interesting even in the embedded case. In contrast, it follows from algebraic facts about $\Gamma$-groups

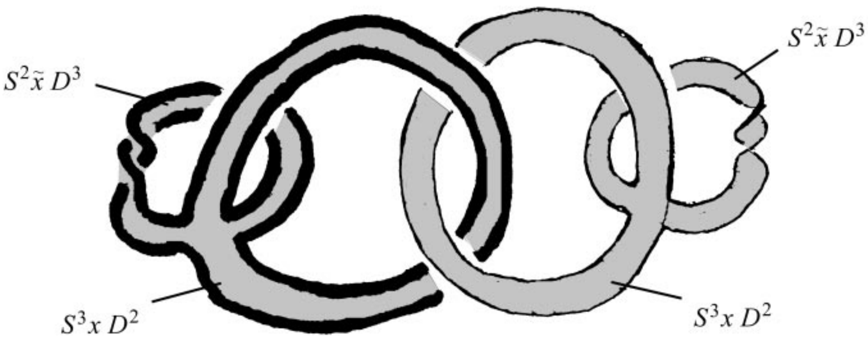


**Fig. 2.** A boundary link $S^4 \amalg S^4 \hookrightarrow S^7$ which is not null-homotopic

that every even-dimensional boundary link is null-concordant and thus also null-homotopic. Our proof of Theorem 1. actually gives a geometric argument that even-dimensional boundary links are null-concordant. It coincides with Kervaire's original argument [8] for knots.

It remains an open problem whether any even dimensional link is null-concordant and it is a very good test question whether they are all null-homotopic. It should also be mentioned that it took until 1991 before Cochran and Orr [3] found examples of links in any odd dimension that are not concordant to boundary links (and with vanishing $\mu$-invariants in dimension 3).

In our proof of Theorem 1., symmetric surgery is only needed for odd dimensions $N$, the even case being easier (as for embedded links). For $N = 2$ and two component link maps, Theorem 1. was proven in [10].

Surprisingly, the situation becomes more complicated in codimension $> 2$:

*Example 1.* There is a boundary link $L : S^4 \amalg S^4 \hookrightarrow S^7$ which is *not* homotopically trivial. In fact, $L$ has non-trivial generalized linking number $\alpha(L) \in \pi_2^{st}$.

The 5-dimensional Seifert surfaces for this link are constructed from plumbings of a 2-sphere and a 3-sphere, where one uses the nontrivial $D^3$-bundles to thicken the 2-spheres. Moreover, the two 3-spheres form a Hopf link in $S^7$, see Fig. 2. Note that in this example the Seifert surfaces do not allow a spin structure. In fact, we have the following result for arbitrary codimensions $\geq 2$:

**Theorem 2.** *Let $L : \amalg_i S^{n_i} \longrightarrow S^{N+2}$, $n_i \leq N$, be a boundary link map with parallelizable Seifert surfaces. Then $L$ is homotopically trivial*

The Seifert surfaces of a boundary link map always refer to the ones which are mapped in disjointly. Now Theorem 1. follows from Theorem 2. together with the following result.

**Proposition 3.** *A boundary link map $L : \amalg_i S^N \to S^{N+2}$ is link homotopic to a boundary link map $L'$ with parallelizable Seifert surfaces.*

The paper is organized as follows. In Sect. 2 we prove Theorem 2. modulo the symmetric surgery part. In particular, this contains the cases $N$ even and $N$ odd, $n_i < N$. In Sect. 3 we introduce symmetric surgery and finish the proof of Theorem 2.. In Sect. 4 we prove Proposition 3. and in the last Sect. 5 we describe Example 1 and prove that it is the smallest one:

**Theorem 4.** *If $n_i \leq 3$ and $n_i \leq N$ then any boundary link map $L : \amalg_i S^{n_i} \to S^{N+2}$ is homotopically trivial.*

In fact, our Example 1 is the unique smallest possible example since the generalized linking number $\alpha : LM^7_{4,4} \to \pi_2^{st} \cong \mathbb{Z}/2$ detects link maps with one embedded component $L_1$: The inclusion of the meridian 2-sphere to $L_1$ induces a homotopy equivalence $S^2 \simeq S^7 \setminus L_1(S^4)$ and thus the second component $L_2$ is an element of $\pi_4(S^2) \cong \pi_2^{st}$.

Here $LM^n_{p,q}$ denotes the set of link homotopy classes of link maps $S^p \amalg S^q \to S^n$. It was shown in [11] that the connected sum operation gives a well-defined addition on this set if $p, q \leq n-2$. Up to link concordance, reflection in a hypersphere provides an inverse for any given link map. Since link concordance implies link homotopy [15] it follows that the sets $LM^n_{p,q}$ actually form abelian groups if $p, q \leq n-2$. For example, the groups $LM^{q+2}_{1,q}$ are infinite cyclic and detected by the linking number for all $q \geq 1$. In the metastable range, Koschorke [12] has an exact sequence which relates the groups $LM^n_{p,q}$ to classical homotopy theory and thus makes many calculations possible. Very recently, the infinitely generated group $LM^4_{2,2}$ was computed, see [10] [16] [13].

One of the nice features of working with link maps up to link homotopy is that the categories are irrelevant: One may always assume that the maps in question are smooth or PL. Without further notice, we always work in the smooth setting.

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## 2. Parallelizable Seifert surfaces

In this section we prove Theorem 2. up to the point where symmetric surgery is needed. This will only be the case in the presence of a codimension 2 component and $N$ odd.

The idea of the proof is simple: We try to replace the disjoint Seifert surfaces in $S^{N+2}$ by disjoint disks in $D^{N+3}$. Then the result follows from "link concordance implies link homotopy" [15]. In fact, we will construct link maps of cobordisms, rel boundary, between the Seifert surfaces and disks. These cobordisms will have handles only up to the middle dimension which will guarantee the disjointness properties. Thus the problem splits into two steps: First we have to find suitable abstract cobordisms between the Seifert surfaces and disks. This is where we use the assumption that the Seifert surfaces are parallelizable. Then these cobordisms will be mapped disjointly into $D^{N+3}$.

