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On the signature of four-manifolds with universal covering spin

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I Introduction

In this note we study closed oriented 4-manifolds whose universal covering is spin and ask whether there are restrictions on the divisibility of the signature. Since any natural number appears as the signature of a connected sum of $\mathbb{CP}^2$'s, without the assumption on the universal covering there cannot exist any restrictions. Certainly, the most famous such restriction was proved by Rohlin in [10], where he showed that the signature σ of a smooth 4-dimensional spin manifold is divisible by 16 (compare part (2) of our Main Theorem for a new proof). The *Kummer surface* K shows that this is the best possible general result. Dividing by a certain free holomorphic involution on K, one obtains the *Enriques surface* (compare [1]) which by construction has signature 8 and fundamental group $\mathbb{Z}/2$. Furthermore, Hitchin showed in [5] that there exists an antiholomorphic free involution on the Enriques surface. We will refer to the quotient as the *Hitchin manifold* which then has signature 4 and fundamental group $\mathbb{Z}/2 \times \mathbb{Z}/2$. Rohlin's theorem admits a nice generalization to nonspin 4-manifolds, compare [4, Theorem 6.3]:

$$\sigma(M) = F \circ F - 2 \cdot \beta(F) \bmod 16 .$$

Here F is a (not necessarily orientable) surface in M which is dual to w_2M. This implies that there exists a Pin^--structure on $M \setminus F$ which induces a Pin^--structure on F and hereby a quadratic refinement of the $\mathbb{Z}/2$-intersection form on F. Thus the $\mathbb{Z}/8$-valued Brown-Arf invariant $\beta(F)$ can be defined.

Starting with a surface F, the above formula suggests that one can construct further examples of 4-manifolds with small signature and universal covering spin. But so far, all attempts using this method failed and therefore we tried to find a different way of attacking the problem. This was motivated by the questions of several people at the Oberwolfach Topology Conference in 1990 who wanted to know whether the signature of all closed oriented 4-manifolds whose universal covering is spin is divisible by 4 respectively for which fundamental groups it is always divisible by 8 or 16.

In our Main Theorem below we will answer these questions but first we want to introduce some useful notations. Let M be a manifold with fundamental group π and universal covering spin. Choosing a 2-equivalence $u: M \to K(\pi, 1)$, the homotopy fibration $\tilde{M} \xrightarrow{p} M \xrightarrow{u} K(\pi, 1)$ induces an exact sequence

$$0 \to H^2(\pi; \mathbb{Z}/2) \xrightarrow{u^*} H^2(M; \mathbb{Z}/2) \xrightarrow{p^*} H^2(\tilde{M}; \mathbb{Z}/2) .$$

Since $0 = w_2\tilde{M} = p^*(w_2 M)$, there exists a unique element $w \in H^2(\pi; \mathbb{Z}/2)$ with $w_2 M = u^*(w)$. Moreover, the pair (π, w) is determined by M up to automorphisms of π. Calling the isomorphism class of this pair the *w_2-type* of M, we can now formulate our question more precisely.

What are the possible signatures of closed oriented 4-manifolds with a given w_2-type (π, w)?

Note that in our formulation a manifold can have a w_2-type (π, w) only if its universal covering is spin. By taking connected sums along 1-skeletons of such manifolds (see [9]), one shows that the possible signatures form an ideal in $\mathbb{Z}$. We define the natural numbers $\sigma(\pi, w)$ (respectively $\sigma^{TOP}(\pi, w)$) by the requirement that they generate the ideal

$$\{\sigma(M) \in \mathbb{Z} \mid M \text{ is a (smooth) closed oriented 4-manifold with } w_2\text{-type } (\pi, w)\} .$$

We remark that in the topological setting Freedman's $|E_8|$-manifold [3] is an example of a 4-dimensional spin manifold with signature 8, so that one has to be careful about the category. However, part (9) of the following result shows that the answer for both categories agrees for all nonspin w_2-types.

Main Theorem. *Let π be a finitely presentable group and $w \in H^2(\pi; \mathbb{Z}/2)$. Then*

(1) *$\sigma(\pi, w)$ divides* 16.

(2) *$\sigma(\pi, 0) = 16$.*

(3) *If $w \neq 0$ then $\sigma(\pi, w)$ divides* 8.

(4) *$\sigma(\pi, w) = 8$ if $0 \neq w \in \operatorname{Ext}(H_1\pi; \mathbb{Z}/2) \subsetneq H^2(\pi; \mathbb{Z}/2)$.*

(5) *If $Sq^1 w \notin \{a \cup w \mid a \in H^1(\pi; \mathbb{Z}/2)\}$ then $\sigma(\pi, w)$ divides* 4.

(6) *If the multiplication by w is injective on $H^1(\pi; \mathbb{Z}/2)$ then $\sigma(\pi, w)$ divides 2 if and only if $\wp(w) \notin i_{4,2}(\{b^2 + b \cup w \mid b \in H^2(\pi; \mathbb{Z}/2)\}) + \operatorname{Ker}(k_4: H^4(\pi; \mathbb{Z}/4) \to \operatorname{Hom}(H_4(\pi), \mathbb{Z}/4))$ where $i_{4,2}: \mathbb{Z}/2 \subsetneq \mathbb{Z}/4$ and $\wp$ is the Pontrjagin-square* [15, *Sect.* 4].

(7) *If the multiplication by w is injective on $H^1(\pi; \mathbb{Z}/2)$ then $\sigma(\pi, w) = 1$ if and only if $w^2 \notin \{b^2 + b \cup w \mid b \in H^2(\pi; \mathbb{Z}/2)\} + \operatorname{Ker}(k_2: H^4(\pi; \mathbb{Z}/2) \to \operatorname{Hom}(H_4(\pi), \mathbb{Z}/2))$.*

(8) *$\sigma^{TOP}(\pi, 0) = 8$.*

(9) *$\sigma^{TOP}(\pi, w) = \sigma(\pi, w)$ if $w \neq 0$.*

As an immediate corollary to this result, we will obtain the following examples. Note that the Hitchin manifold very nicely fits into example (b).