Let $W^{n+1}$ be one of the parallelizable Seifert surfaces for our link map $L$. As explained in the classic [9] one may do framed surgeries on k-spheres in the interior of $W$ to get a highly connected manifold $W'$. Here the framing refers to a trivialization of the *stable normal* bundle which is carried along in all constructions. More precisely, one can add k-handles to $W \times 1 \subset W \times I$, $k = 1, \dots, [n/2]$, to obtain a framed $(n+2)$-manifold $N$ with lower boundary $W$ and upper boundary a $[n/2]$-connected manifold $V$. (In addition, $N$ has a "vertical" boundary $S^n \times I$ since $\partial W = S^n = \partial V$.)

Then one runs into the surgery obstruction in $L_{n+1}(e)$ for making $V$ contractible by surgeries in the middle dimension. Note that for $n \geq 4$ the h-cobordism theorem implies that a contractible manifold with boundary an n-sphere must be an $(n+1)$-disk (see Remark 2 for the cases $n < 4$).

The surgery obstruction groups are 4-periodic in the dimension and in our simply-connected case they were described in [9]: For even $n$ the groups $L_{n+1}(e)$ are trivial. If $n+1$ is divisible by 4 then $L_{n+1}(e) \cong \mathbb{Z}$ and the surgery obstruction is the signature. Finally, if $n+1 \equiv 2 \mod 4$ then $L_{n+1}(e) \cong \mathbb{Z}/2$ and the surgery obstruction is the Kervaire invariant.

A priori, these surgery obstructions depend on the highly connected manifold $V$ but it turns out that they can be read off from the original manifold $W$. In our case this is easy to see since the signature and the Kervaire invariant are unchanged under framed cobordisms. These invariants are not importamt in our context of link homotopy since they involve only one component at a time. More precisely, we may locally add to each component of $L$ the oppositely oriented knot map it represents, together with a copy of the oppositely oriented Seifert surface $-W$. Since knot maps are null homotopic, this addition can be achieved by a link homotopy taking place in small disjoint $(N+2)$-balls, one for each component. It has the effect that the new Seifert surfaces have trivial surgery obstructions. Therefore, there is a sequence of surgeries on framed $k$-spheres in the interior of $W$, $k = 1, \dots, [n/2]+1$, leading from $W$ to $D^{n+1}$.

*Remark 2.* We should say some words about low dimensions, where surgery does not usually work: If the Seifert surfaces are 2-dimensional then there is always a $1/2$-basis of framed circles which lead to the 2-disk. In dimension 3 one knows that any closed oriented 3-manifold is obtained from $S^3$ by surgeries on an evenly framed link [7]. This is exactly the statement which we need for $W^3$ above. Finally, consider a simply-connected $4$-dimensional Seifert surface $W$. We still assume that $W$ is parallelizable and has zero signature. By [17] this implies that for some $r, s \in \mathbb{N}$ there is a diffeomorphism

$$W \# r(S^2 \times S^2) \cong s(S^2 \times S^2) \smallsetminus \text{ small open } 4\text{-ball } D_0.$$

The interior connected sum of $W$ with copies of $S^2 \times S^2$ can be achieved locally by adding copies of the unknotted pair $(S^{N+2}, S^2 \times S^2)$. This does not change the original link $L$ and it enables us to assume that $W$ is a standard manifold. We may clearly pick framed 2-spheres in $W = s(S^2 \times S^2) \smallsetminus D_0$ such that surgeries lead to $D^4$.

In the remainder of this section we will outline a procedure how to ambiently realize these abstract surgeries in an $(N+3)$-ball bounding our $S^{N+2}$. The necessary disjointness properties will follow from general position, except for $N$ odd and $k$ equal to the middle dimension. That's were symmetric surgery is needed.

We assume that the Seifert surfaces $W_i^{n_i+1}$ of the link map $L$ have the property above. The next step is to improve the maps $W_i \to S^{N+2}$. Since $W_i$ are parallelizable, it follows from Hirsch-Smale immersion theory [6] that there are immersions $W_i \looparrowright S^{N+2}$ with trivial normal bundles and arbitrary close to the original maps in the $C^0$-topology. Hence we may chose these immersions to be still disjoint from each other and such that the new link map is link homotopic to the original one. For each component $W_i$, pick a $(N - n_i + 1)$-frame $\nu_i$ for the normal bundle of $W_i^{n_i+1}$ in $S^{N+2}$. Together with the standard normal vector field for $S^{N+2} \subset \mathbb{R}^{N+3}$ the frame $\nu_i$ is a framing of the stable normal bundle of $W_i$. We can therefore use it to control the framings of our surgeries.

*Remark 3.* Since $W_i$ is homotopy equivalent to an $n$-complex the tangent bundle is stably trivial if and only if it is trivial, i.e. $W_i$ is parallelizable. It is always true that the stable tangent bundle is trivial if and only if the stable normal bundle is trivial, since they are stable inverses of each other. Note however, that a trivialization of the tangent bundle is more information than just a stable trivialization, i.e. a framing. This distinction will only become important for surgeries in the middle dimension, see Remark 4.

We may clearly assume that the $W_i$ are connected (otherwise disregard the closed components). Therefore, the first abstract surgery which we want to realize ambiently would be on an embbedding $f : S^1 \hookrightarrow W_i$. Note that the composition

$$S^1 \hookrightarrow W_i \looparrowright S^{N+2}$$

is an immersion but it might have self-intersections. It clearly extends to map $F : D^2 \to D^{N+3}$ which sends the interior of $D^2$ into the interior of $D^{N+3}$.

By jet transversality [5, II.5] this may be assumed to be an immersion (and actually an embedding away from the boundary but this won't be used). All we need is that $F$ has a normal bundle. Then we have to extend the normal frame $\nu_i$ from $f(S^1)$ to a normal frame on $F(D^2)$.

**Lemma 1.** *This extension of the normal frame $\nu_i$ exists.*

*Proof.* The extension problem is as follows: Given the trivial bundle $D^2 \times \mathbb{R}^{N+1}$ and an $(N - n_i + 1)$-frame on $S^1 \subset D^2$. Does it extend to an $(N - n_i + 1)$-frame on $D^2$ ? The most elementary way to prove that the answer is "yes" is to inductively construct the frame, one vector field at a time: A non-vanishing vector field in a trivial bundle is just a map into the unit sphere of the fibre. Thus the first vector field extends because $\pi_1(S^N) = 0$. Then we work in the orthogonal complement of that first vector field. Thus the second vector field exists because $\pi_1(S^{N-1}) = 0$. Continuing in this manner the last condition is $\pi_1(S^{(N+1)-(N-n_i+1)}) = 0$. Actually, all the conditions that we need follow from

$$(N+1) - (N - n_i + 1) > 1 \quad \iff \quad n_i > 1$$

which is clearly satisfied (otherwise we wouldn't be doing surgery on a circle). □

We now attach an ambient 2-handle to $W_i$ as follows: First thicken $W_i$ slightly in the radial direction into $D^{N+3}$. Then take the union with the disk-bundle of the orthogonal complement of the vector-bundle spanned by the above extension of $\nu_i$ over $F(D^2)$. This disk-bundle has fibre dimension $n_i$ and thus the result is an immersion of a framed $(n_i + 2)$-manifold $N_i \looparrowright D^{N+3}$ with a vertical boundary $S^{n_i} \times I$ and two horizontal boundary components. The "upper" one is $W_i$ and we denote the "lower" one by $V_i$, the result of the surgery on the circle $f(S^1)$ in $W_i$. By construction, the normal frame $\nu_i$ extends to a normal frame for $N_i$ in $D^{N+3}$ which we still denote by $\nu_i$. In particular, $N_i$ is a framed cobordism (rel $\partial$) between $W_i$ and $V_i$.

In the next step we want to do surgery on a 2-sphere in $V_i$. Note that the setup is slightly different from the first step because $V_i$ is not immersed in $S^{N+2}$ any more, nor are we sure that it lies in some level sphere of $D^{N+3}$. However, the above argument nevertheless goes through. We will give the inductive argument for $f : S^k \hookrightarrow V_i$ for some $1 < k < \frac{n_i+1}{2}$. The first step is to extend the composition

$$S^k \hookrightarrow V_i \looparrowright D^{N+3}$$

to a map $F : D^{k+1} \to D^{N+3}$ which in the interior is disjoint from all previously constructed $N_i$. This extension exists by general position using the fact that

$$2(k+1) < n_i + 3 \leq N + 3$$

and that $N_i$ has only handles of index $\leq k+1$ (since we might already have realized an ambient surgery on another $S^k \hookrightarrow V_i$). By jet transversality we may again assume that $F$ is an immersion. The extension problem of the normal data is now as follows: In the trivial bundle $D^{k+1} \times \mathbb{R}^{N+2-k}$ we have given the $(N-n_i+1)$-frame $\nu_i$ restricted to $S^k$ and we want to extend it to an $(N-n_i+1)$-frame on $D^{k+1}$. Exactly as in Lemma 1. one shows that there is a solution to this extension problem if $\pi_k(S^{(N+2-k)-(N-n_i+1)}) = 0$. This certainly follows from

$$(N+2-k)-(N-n_i+1) > k \quad \Longleftrightarrow \quad 2k < n_i + 1$$

which is exactly our assumption that we are doing surgery below the middle dimension. We can thus attach an ambient k-handle to the previously constructed cobordism $N_i$ as follows: Just take the union of $N_i$ with the disk-bundle of the orthogonal complement of the vector-bundle spanned by the above extension of $\nu_i$ over $F(D^{k+1})$. The result is an immersion of a framed $(n_i+2)$-manifold $N_i' \looparrowright D^{N+3}$ with a "vertical" boundary $S^{n_i} \times I$ and two "horizontal" boundary components. One component is $V_i$ and the other is the result of the surgery on the circle $f(S^k)$ in $V_i$. By construction, the normal frame $\nu_i$ extends to a normal frame for $N_i'$ in $D^{N+3}$.

Summarizing the previous steps, we now have disjoint immersions $N_i^{n_i+2} \looparrowright D^{N+3}$ which are constructed from the original Seifert surfaces $W_i^{n_i+1}$ by attaching handles of index $\leq k_i+1$, where $k_i$ is the largest integer $< \frac{n_i+1}{2}$. Together with the vertical boundaries $S^{n_i} \times I$ the lower horizontal boundaries $V_i$ of $N_i$ bound the original link components $L_i : S^{n_i} \to S^{N+2}$. The surgeries were done in order to kill the homotopy groups of $W_i$ and therefore, the $V_i$ are $k_i$-connected.

If $n_i$ is even then $k_i = n_i/2$ which implies by Poincaré duality that $V_i$ is contractible. By the discussion about abstract surgeries at the beginning of this section, we actually know that $V_i$ is an $(n_i+1)$-ball.

*Remark 4.* It should be pointed out that the last surgeries on $k_i$-spheres in $V_i$ are in the middle dimension. In particular, the stable framing $\nu_i$ does not quite determine the (unstable) framing of the normal bundle of $S^{k_i} \hookrightarrow V_i$. In fact, there is an exact sequence

$$\mathbb{Z} \cong \pi_{k_i+1} S^{k_i+1} \longrightarrow \pi_{k_i} SO(k_i+1) \overset{j_*}{\twoheadrightarrow} \pi_{k_i} SO(k_i+2) \cong \pi_{k_i} SO$$

and in the proof that odd-dimensional $L$-groups vanish one really uses the freedom of being able to pick arbitrary attaching maps in the kernel of $j_*$. However, by changing the last vector field in the frame $\nu_i$ along $D^{k_i+1}$ (corresponding to an element in $\pi_{k_i+1} S^{k_i+1}$) we may arrange that any such unstable framing extends over our ambient handle.