Examples. (a) If π is a finite group with cyclic or quaternion 2-Sylow subgroup, then $\sigma(\pi, w) = 8\ \forall w \neq 0$.

$$\text{(b) } \sigma(\mathbb{Z}/2 \times \mathbb{Z}/2, w) = \begin{cases} 4 & \text{if } w = x_1^2 + x_1 \cdot x_2 + x_2^2 \ , \\ 16 & \text{if } w = 0 \ , \\ 8 & \text{else} \ , \end{cases}$$

where $\{x_1, x_2\}$ is the usual basis for $H^1(\mathbb{Z}/2 \times \mathbb{Z}/2; \mathbb{Z}/2)$.

(c) $\sigma((\mathbb{Z}/4)^6, w) = 2$ for a certain class w.

(d) Let $\pi := \mathbb{Z}/16 \rtimes \mathbb{Z}/8$ be a semidirect product with action of $\mathbb{Z}/8$ on $\mathbb{Z}/16$ given by $t \mapsto t^5$. Then there exists a class $w \in H^2(\pi; \mathbb{Z}/2)$ such that $\sigma(\pi, w) = 1$.

Let me now briefly outline the proof of the Main Theorem. First recall that the signature is an invariant of the oriented bordism class of a manifold. Since we are interested only in the signatures of manifolds with a fixed w_2-type (π, w), we define the concept of (π, w)-*bordism groups* as follows. The class w gives a fibration $w: K(\pi, 1) \to K(\mathbb{Z}/2, 2)$ and we can form the pullback

$$\begin{tikzcd} B = B(\pi, w) \arrow[r] \arrow[d, "\xi(\pi, w)"'] & K(\pi, 1) \arrow[d, "w"] \\ BSO \arrow[r, "w_2(\gamma)"] & K(\mathbb{Z}/2, 2) \end{tikzcd}$$

where γ denotes the stable universal bundle over BSO. Now the bordism groups

$$\Omega_n(\pi, w) := \Omega_n(\xi(\pi, w))$$

can be defined as in [12]. They consist of bordism classes of triangles

$$\begin{tikzcd} & B \arrow[d, "\xi(\pi, w)"] \\ M \arrow[ur, "\bar{\nu}"] \arrow[r, "\nu"'] & BSO \end{tikzcd}$$

where M is a smooth closed oriented manifold and ν is the stable normal Gauß map of M given by some embedding into Euclidean space. The map $\bar{\nu}$ is called a $\xi(\pi, w)$-*structure* on M. Using obstruction respectively surgery techniques one proves the following result.

Proposition [9]. *Let $\xi(\pi, w)$ be a fibration as above. Then*

(1) *For any manifold with w_2-type (π, w) there exists a $\xi(\pi, w)$-structure $\bar{\nu}$ which is a 2-equivalence.*

(2) *Every bordism class in $\Omega_n(\pi, w)$ is represented by a manifold with w_2-type (π, w).*

It follows that the ideal $\sigma(\pi, w) \cdot \mathbb{Z}$ in question is just the image of the signature homomorphism

$$\sigma: \Omega_4(\pi, w) \to \mathbb{Z} \ .$$

In the next part we will develop a technique for computing these bordism groups and as an application are able to prove the Main Theorem. We finish this introduction by remarking that this note is part of my PhD thesis [14].

II The James spectral sequence

Our aim is to construct a spectral sequence with $E^2_{p,q} \cong H_p(K(\pi, 1); \Omega_q^{Spin})$ which converges to $\Omega_{p+q}(\pi, w)$. We first use the Pontrjagin-Thom isomorphisms (see e.g. [12])

$$\Omega_n(\pi, w) \cong \pi_n(M\xi(\pi, w)) \text{ and } \Omega_n^{Spin} \cong \pi_n(M\,Spin)$$

to translate the bordism groups into (stable) homotopy groups of the corresponding Thom spectra. Assuming $w_2(\gamma)$ to be a fibration, the definition of $\xi(\pi, w)$ as a pullback gives a commutative diagram

$$(*)\qquad \begin{array}{ccccc} BSpin & \xrightarrow{i} & B & \xrightarrow{f} & K(\pi, 1) \\ \| & & \downarrow{\scriptstyle \xi(\pi, w)} & & \downarrow{\scriptstyle w} \\ BSpin & \longrightarrow & BSO & \xrightarrow{w_2(\gamma)} & K(\mathbb{Z}/2, 2) \end{array}$$

in which f is a fibration and $\xi(\pi, w) \circ i$ is the universal bundle over $BSpin$. Moreover, the orientability of $\xi(\pi, w)$ implies that the homotopy equivalences of the fiber $BSpin$ induced by elements of π are all homotopic to the identity. Therefore, the existence of a spectral sequence as above follows from the following more general result by applying it to the fibration f and using stable homotopy as the generalized homology theory.

Theorem. *Let h be a generalized homology theory which is connected, i.e. $\pi_i(h) = 0$ $\forall i < 0$. Furthermore, let $F \to B \xrightarrow{f} K$ be an h-orientable fibration and $\xi : B \to BSO$ a stable vector bundle. Then there exists a spectral sequence*

$$E^2_{p,q} \cong H_p(K; h_q(M(\xi|F))) \Rightarrow h_{p+q}(M\xi)$$

(*which we shall call the* James spectral sequence *for the fibration f, because James juggled around with Thom spaces in his book* [6] *in a similar way we are going to do it.*)

Remark. All spectral sequences we will consider will be 1.quadrant spectral sequences, so there will not occur any problems concerning their convergence. This is the reason why we assumed the generalized homology theory h to be connected.

Proof. Since $SO = \bigcup_{n \in \mathbb{N}} SO(n)$ is a topological group, there is a contractible space ESO on which SO acts freely and a model for BSO is the orbit space ESO/SO. As a subgroup of SO each $SO(n)$ also acts freely on this space ESO and if we define $BSO(n) := ESO/SO(n)$ then the maps $i_n : BSO(n) \to BSO$ are fiber bundles with fibers $SO/SO(n)$. Similarly all maps $i_n^{n+1} : BSO(n) \to BSO(n+1)$ are fiber bundles with fibers $SO(n+1)/SO(n)$. From the originally given stable vector bundle ξ over B we now construct a sequence of fiber bundles over B by the pullback

$$\begin{array}{ccc} B_n & \xrightarrow{b_n} & B \\ \downarrow{\scriptstyle \xi_n} & & \downarrow{\scriptstyle \xi} \\ BSO(n) & \xrightarrow{i_n} & BSO\,. \end{array}$$

Composing the maps b_n with the original fibration $f: B \to K$ we get a sequence of fibrations $f_n: B_n \to K$ with fibers F_n together with vector bundles ξ_n over each B_n such that the following diagrams commute:

$$\begin{array}{ccccc} F_n & \longrightarrow & B_n & \xrightarrow{f_n} & K \\ \downarrow & & \downarrow & & \| \\ F_{n+1} & \longrightarrow & B_{n+1} & \xrightarrow{f_{n+1}} & K \end{array} \qquad \begin{array}{ccc} B_n & \xrightarrow{\xi_n} & BSO(n) \\ \downarrow & & \downarrow{\scriptstyle i_n^{n+1}} \\ B_{n+1} & \xrightarrow{\xi_{n+1}} & BSO(n+1) \end{array} .$$

By definition, the Thom spectrum $M\xi$ consists of the family of Thom spaces $\{T(\xi_n), s_n: S^1 \wedge T(\xi_n) \to T(\xi_{n+1})\}$ and similarly $M\xi|F = \{T(\xi_n|F_n), s_n|S^1 \wedge T(\xi_n|F_n)\}$.

We will obtain the desired spectral sequence converging to $h_*(M\xi)$ as a direct limit of relative Serre spectral sequences as follows: For any $n \in \mathbb{N}$, the disk-sphere bundle pair $(D(\xi_n), S(\xi_n))$ is a relative fibration over B_n with relative fiber (D^n, S^{n-1}). Composed with $f_n: B_n \to K$ this becomes a relative fibration over K with relative fiber $(D(\xi_n|F_n), S(\xi_n|F_n))$. This fibration is h-orientable because we are considering oriented vector bundles and the original fibration f was assumed to be h-orientable. Thus there is a relative Serre spectral sequence (see [13, Chap. 15, Remark 2])

$${}^nE: H_p(K; h_q(D(\xi_n|F_n), S(\xi_n|F_n))) \Rightarrow h_{p+q}(D(\xi_n), S(\xi_n)) .$$

Replacing ξ_n by $\xi_n \oplus \varepsilon_{\mathbb{R}}$ we obtain a spectral sequence which is isomorphic to the above via the suspension isomorphism for the generalized homology theory h. We shall identify these two spectral sequences and obtain from the vector bundle homomorphism $\xi_n \oplus \varepsilon_{\mathbb{R}} \to \xi_{n+1}$ homomorphisms of spectral sequences ${}^nE \to {}^{n+1}E$. More exactly, we obtain a family of commutative diagrams

$$\begin{array}{ccc} {}^nE^i_{p,q} & \longrightarrow & {}^{n+1}E^i_{p,q} \\ {\scriptstyle {}^nd^i_r}\downarrow & & \downarrow{\scriptstyle {}^{n+1}d^i_r} \\ E^i_{p-r,q+r-1} & \longrightarrow & {}^{n+1}E^i_{p-r,q+r-1} \end{array}$$

and we can define a spectral sequence E by the direct limit, i.e. we set

$$(E^i_{p,q}, d^i_r) := \left(\varinjlim_n {}^nE^i_{p,q}, \varinjlim_n {}^nd^i_r \right) .$$

The exactness of the direct limit functor gives the isomorphisms $E^{i+1}_{*,*} \cong H(E^i_{*,*}, d^i)$ which show that we have in fact defined a spectral sequence. By definition we have

$$\begin{aligned} E^2_{p,q} &= \varinjlim_n H_p(K; h_q(D(\xi_n|F_n), S(\xi_n|F_n))) \\ &\cong H_p(K; \varinjlim_n h_q(T(\xi_n|F_n))) \\ &\cong H_p(K; h_q(M\xi|F)) . \end{aligned}$$

Since all ${}^nE$ are 1.quadrant spectral sequences we obtain $E^\infty_{p,q} = \varinjlim_n {}^nE^\infty_{p,q}$ with graded object

$$\varinjlim_n h_{p+q}(D(\xi_n|F_n), S(\xi_n|F_n)) \cong \varinjlim_n h_{p+q}(T(\xi_n)) \cong h_{p+q}(M\xi) . \quad \square$$

Remark. By construction, the James spectral sequence is natural with respect to commutative diagrams of fibrations

$$\begin{tikzcd} F' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d, "\varphi"'] & K' \arrow[d] \\ F \arrow[r] & B \arrow[r] & K \end{tikzcd}$$

such that $\xi' = \xi \circ \varphi$.