If $n_i$ is odd then $k_i = \frac{n_i-1}{2}$ and thus by the same discussion there are abstract surgeries on framed $(k_i+1)$-spheres in $V_i$ which lead to $(n_i+1)$-balls $B_i$. We will next check that in some cases a variant of the above procedure still goes through to do these surgeries ambiently.

**Lemma 2.** *If $N$ is even then ambient $k_i$-surgeries on $V_i$ can be done such that the resulting balls $B_i$ are mapped disjointly into $D^{N+3}$ and thus the link $L$ is null-homotopic.*

*Proof.* We may assume that $n_i$ are odd. If $N$ is even then it follows from $n_i < N$ that

$$k_i + 1 = \frac{n_i+1}{2} \leq \frac{N}{2}.$$

To do the first step of the above construction we need to find an extension $F : D^{k_i+2} \to D^{N+3}$ of a composition

$$f : S^{k_i+1} \hookrightarrow V_i \looparrowright D^{N+3}.$$

Moreover, the interior of $F$ should be disjoint from all $N_i$. Since $N_i$ is built out of handles of index $\leq k_i+1$ the disjointness follows from general position since

$$(k_i+2)+(k_i+1) = 2(k_i+1)+1 \leq N+1.$$

In fact, the extra leverage of one dimension can be used to make all necessary extension $F = F_j$ disjoint from each other. In the dimension count for the normal data $\nu_i$ we see that we might not be able to extend the framings in this middle dimension. However, since this is the very last step of the argument, we don't need this extension: All we need is a *map* $\bar{F} : D^{k_i+2} \times D^{n_i-k_i}$ which extends the given $\bar{f} : S^{k_i+1} \times D^{n_i-k_i} \hookrightarrow V_i$ to $D^{N+3}$ and equals $F$ on the core $D^{k_i+2} \times \{0\}$. But this is a trivial extension problem since $D^{k_i+2} \times D^{n_i-k_i}$ collapses to the subset

$$S^{k_i+1} \times D^{n_i-k_i} \cup D^{k_i+2} \times \{0\}.$$

In particular, away from the boundary $\bar{f}$ in $V_i$, the image of $\bar{F}$ is the same as that of $F$ and thus all the disjointness properties stay satisfied. □

Assume that $N$ is odd. If some component satisfies $n_i < N$ then we get again

$$k_i + 1 = \frac{n_i+1}{2} \leq \frac{N}{2}.$$

Therefore, the same argument as in Lemma 2. shows that for all components of dimensions $n_i < N$ we may complete the last step of the ambient surgery. For the codimension 2 components the argument still shows that the last $(k_i+1)$-handles can be mapped into $D^{N+3}$ in a way such that the interiors miss the previously constructed $N_i$. However, in this case we have $n_i = N$ and thus

$$k_i + 2 = \frac{n_i + 1}{2} + 1 = \frac{N+3}{2}.$$

By general position these last $(k_i + 2)$-handles are immersed and will meet in $D^{N+3}$ in a finite number of points. In the next section we will explain the symmetric surgeries which will remove these intersection points. These surgeries take place in a neighborhood of the $(k_i + 2)$-handles and thus we can work on a pure codimension 2 link map in Sect. 3.

## 3. Symmetric surgery

We first describe the model spaces involved. For the 4-dimensional case compare [4, §2.3]. In order to avoid smoothening a lot of corners, we will just describe the topology of the spaces and not their smooth structure. The following evident lemma will be used to show that most of our models are homeomorphic to balls (which can be equipped with a smooth structure if necessary).

**Lemma 3.** *Let $N$ be a manifold and $M \subset \partial N$ a compact codimension 0 submanifold. Then $N$ is homeomorphic to $N \cup_{M \times 0} M \times [0,1]$.*

All the isotopies that will be needed in this section surprisingly follow from the following obvious lemma.

**Lemma 4.** *Let $S^{k-1} \subset S^k = \partial D^{k+1}$ be an unknotted sphere. Then the closures of the components of $S^k \setminus S^{k-1}$ are two $k$-balls which are isotopic (rel $\partial$) in $D^{k+1}$.*

Let $D_e$ and $D_f$ be two standard n-balls. Consider the following $(2n+1)$-ball:

$$B := D_e \times [-2, 2] \times D_f \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n.$$

Denote by $S_e$ the n-sphere which is the boundary of $h_e := D_e \times [-2, 0] \times 0$ and similarly $S_f := \partial h_f$ where $h_f := 0 \times [0, 2] \times D_f$. Define the following subsets of $B$, see Fig. 3:

$$H := (S_e \times D_f) \cup_{D_e \times 0 \times D_f} (D_e \times S_f)$$

and

$$C := (h_e \times \partial D_f) \cup_{D_e \times 0 \times \partial D_f} (D_e \times 0 \times D_f) \cup_{\partial D_e \times 0 \times D_f} (\partial D_e \times h_f).$$

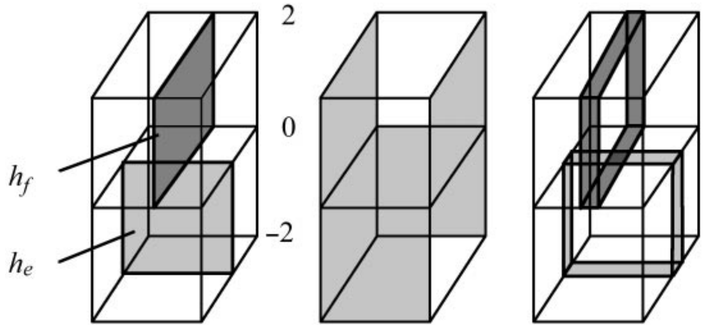


**Fig. 3.** The model spaces $B$, $C$ and an isotopic version of $H$

*Remark 5.* In Fig. 3 we show $h_e$ and $h_f$ inside the ball $B$. Clearly a $(2n+1)$-dimensional thickening of $h_e \cup h_f$ in $B$ is isotopic to the whole ball $B$ (rel $h_e \cup h_f$). It is the image of $H$ under this isotopy which we have drawn on the right hand side of Fig. 3. This has purely artistic reasons. Note however, that the logic in later applications is reversed: We'll first find $h_e$ and $h_f$, meeting in a single point in some manifold and then we thicken the union to our model ball $B$, using the isotopy above.