We next want to determine those differentials d_2 in the James spectral sequence which are interesting for the groups $\Omega_4(\pi, w)$. In this case the generalized homology theory is stable homotopy and the fibration $f: B \to K(\pi, 1)$ is the pullback as in diagram (*). Note that the class $w \in H^2(\pi; \mathbb{Z}/2)$ did not enter into the E^2-term of the James spectral sequence so that we strongly expect it to enter into the differentials. For the result below we recall the following facts. Let $\iota: S^0 \to M\,Spin$ be the unit of the ring spectrum $M\,Spin$ coming from the inclusions of the bottom cell $D^n/S^{n-1} \subsetneq T(\xi_n)$. One knows that $\iota_*: \pi_i(S^0) \to \pi_i(M\,Spin)$ is an isomorphism for $i \leqq 2$ and thus induces isomorphisms

$$\Omega_0^{Spin} \cong \mathbb{Z} \quad \text{and} \quad \Omega_i^{Spin} \cong \mathbb{Z}/2 \text{ for } i = 1, 2\,.$$

Proposition 1. *Let $Sq_w^2: H^{p-2}(\pi; \mathbb{Z}/2) \to H^p(\pi; \mathbb{Z}/2)$ denote the homomorphism given by $Sq_w^2(x) := Sq^2(x) + x \cup w$. Then the following assertions hold:*

(1) *For $p \leqq 4$, the differential $d_2: H_p(\pi; \Omega_1^{Spin}) \to H_{p-2}(\pi; \Omega_2^{Spin})$ is the dual of Sq_w^2.*

(2) *For $p \leqq 5$, the differential $d_2: H_p(\pi; \Omega_0^{Spin}) \to H_{p-2}(\pi; \Omega_1^{Spin})$ is reduction* mod 2 *composed with the dual of Sq_w^2.*

Proof. Set $\xi := \xi(\pi, w)$ and note that for the fibration $\{pt\} \to B \xrightarrow{\text{id}} B$ the James spectral sequence

$$H_p(B; h_q(M\xi | \{pt\})) \Rightarrow h_{p+q}(M\xi)$$

translates by construction into the Atiyah-Hirzebruch spectral sequence for $M\xi$ if one uses the Thom isomorphism $H_p(M\xi) \cong H_p(B)$ and the fact that $M\xi | \{pt\} \simeq S^0$ as spectra. By the lemma below, the differentials d_2 in the Atiyah-Hirzebruch spectral sequence are given by dual of Sq^2 on $H^*(M\xi; \mathbb{Z}/2)$, respectively the composition with the reduction mod 2. But under the Thom isomorphism these maps become $Sq^2_{w_2(\xi)}$ on $H^*(B; \mathbb{Z}/2)$. Now we use naturality of the James spectral sequence for the fibrations

$$\begin{tikzcd} \{pt\} \arrow[r] \arrow[d] & B \arrow[r, "\text{id}"] \arrow[d, "\text{id}"] & B \arrow[d, "f"] \\ BSpin \arrow[r] & B \arrow[r, "f"] & K(\pi, 1) \end{tikzcd}$$

to get for all $x \in H_p(\pi; \mathbb{Z}/2)$, $p \leqq 4$:

$$d_2(x) = d_2(f_*(y)) = (Sq^2_{w_2(\xi)})^*(f_*(y)) = (Sq_w^2)^*(x)$$

and the corresponding result for $x \in H_p(\pi; \mathbb{Z})$, $p \leqq 5$.

Here we used that f_* is onto for $p \leqq 4$ which follows from the Hurewicz Theorem since f is a 4-equivalence. Finally, we also used that $f_*: H_5(B; \mathbb{Z}) \to H_5(\pi; \mathbb{Z})$ is onto. This is equivalent to the vanishing of the differential

$d_5: H_5(\pi; \mathbb{Z}) \to H_4(BSpin) \cong \mathbb{Z}$ in the Serre spectral sequence for the fibration f. But since f is the pullback of the fibration $BSpin \to BSO \to K(\mathbb{Z}/2, 2)$ and the corresponding differential d_5 vanishes ($H_5(K(\mathbb{Z}/2, 2); \mathbb{Z})$ is finite !), we are done by the naturality of the Serre spectral sequence. □

Remark. It is conceivable that the assertions in Proposition 1 also hold for arbitrary p although our proof only works in the range described.

Lemma. *Let X be a spectrum and $H_p(X; \Omega_q^{Spin}) \Rightarrow \Omega_{p+q}^{Spin}(X)$ the corresponding Atiyah-Hirzebruch spectral sequence. Then*

(1) *The differential $d_2: H_p(X; \Omega_1^{Spin}) \to H_{p-2}(X; \Omega_2^{Spin})$ is the dual of Sq^2: $H^{p-2}(X; \mathbb{Z}/2) \to H^p(X; \mathbb{Z}/2)$.*

(2) *The differential $d_2: H_p(X; \Omega_0^{Spin}) \to H_{p-2}(X; \Omega_1^{Spin})$ is reduction* mod 2 *composed with the dual of Sq^2.*

Proof. Using the inclusion of the botton cell $\iota: S^0 \to M\,Spin$, the naturality of the Atiyah-Hirzebruch spectral sequence shows that in the range in question we can as well compute the differentials d_2 for the spectral sequence

$$H_p(X; \pi_q^{st}) \Rightarrow \pi_{p+q}^{st}(X)\,.$$

Now the differentials d_2 are stable homology operations and thus are induced from elements in

$[H\mathbb{Z}/2, \Sigma^2 H\mathbb{Z}/2] \cong H^2(H\mathbb{Z}/2; \mathbb{Z}/2) = \langle Sq^2 \rangle \cong \mathbb{Z}/2$ in part (1) and

$[H\mathbb{Z}, \Sigma^2 H\mathbb{Z}/2] \cong H^2(H\mathbb{Z}; \mathbb{Z}/2) = \langle Sq^2 \circ r_2 \rangle \cong \mathbb{Z}/2$ in part (2).