By two applications of Lemma 3. we see that $C$ is a $2n$-ball which is the model for the *contraction*. Moreover, $H$ is homotopy equivalent to the "hyperbolic" wedge $S_e \vee S_f$ and has the same boundary as $C$, i.e. a $(2n-1)$-sphere. Note that $B$ is obtained from $H$ by filling back in the two $(n+1)$-handles $h_e \times D_f$ and $D_e \times h_f$. If one fills in only one of the two handles, say $h_e \times D_f$, then one obtains the model for a surgery on $S_e \subset H$. This surgery changes $H$ to the ball (by Lemma 3.)

$$\partial_e B := (h_e \times \partial D_f) \cup_{D_e \times 0 \times \partial D_f} (D_e \times (S_f \smallsetminus \mathring{D}_f)).$$

It has the same boundary as $C$ and $H$. Moreover, an application of Lemma 4. inside the ball $D_e \times [-2, 0] \times D_f$ shows that $\partial_e B$ and $C$ are isotopic (rel $\partial$).

Symmetrically, surgery on $S_f$ leads to the ball $\partial_f B$ which is again isotopic (rel $\partial$) to $C$. In addition, one has

$$\partial_e B \cap \partial_f B = \partial H \text{ and } \partial B = \partial_e B \cup_{\partial H} \partial_f B \ ,$$

see Fig. 4.

*Remark 6.* The above discussion points out the symmetry between the two possible surgeries on $S_e$ respectively $S_f$. It is an elementary explanation of the Morse cancellation of the two handles $h_e$ and $h_f$.

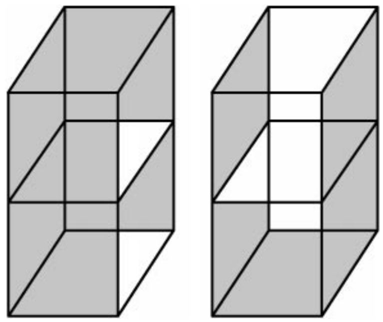

**Fig. 4.** The surgeries $\partial_e B$ and $\partial_f B$

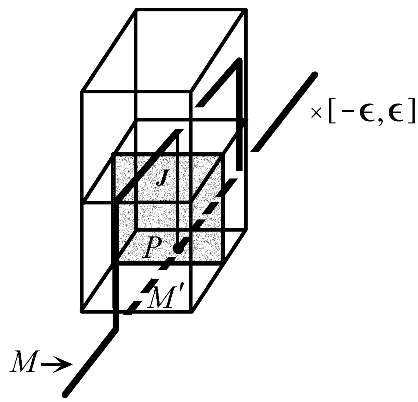


**Fig. 5.** Pushing a sheet $M'$ off the contraction

The model for the *pushoff* part of the construction takes place in a neighborhood of $B \times \mathbb{R}$ in $\mathbb{R}^{2n+1} \times \mathbb{R}$. Pick a point $p$ in the interior of $h_e$. It is the transverse intersection point between $h_e$ and the sheet $M' := p \times D'_f \times [-\epsilon, \epsilon]$ for any small $\epsilon > 0$ and an $n$-disk $D'_f \subset \mathbb{R}^n$ of radius slightly bigger than 1. We want to replace the sheet $M'$ by a sheet $M$ disjoint from the contraction $C$ and with $\partial M = \partial M'$. For some small $\delta > 0$, choose an arc $J$ in $D_e \times (-2, \delta] \times 0$ from the point $p$ to a point $q = (q_1, \delta, 0)$ with $q_1$ in the interior of $D_e$. Consider the $(n+2)$-ball

$$J \times D'_f \times [-\epsilon, \epsilon]$$

We define the new sheet $M$ by asking that the boundary of this ball is the union of $M$ and $M'$ along their common boundary $\partial M'$. By Lemma 4. this ball actually defines an isotopy from $M'$ to $M$ (rel $\partial$) which we refer to as *pushing* $M'$ *off the contraction* $C$. In fact, one easily checks that $M$ is disjoint from $C$, see Fig. 5. Symmetrically, one may push any sheet intersecting $h_f$ transversely in a point off the contraction. Note however, that two pushed off sheets, one off $h_e$ the other off $h_f$, will intersect in two additional points.

The usual model for surgery starts with an embedding $\varphi : S^p \times D^q \hookrightarrow V$. Then one attaches an $(p+1)$-handle to the image of $\varphi$ in $V \times 0 \subset V \times [0,1]$. This gives the *trace of the surgery* with upper boundary $V$ and lower boundary a manifold obtained from $V$ by cutting out the interior of the image of $\varphi$ and replacing it by $D^{p+1} \times S^{q-1}$.

Let $V$ have dimension $2n$ and let $p = q = n$. Then we are doing surgery in the middle dimension and there is an important additional fact to use: Whenever the surgery obstruction vanishes, the spheres $\varphi : S^n \times D^n \hookrightarrow V$ may be assumed to have framed embedded *dual spheres*. In other words, the embedding $\varphi$ can be extended to an embedding of our hyperbolic model $H \hookrightarrow V^{2n}$. In dimensions $2n > 4$ this follows from the fact that a quadratic form represents zero in $L_{2n}(e)$ if and only if it is hyperbolic. For small dimensions $(n = 1, 2)$, see Remark 2.