Here for any abelian group A, HA denotes the spectrum associated to ordinary homology with coefficients in A. To finish the proof we have to show that in both cases $d_2 \neq 0$. For this just take $X := \Sigma^{p-2} H\mathbb{Z}/2$ and recall that $\pi_i(\Sigma^{p-2} H\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } i = p-2, \\ 0 & \text{else.} \end{cases}$ □

We finish this section by describing the edge-homomorphisms of the James spectral sequence in the case that the homology theory h is stable homotopy. The edge-homomorphism coming from the base-line is $\mathfrak{ed}: \pi_n(M\xi) \to H_n(K; \pi_0(M\xi|F))$ where by the Hurewicz- and Thom isomorphisms $\pi_0(M\xi|F) \cong \mathbb{Z}$ (assuming that F is connected) and thus

$$\mathfrak{ed}: \Omega_n(\xi) \cong \pi_n(M\xi) \to H_n(K)\,.$$

Proposition 2. *Let $[\tilde{v}: M \to B] \in \Omega_n(\xi)$, i.e. $\xi \circ \tilde{v}$ is a stable normal Gauß map for M. Then the above edge-homomorphism is given by*

$$\mathfrak{ed}[\tilde{v}: M \to B] = f_* \circ \tilde{v}_*[M] \in H_n(K)\,,$$

where $[M] \in H_n(M)$ is the fundamental class given by the orientation determined by $\tilde{v}$.

Proof. Using the naturality of the James spectral sequence for the fibrations

$$\begin{array}{ccccc} \{pt\} & \longrightarrow & B & \xrightarrow{\text{id}} & B \\ \downarrow & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle f} \\ F & \longrightarrow & B & \xrightarrow{f} & K \end{array}$$

we are reduced to showing $\mathfrak{ed}[\tilde{\nu}: M \to B] = \tilde{\nu}_*[M]$ in the spectral sequence for the upper fibration. Let us now choose an embedding $M \subsetneq S^{n+k}$ for sufficiently large k. Then we obtain a commutative diagram

$$\begin{tikzcd} H_{n+k}(S^{n+k}) \arrow[r] & H_{n+k}(T(\nu(M \subsetneq S^{n+k}))) \arrow[r, "T\tilde{\nu}_*"] \arrow[d, "\cong"'] & H_{n+k}(T(\xi_k)) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H_n(M\xi) \\ & H_n(M) \arrow[r, "\tilde{\nu}_*"] & H_n(B) & \end{tikzcd}$$

where a generator of $H_{n+k}(S^{n+k})$ is mapped to $[M] \in H_n(M)$. This proves that the diagram

$$(**) \qquad \begin{tikzcd} \pi_n(M\xi) \arrow[r, "\text{Hurewicz}"] \arrow[d, "\text{Pontrjagin-Thom}"', "\cong"] & H_n(M\xi) \arrow[d, "\cong"', "\text{Thom}"] \\ \Omega_n(\xi) \arrow[r, "\phi"] & H_n(B) \end{tikzcd}$$

commutes if we set $\phi[\tilde{\nu}: M \to B] := \tilde{\nu}_*[M]$.

Since the James spectral sequence for a fibration where the fiber is a point is isomorphic to the Atiyah-Hirzebruch spectral sequence under the Thom isomorphism, diagram (**) shows that it suffices to show that the Hurewicz homomorphism is the edge-homomorphism for the Atiyah-Hirzebruch spectral sequence for $M\xi$. But this follows from the fact that this edge-homomorphism is a stable homology operation from stable homotopy to ordinary homology, i.e. an element of $[S^0, H\mathbb{Z}] \cong \mathbb{Z}$. Moreover, the Hurewicz homomorphism generates this group and if we take the spectrum $\Sigma^n H\mathbb{Z}$ as a test case we can conclude that the edge-homomorphism cannot be a nontrivial multiple of this generator. Finally, using the spheres S^n as a second test example, one sees that the sign is correct, too. □

The edge-homomorphism of the James spectral sequence coming from the inclusion of the fiber is a map $\mathfrak{ed}': H_0(K; \pi_n(M\xi|F)) \to \pi_n(M\xi)$ and if we also assume K to be connected then

$$\mathfrak{ed}': \Omega_n(\xi|F) \to \Omega_n(\xi) .$$

Proposition 3. *Let $[\tilde{\nu}: M \to F] \in \Omega_n(\xi|F)$, i.e. $\xi|F \circ \tilde{\nu}$ is a stable normal Gauß map for M. Then the above edge-homomorphism is given by*

$$\mathfrak{ed}'[\tilde{\nu}: M \to F] = [M, i \circ \tilde{\nu}] \in \Omega_n(\xi) ,$$

where $i: F \to B$ is the inclusion of the fiber.

Proof. This is just the naturality of the Pontrjagin-Thom construction, combined with the fact that in the Serre spectral sequence this edge-homomorphism is given by the induced map i_*, see [13, Chap. 15, Remark 5]. □

III Proof of the Main Theorem, smooth case

(1) follows from the existence of the Kummer surface K because a spin manifold admits a $\xi(\pi, w)$-structure for all pairs (π, w).

(2) Our aim is to prove Rohlin's theorem using only the isomorphisms

$$\Omega_i^{SO} \cong \begin{cases} 0 & \text{if } i = 1, 2, 3, \\ \mathbb{Z} & \text{if } i = 0, 4 \end{cases}$$

and the fact that $\Omega_3^{Spin} = 0$ which was proven in an elementary way in [7]. If we apply the James spectral sequence to the fibration $BSpin \to BSO \to K(\mathbb{Z}/2, 2)$, we can conclude that $\Omega_i^{Spin} \cong \mathbb{Z}/2$ for $i = 1, 2$ and we obtain a filtration

$$\Omega_4^{Spin}/\text{Image}(d_i) \underset{\mathbb{Z}/2}{\subseteq} F_{2,2} \underset{\mathbb{Z}/2}{\subseteq} F_{3,1} \underset{\mathbb{Z}/4}{\subseteq} \Omega_4^{SO} \xrightarrow[\sigma]{\cong} \mathbb{Z}\,.$$