We may thus attach *two* $(n+1)$-handles to $V \times [0,1]$, one to $S_e \subset V \times 0$, the other to $S_f \subset V \times 1$. This gives a manifold $X$ together with an embedding of our model ball $\Phi : B \hookrightarrow X$: As explained in Remark 5, all we need are embeddings of $h_e$ and $h_f$ which meet in a single point (on their boundaries). Such embeddings are provided by the cores of the handles together with product structures $\partial h_e \times [0, \frac{1}{2}]$ respectively $\partial h_f \times [\frac{1}{2}, 1]$ in $V \times [0,1]$. The arguments above show that $X$ is homeomorphic to $Y^{2n} \times [0,1]$ with $Y \times 0$ mapping to surgery on $S_e$, $Y \times 1$ mapping to surgery on $S_f$, and $Y \times \frac{1}{2}$ identified with *symmetric surgery* or *contraction* of $H$. By definition, this is obtained from $V$ by removing the image of $H$ and replacing it by the contraction $C$.

We are now in the position to finish the proof of Theorem 2.. Let us briefly recall the set-up from the end of Sect. 2. The dimension $N$ is odd and we write it as $N = 2n - 1$. Thus our pure codimension 2 link map is $L : \amalg_i S^{2n-1} \looparrowright S^{2n+1}$ and the Seifert surfaces $W_i$ are $2n$-dimensional. Moreover, we have disjoint immersions $N_i^{2n+1} \looparrowright D^{2n+2}$ with a non-vanishing normal vector field $\nu_i$. The manifolds $N_i$ have vertical boundaries $S^{2n-1} \times I$, upper boundaries $W_i$ and lower boundaries $V_i$ which are $(n-1)$-connected. Moreover, there are framed n-spheres $e_j, f_j \subset \amalg_i V_i,\ j = 1, \dots, J$, with geometric intersections

$$e_j \cap e_k = \emptyset = f_j \cap f_k, \quad e_j \cap f_k = \delta_{j,k},$$

and such that surgery on all $e_j$ makes the $V_i$ into $2n$-balls. Alternatively, we will do symmetric surgery on the hyperbolic pairs $e_j, f_j$! Finally, there are immersions $E_j, F_j : D^{n+1} \looparrowright D^{2n+2}$ with boundaries $e_j, f_j$ and with interiors disjoint from all $N_i$. We may assume that the disks $E_j, F_k,\ j,k = 1, \dots, J$, intersect and self-intersect transversely in finitely many points $p_r$.

Pick any ordering $j = 1, 2, \dots, J$. Thicken the first hyperbolic pair $e_1, f_1$ to our model $H$ in $V_i$, identifying $e_1$ to $S_e$ and $f_1$ to $S_f$. We may clearly assume that the composition $H \subset V_i \looparrowright D^{2n+2}$ is an embedding on the central square $D_e \times 0 \times D_f$ since this is just a thickening of the wedge point $e_1 \cap f_1$. We would like to ambiently attach our two handles to $V_i \times [0,1]$ by identifying $E_1$ to $h_e$ and $F_1$ to $h_f$. We may use the direction "into" $N_i$ as the $[0,1]$-direction (and completely disregard $N_i$ otherwise). However, the thickenings of $e_1, f_1$ in $V_i$ might not extend to a thickening of $E_1$ respectively $F_1$ in $D^{2n+2}$. In fact, there is a relative Euler number which is the obstruction to extending the normal vector field $\nu_i$ from $V_i$ to (say) $E_1$. But we can extend to a vector field with only finitely many zeros which we may assume to be distinct from the points $p_r$. In particular, the thickenings of $E_1, F_1$ exist in the neighborhood of disjoint arcs $J_r$ that connect the intersection points $p_r$ to the boundary as in the model pushoff discussion. Moreover, the normal direction $\nu_i$ can be used as the last scoordinate in $[-\epsilon, \epsilon]$ of the sheet $M'$ in the model pushoff.

As in the proof of Lemma 2. we can extend these thickenings to *maps* $\bar{E}_1, \bar{F}_1 : D^{n+1} \times D^{n+1} \to D^{2n+2}$. This leads to a map $\Phi$ of our model $B$ to $D^{2n+2}$.

The next step is to contract $H$ to $C$ in the model $B$ and map it forward into $D^{2n+2}$ by $\Phi$. Then push all disks $E_j, F_j, j \geq 2$, off the contraction. This will introduce more intersections among these disks but makes them disjoint from the contraction. We repeat the above process for the hyperbolic pair $e_2, f_2$. After contracting it, we push all disks $E_j, F_j, j \geq 3$, off the second contraction and continue working on the pair $e_3, f_3$. After finitely many steps, all pairs $e_j, f_j$ are contracted and these contractions map disjointly into $D^{2n+2}$. This gives the desired disjoint maps of $2n$-disks bounding the original link map $L$. □

## 4. Framing Seifert surfaces in codimension 2

*Proof of Proposition 3.* By immersion theory [6] we may assume that the link map $L$ is an immersion since the tangent bundles of spheres are stably trivial. By jet transversality we may also assume that the maps $W_i \to S^{N+2}$ are smooth and immersions with normal crossings away from submanifolds called $S_r$ in [5, Ch.II.5]. Here $r$ is the corank of the differential and so $r = 0$ corresponds to points where $W_i$ is immersed. Moreover, the codimension of $S_r$ in $W_i$ is $r(r+1)$ [5, Thm.II.5.4] which is $> 1$ for $r \geq 1$.

**Lemma 5.** *Under these conditions there is an epimorphism*

$$\phi : \pi_1(S^{N+2} \smallsetminus L) \longrightarrow F_n$$

*which sends meridians of $L$ to $n$ free generators $x_i$ of the free group $F_n$.*

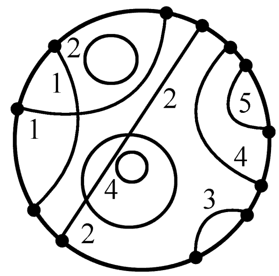


**Fig. 6.** A possible configuration of circles and arcs in $\Delta$

*Proof.* The homomorphism $\phi$ is obtained by putting a circle in $S^{N+2} \smallsetminus L$ into general position with $W_i$ and then reading off the oriented intersections as a word in $x_i^{\pm 1}$. (By the dimension count above, a circle only intersects embedded sheets of $W_i$ transversely in points.) To prove that $\phi$ is well-defined it suffices to show that a circle which bounds a disk in $S^{N+2} \smallsetminus L$ reads off the trivial word: Put the 2-disk $\Delta$ into general position with the $W_i$. Since the singularities of $W_i$ are of codimension $> 2$ in $S^{N+2}$ we may assume that $\Delta$ misses them. Actually, there is one sort of codimension 2 singularities, namely the self-intersection of two embedded sheets of one $W_i$. Therefore, the intersection

$$\Delta \cap (\bigcup_i W_i)$$

consists of a union of oriented circles and arcs, labelled by the indices $i = 1, \ldots, n$, see Fig. 6.

Moreover, only circles and arcs labelled by the same index may cross (in general position) because the $W_i$ are disjoint. Then an innermost argument shows that the word on the boundary of $\Delta$ is trivial. □

Let $\mathcal{W}$ denote a wedge of $n$ circles. Since $\mathcal{W}$ is a $K(F_n, 1)$, Lemma 5. gives a continuous map $\Phi : S^{N+2} \smallsetminus L \to \mathcal{W}$ which induces $\phi$ on the fundamental group. Let $N_i$ be a small open regular neighborhood of the i-th component $L_i$. We assume that $N_i$ do not meet each other and denote by $N$ the disjoint union of all $N_i$. Then we have a continuous map $\Phi : S^{N+2} \smallsetminus N \to \mathcal{W}$ which we smoothen in the complement of the inverse image of the wedge point in $\mathcal{W}$. Consider "anti-basepoints" $p_i \in \mathcal{W}$, i.e. regular values of $\Phi$, one for each circle factor $i = 1, \ldots, n$. Then $\Phi^{-1}(p_i)$ are disjoint framed codimension 1 submanifolds of $S^{N+2} \smallsetminus N$ with

$$M_i := \partial \Phi^{-1}(p_i) = \Phi^{-1}(p_i) \cap \partial N_i.$$

We consider the framed codimension 1 submanifold $M_i$ of $\partial N_i = \partial \bar{N}_i$ as an element

$$[M_i] \in [\partial N_i, MO(1)] \cong [\partial N_i, S^1] \cong H^1(\partial N_i; \mathbb{Z}) \cong \text{Hom}\,(\pi_1(N_i), \mathbb{Z}),$$

the group of framed cobordism classes of codimension 1 submanifolds in $\partial N_i$. By definition, $[M_i]$ is the image of a generator under the composition

$$(*) \qquad \mathbb{Z} \cong H_N(N_i) \cong H^1(S^{N+2} \setminus N_i) \xrightarrow{res} H^1(\partial N_i).$$

Assume that there are framed immersions $V_i \looparrowright N_i$ such that $\partial V_i = L_i \amalg -M_i'$ with $[M_i'] = [M_i] \in H^1(\partial N_i)$. Then the proof of Proposition 3. is finished by taking as the disjoint parallelizable Seifert surfaces for $L$ the unions of $V_i$ and $\Phi^{-1}(p_i)$ along framed cobordisms between $M_i'$ and $M_i$ in small collars $\partial N_i \times I$.

Now observe that the existence of the immersions $V_i$ is a question about one component at a time because one may take $V_i$ to be the transverse intersection of $N_i$ with any framed immersed Seifert surface for the component $L_i$. As before this will have the desired property that the framed cobordism class of $V_i \cap \partial N_i$ is the image of a generator under the composition (*) above.

Thus Proposition 3. is implied by the observation that a more careful application of immersion theory indeed shows that the original maps $L_i : S^N \to S^{N+2}$ are arbitrary close in the $C^0$-topology to immersions which extend to immersions $D^{N+1} \looparrowright D^{N+2}$. More precisely, let $U_i$ be disjoint open neighborhoods of $L_i(S^N)$. Immersion theory says that any regular homotopy class of bundle monomorphisms $TS^N \rightarrowtail L_i^*(TU_i)$ is realized by an immersion homotopic to $L_i$ in $U_i$. We consider the composition

$$\begin{aligned} TS^N &\rightarrowtail TS^N \oplus 1 \cong S^N \times \mathbb{R}^{N+1} \\ &\overset{i}{\rightarrowtail} S^N \times \mathbb{R}^{N+2} \cong L_i^*(TU_i) \longrightarrow L_i^*(TS^{N+2}), \end{aligned}$$

where $i$ is chosen to be *constant* in the $S^N$-factor. Picking any extension of $L_i$ to a map $D^{N+1} \to S^{N+2}$, this composition extends to a bundle monomorphism over $D^{N+1}$. $\square$

## 5. The 7-dimensional example

We first describe Example 1 from Fig. 2 in more detail. Consider a (necessarily unknotted) 2-sphere $S^2 \subset S^7$. Write the trivial normal bundle as

$$\nu(S^2, S^7) = H \oplus H \oplus 1,$$

where $H$ is the (two-dimensional) Hopf-bundle over $S^2$. Let $E$ be the disk-bundle of $H \oplus 1$. We may plumb E together with an unknotted $S^3 \times D^2$ in $S^7$. This gives a 5-manifold $M^5 \subset S^7$ whose boundary is an unknotted 4-sphere. In fact, we may do ambient surgery on the 3-sphere in M to get a 5-disk $B^5 \subset S^7$ with $\partial B^5 = \partial M^5$ . Note however, that we may not surger the 2-sphere in $M^5$ because it has a non-trivial normal bundle!