Because $\text{Image}(d_i) \subseteq \Omega_4^{Spin}$ is a torsion group and thus the signature vanishes on this subgroup, the divisibility of the signature on Ω_4^{Spin} equals the product of the orders of the three subquotients in the filtration. But since the homology of $K(\mathbb{Z}/2, 2)$ is given by

$$H_i(K(\mathbb{Z}/2, 2); \mathbb{Z}/2) \cong \mathbb{Z}/2 \quad \text{for } i = 2, 3$$

$$H_4(K(\mathbb{Z}/2, 2); \mathbb{Z}) \cong \mathbb{Z}/4$$

the above subquotients are as claimed once we show that there are no differentials involved. All differentials leaving from $H_5(K(\mathbb{Z}/2, 2); \mathbb{Z}) \cong \mathbb{Z}/2$ are trivial because the edge-homomorphism $\Omega_5^{SO} \to H_5(K(\mathbb{Z}/2, 2); \mathbb{Z})$ is onto. This follows from the fact that the nontrivial element $z \in H_5(K(\mathbb{Z}/2, 2); \mathbb{Z})$ is realized by the oriented 5-manifold $M := SU(3)/SO(3)$. To verify the last assertion, note that M is simply-connected and non spin with the following cohomology groups:

i	0	1	2	3	4	5
$H^i(M; \mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$
$H^i(M; \mathbb{Z}/2)$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$

Then Poincaré duality and the long exact sequence for the coefficient sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$ together with the fact that its boundary map reduces mod 2 to Sq^1 implies that

$$0 \neq \langle w_2(M) \cup Sq^1(w_2(M)), [M] \rangle = \langle \iota_2 \cup Sq^1(\iota_2), (w_2)_*[M] \rangle,$$

$$\text{i.e. } (w_2)_*[M] = z\,.$$

The only other possible differentials in dimension 4 are

$$d_2 : E^2_{4,i} \to E^2_{2,i+1}, \quad i = 0, 1\,.$$

But using Proposition 1 these are given by $(Sq^2_{\iota_2})^*$ respectively by $(Sq^2_{\iota_2})^* \circ r_2$. Now in our situation $(Sq^2_{\iota_2})^* \equiv 0$ because $Sq^2_{\iota_2}(\iota_2) = Sq^2(\iota_2) + \iota_2 \cup \iota_2 = 0$ and thus Rohlin's theorem follows.

(3) The commutative diagram of fibration (*) implies that

$$w_* : H_2(\pi; \mathbb{Z}/2) \to H_2(K(\mathbb{Z}/2, 2); \mathbb{Z}/2) \cong \mathbb{Z}/2$$

is onto if $w \neq 0$.

We now apply the James spectral sequence to both fibrations. The lower fibration was discussed in (2) and for the upper fibration, the James spectral sequence gives a filtration

$$\Omega_4^{Spin} \subseteq F_{2,2}(\pi, w) \subseteq F_{3,1}(\pi, w) \subseteq \Omega_4(\pi, w) .$$

Here certainly differentials can exist and thus the subquotients of this filtration are only subquotients of $E^2_{p,4-p}(\pi, w)$. Now consider an element $x \in H^2(\pi; \mathbb{Z}/2) \cong E^2_{2,2}(\pi, w)$ with $w_*(x) \neq 0$. Then x cannot be hit by a differential since $x = d_i(y)$ would give the contradiction $w_*(x) = d_i(\xi(\pi, w)_*(y))$ to the arguments in (2). Therefore, x survives to infinity to give an element $\bar{x} \in F_{2,2}(\pi, w) \subseteq \Omega_4(\pi, w)$ mapping to $\xi(\pi, w)_*(\bar{x}) \in F_{2,2} \setminus \Omega_4^{Spin}$. Thus the corresponding manifold has signature 8 (mod 16).

(4) If M is a manifold with w_2-type (π, w) and $u: M \to K(\pi, 1)$ is a 2-equivalence then the universal coefficient sequence shows that under our assumption one has:

$$\langle w_2(M), x\rangle = \langle u^*(w), x\rangle = \langle w, u_*(x)\rangle = 0 \quad \forall x \in H_2(M; \mathbb{Z})$$

which implies that the intersection form on $H_2(M; \mathbb{Z})$ is even. But since $\sigma(M)$ is just the signature of this form, it follows that $\sigma(M)$ is divisible by 8. Since by (3) there also exists a manifold with signature 8 in this w_2-type, we are done.

(5) To prove $\sigma(\pi, w)|4$, we want to find an element in $F_{3,1}(\pi, w) \subseteq \Omega_4(\pi, w)$ which maps to $F_{3,1} \setminus F_{2,2}$ under $\xi(\pi, w)_*$.

Take an arbitrary element $x \in H_3(\pi; \mathbb{Z}/2)$. Then $w_*(x) \neq 0$ if and only if $\langle Sq^1 w, x\rangle \neq 0$ because $H^3(K(\mathbb{Z}/2, 2); \mathbb{Z}/2)$ is generated by $Sq^1(\iota_2)$. Such an element x cannot be hit by a differential since again this would contradict our knowledge about the spectral sequence for the fibration with total space BSO. Therefore, x survives to infinity if and only if $d_2(x) = 0$ which is equivalent to

$$\langle a \cup w, x\rangle = 0 \quad \forall a \in H^1(\pi; \mathbb{Z}/2)$$

because by Theorem 1 the dual of d_2 is multiplication by w (Sq^2 vanishes on 1-dimensional classes). Obviously our assumption is equivalent to the existence of an element x satisfying the above two conditions.

(6) works exactly the same way as (7) so we omit the details here and rather prove the more interesting part (7).

(7) We have a commutative diagram

$$\begin{tikzcd}
\Omega_4(\pi, w) \arrow[r, "\xi(\pi, w)_*"] \arrow[d, "q"'] & \Omega_4^{SO} \arrow[r, "\sigma"] \arrow[d, "q'"'] & \mathbb{Z} \arrow[d, "r_4"'] \\
H_4(\pi) \arrow[r, "w_*"] & H_4(K(\mathbb{Z}/2, 2)) \arrow[r, "\cong"] & \mathbb{Z}/4
\end{tikzcd}$$

which shows that $w_* \circ q$ is the signature mod 4. Recall that $H^4(K(\mathbb{Z}/2, 2); \mathbb{Z}/4)$ is generated by the Pontrjagin square $\wp$ and thus as a map into $\mathbb{Z}/4$, w_* is given by

$$w_*(x) = \langle \wp, w_*(x)\rangle = \langle w^*(\wp), x\rangle = \langle \wp(w), x\rangle, \; x \in H_4(\pi) .$$

Reducing further mod 2 gives $\langle w^2, x\rangle$. Moreover, x lies in the image of q if and only if $d_i(x) = 0$ for $i = 2, 3$.

By assumption multiplication by w is injective on $H^1(\pi; \mathbb{Z}/2)$ which is equivalent to the surjectivity of $d_2: E^2_{3,1}(\pi, w) \to E^2_{1,2}(\pi, w)$. But this implies the vanishing of d_3 on $E^3_{4,0}(\pi, w)$.

Thus if the image of

$$w_*:\{x \in H_4(\pi) | d_2(x) = 0\} \to \mathbb{Z}/4$$

is $n \cdot \mathbb{Z}/4$ with $n \in \{0, 1, 2\}$ then $\sigma(\pi, w) \equiv n \bmod 4$.

In particular, $\sigma(\pi, w) \equiv 1 \bmod 4$ if and only if there exists an $x \in H_4(\pi)$ with $\langle w^2, x\rangle \neq 0$ and $d_2(x) = 0 (\Leftrightarrow \langle b^2 + b \cup w, x\rangle = 0 \,\forall b \in H^2(\pi; \mathbb{Z}/2)$ by Theorem 1).

Claim: These two conditions are equivalent to our second assumption

$$w^2 \notin \{b^2 + b \cup w | b \in H^2(\pi; \mathbb{Z}/2)\} + \operatorname{Ker}(k_2 : H^4(\pi; \mathbb{Z}/2) \to \operatorname{Hom}(H_4(\pi), \mathbb{Z}/2)) .$$

To see this equivalence, first observe that $\operatorname{Ker}(k_2)$ is by definition the subgroup of $H^4(\pi; \mathbb{Z}/2)$ which annihilates all of $H_4(\pi)$. Dividing out $\operatorname{Ker}(k_2)$ and defining $A := H_4(\pi)/2 \cdot H_4(\pi)$,

$$U := k_2(\{b^2 + b \cup w | b \in H^2(\pi; \mathbb{Z}/2)\}) \subseteq \operatorname{Hom}(H_4(\pi), \mathbb{Z}/2) = \operatorname{Hom}(A, \mathbb{Z}/2)$$

and $\mu := k_2(w^2) \in \operatorname{Hom}(A, \mathbb{Z}/2)$, we get the following statement which is equivalent to the claim:

$$\exists x \in A \text{ with } \mu(x) \neq 0 \text{ and } u(x) = 0 \quad \forall u \in U \Leftrightarrow \mu \notin U .$$

In other words, we want to prove $U = \operatorname{Ann}(\operatorname{Ann}(U))$ if we define the annihilator by

$$\operatorname{Ann}(U) := \{y \in A | u(y) = 0 \,\forall u \in U\} .$$

Clearly we have the inclusion $U \subseteq \operatorname{Ann}(\operatorname{Ann}(U))$ and since both sides are finite dimensional $\mathbb{Z}/2$-vector spaces for the equality we have to show that their dimensions agree. But this follows directly from the nondegeneracy of the bilinear form

$$\operatorname{Hom}(A, \mathbb{Z}/2) \times A \to \mathbb{Z}/2$$
$$(u, a) \mapsto u(a) . \quad \square$$

We finish this section by verifying the statements for the examples (a)–(d) from the introduction:

(a) follows directly from part (4) of the Main Theorem because the groups in question fulfill the assumption there.

(b) If $w \neq x_1^2 + x_1 \cdot x_2 + x_2^2 =: y$, there exists an inclusion $i: \mathbb{Z}/2 \subsetneq \mathbb{Z}/2 \times \mathbb{Z}/2$ such that $i^*(w) = 0$.

Now if M is a 4-manifold with w_2-type (π, w) then the double covering corresponding to $i(\mathbb{Z}/2)$ is a spin manifold and thus its signature is divisible by 16. Therefore, $\sigma(M) \equiv 0 \bmod 8$ and the assertion follows from part (3) of the Main Theorem. The case $w = y$ is handled by part (5) of the theorem because one computes that

$$Sq^1(y) = x_1^2 \cdot x_2 + x_1 \cdot x_2^2 \notin \{a \cdot y | a \in H^1(\mathbb{Z}/2 \times \mathbb{Z}/2; \mathbb{Z}/2)\} .$$

(c) is a straightforward computation in the cohomology of the abelian group $(\mathbb{Z}/4)^6$ together with part (6) of the Main Theorem.

(d) We want to apply part (7) of the Main Theorem and will make use of the following computation of the $\mathbb{Z}/2$-cohomology ring of the group π given in [2]:

$$H^*(\pi; \mathbb{Z}/2) \cong \mathbb{Z}/2[a, b, v, w]/(a^2 = b^2 = 0)$$

with $\deg(a) = \deg(b) = 1$ and $\deg(v) = \deg(w) = 2$ and

$$b = p^*(\beta), v = p^*(\nu) \text{ and } i^*(a) = \alpha, i^*(w) = \mu .$$

Here

$$\mathbb{Z}/16 \cong \langle t \rangle \xrightarrow{i} \pi \xrightarrow{p} \mathbb{Z}/8$$

is the split extension in question and α, μ respectively β, ν are the generators for the corresponding cyclic groups.