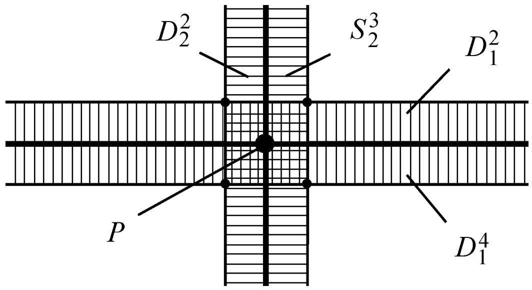


**Fig. 7.** A neighborhood of the intersection point $p$

To construct the link $L : S^4 \amalg S^4 \hookrightarrow S^7$ in Example 1 we take two disjoint copies $M_1, M_2$ of the 5-manifold $M$ above such that the two 3-spheres form a Hopf-link in $S^7$. Then $L := \partial M_1 \amalg \partial M_2$ is a boundary link but it is not homotopically trivial. The *generalized linking number* [11]

$$\alpha : LM^7_{4,4} \longrightarrow \Omega_2^{fr} \cong \mathbb{Z}/2$$

satisfies $\alpha(L) \neq 0$ which can be seen as follows. We bound the first component $L_1$ by the 5-ball $B_1^5$ as above. This is a framed manifold and thus by definition

$$\alpha(L) = B_1^5 \cap L_2(S^4) \in \Omega_2^{fr}$$

where the (normal) framing on the intersection is the union of the unique framing on $B_1^5$ and the normal framing on $L_2(S^4)$. By convention, this framing is the stable difference of two (once and for all fixed) trivializations of $TS^k \oplus 1$ for $k = 4, 7$.

Since the two 3-spheres $S_1^3, S_2^3$ in the definition of $L$ form a Hopf-link, we may assume that the 4-disk $D_1^4$ with $\partial D_1^4 = S_1^3$ which was used to get from $M_1$ to $B_1^5$, intersects $S_2^3$ in a single point $p \in S^7$. A neighborhood of $p$ is pictured in Fig. 7.

It contains the 5-ball $U_2 \times D_2^2 \subset M_2$ where $U_2$ is a small 3-ball around $p$ in $S_2^3$. It also contains the 6-ball $U_1 \times D_1^2$ where $U_1$ is a small 4-ball around $p$ in $D_1^4$ and the $D_1^2$-direction comes from an extension of the normal bundle of $S_1^3 \subset M_1$ over $D_1^4$ (whose boundary is used in the surgery from $M_1$ to $B_1^5$). This implies that $\alpha(L)$ is a 2-torus (4 points in Fig. 7):

$$\alpha(L) = \partial D_1^2 \times \partial D_2^2.$$

Moreover, the normal framing is the product of two framings on $\partial D_i^2$ which we claim are both nontrivial. First consider the case $i = 2$: By definition, the framing $f_2$ on $\partial D_2^2$ comes from the one on $L_2(S^4) = \partial M_2$. In particular, it is not the framing $f_0$ which extends to $D_2^2$ because this 2-disk does not lie in $\partial M_2$. Clearly, $(\partial D_2^2, f_0) = 0$ in $\Omega_1^{fr}$. Moreover, $f_2$ comes from the framing of the 2-disk $B_2^2 \subset \partial M_2$ with $\partial B_2^2 \cup D_2^2$ a non-vanishing section in the disk-bundle $E = H \oplus 1$ (which was used to define $M_2$). The two normal framings $f_0$ and $f_2$ differ by the normal bundle of $M_2$ restricted to this 2-sphere which is the Hopf bundle $H$. Thus

$$0 \neq (\partial D_2^2, f_2) \in \Omega_1^{fr}.$$

An argument very similar to the above shows that the same is true for $(\partial D_1^2, f_1)$ and thus $\alpha(L) \neq 0$. □

We will next prove Theorem 4. which states that the example above takes place in the smallest possible dimension. First recall that orientable manifolds (with non-empty boundary) of dimension $\leq 3$ are actually parallelizable. Together with Theorem 2. this implies that we just have to show that the arguments in Sect. 2 go through for 4-dimensional Seifert surfaces mapped into $S^{N+2}, N \geq 4$.

**Lemma 6.** *Let $W^4$ be a closed oriented simply-connected 4-manifold. Then there exist $r, s \in \mathbb{N}$ such that*

$$W \# r(\mathbb{CP}^2) \# s(-\mathbb{CP}^2)$$

*is the connected sum of $S^2$-bundles over $S^2$.*

This lemma can be found in [17]. It implies Theorem 4. because after the first step of Sect. 2 we may assume that the Seifert surfaces $W_i$ are simply-connected (and oriented). Moreover, by Lemma 7. below we we may change our link map by adding local knots with punctured $\mathbb{CP}^2$ as Seifert surfaces. As usual, this does not change the link homotopy class of the link map. By Lemma 6. we obtain very special $W_i^4$ which may be surgered to 4-balls by framed surgeries on the 2-spheres which are the fibers of the above $S^2$-bundles. Note that this does not depend on whether $W_i$ is parallelizable or not because we don't use the 2-spheres with non-trivial normal bundle. In particular, we do not need symmetric surgery in these codimensions $\geq 3$.

**Lemma 7.** *Punctured $\mathbb{CP}^2$ embeds into $S^6$.*

*Proof.* Take an (unknotted) $S^2 \subset S^6$. Write the trivial normal bundle as $H \oplus H$, where $H$ is the Hopf-bundle. Then punctured $\mathbb{CP}^2$ sits in $S^6$ as the disk-bundle of one copy of $H$. □

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