Obviously multiplication by w is injective on $H^1(\pi; \mathbb{Z}/2)$ and thus we have to show that

$$w^2 \notin \{y \cdot w + y^2 \mid y \in H^2(\pi; \mathbb{Z}/2)\} + \mathrm{Image}(r_2) .$$

Assume that w^2 can be written as

$$w^2 = l_1(y \cdot w + y^2) + l_2 \cdot r_2(x), \quad l_i \in \mathbb{Z}/2 \text{ with } x \in H^4(\pi) \text{ and}$$

$$y = \lambda_1 \cdot w + \lambda_2 \cdot v + \lambda_3 \cdot a \cdot b, \quad \lambda_i \in \mathbb{Z}/2 .$$

Now observe that the map i^* factors through the fixed point set of the action $t \mapsto t^5$, i.e. through the group $H^4(\langle t \rangle)^{\mathbb{Z}/8}$.

If the element n generates $H^2(\langle t \rangle)$ then n^2 generates the group $H^4(\langle t \rangle)$ and thus we see that $\mathbb{Z}/8$ acts as $n \mapsto 5 \cdot n$ respectively as $n^2 \mapsto 25 \cdot n^2 = 9 \cdot n^2$. This shows that

$$H^4(\langle t \rangle)^{\mathbb{Z}/8} = \langle 2 \cdot n^2 \rangle.$$

and therefore

$$r_2(i^*(x)) = 0 \ \forall x \in H^4(\pi) .$$

If we use the relations

$$i^*(v) = i^*(b) = 0 \quad \text{and thus } i^*(y) = \lambda_1 \cdot w$$

we get the following contradiction:

$$0 \neq \mu^2 = i^*(w^2) = l_1 \cdot (i^*(y) \cdot i^*(w) + i^*(y)^2) = l_1 \cdot (\lambda_1 \cdot w^2 + \lambda_1^2 \cdot w^2) = 0 . \quad \square$$

IV The topological case

A topological manifold has a stable normal Gauß map $\nu: M \to BTOP$, where

$$TOP = \bigcup_{n \geq 0} TOP(n)$$

and $TOP(n)$ is the topological group of all base point preserving self-homeomorphisms of $\mathbb{R}^n$. Then $BTOP$ is the classifying space of stable fiber bundles with fiber $\mathbb{R}^n$ and specified zero-section. There are the obvious inclusion maps $O(n) \to TOP(n)$ which induce a fibration $BO \to BTOP$ with fiber TOP/O.

Similarly, if BPL is the classifying space for stable piecewise linear bundles then there is a fibration $BPL \to BTOP$ with fiber TOP/PL. The fundamental result of [8] says that this fibration is a principal fibration, induced by an H-map

$$\mathfrak{ks}: BTOP \to K(\mathbb{Z}/2, 4)$$

which the authors call the triangulation obstruction and which is today known as the *Kirby–Siebenmann invariant.* Up to dimension 6, the spaces BPL and BO are equal and thus the Kirby–Siebenmann invariant gives in the 4-dimensional case a unique $\mathbb{Z}/2$-valued obstruction for the existence of a lift of the topological stable normal Gauß map $M^4 \to BTOP$ over BO. It was not until the striking results of Freedman in [3] that one could prove the nontriviality of this obstruction. Since for spin manifolds Kirby and Siebenmann had proven in [8, p. 325, Theorem 13.1] the formula

$$(***) \qquad \mathfrak{ks}(M) \equiv \frac{\sigma(M)}{8} \, (\mathrm{mod}\, 2) ,$$

this follows from the existence of Freedman's 4-manifold $|E_8|$. Using this manifold, it follows directly from the work of [8, p. 322–325] that for $G = O, SO$ or $Spin$ the natural maps

$$\Omega_i^G \to \Omega_i^{GTOP}$$

are isomorphisms for $i \leqq 3$ and injective with cokernel $\mathbb{Z}/2$ for $i = 4$. Moreover, the signature divided by 8 gives an isomorphism of $\Omega_4^{Spin\,TOP}$ onto $\mathbb{Z}$, whereas $\Omega_4^{STOP} \cong \mathbb{Z} \times \mathbb{Z}/2$ via $(\sigma, \mathfrak{ks})$. Our aim is to extend these results to the bordism groups $\Omega_4^{TOP}(\pi, w) := \Omega_4^{TOP}(\xi'(\pi, w))$ where $\xi'(\pi, w)$ is defined via the pullback

$$\begin{array}{ccc} B' & \longrightarrow & K(\pi, 1) \\ {\scriptstyle \xi'(\pi, w)} \downarrow & & \downarrow {\scriptstyle w} \\ BSTOP & \xrightarrow{w_2} & K(\mathbb{Z}/2, 2) . \end{array}$$

Note that we obtain the 'linear' fibrations $\xi(\pi, w)$ from the introduction by pulling back the topological ones via the natural 3-equivalence $BSO \to BSTOP$.

Proposition 4. *Let $\xi'(\pi, w) : B' \to BSTOP$ be a fibration as above. Then there is an exact sequence*

$$(****) \qquad 0 \to \Omega_4(\pi, w) \to \Omega_4^{TOP}(\pi, w) \xrightarrow{\mathfrak{ks}} \mathbb{Z}/2 \to 0$$

which splits if and only if $w \neq 0$.

Proof. The exactness of the sequence follows directly from the above information about Ω_i^{STOP} respectively $\Omega_i^{SpinTOP}$, $i \leqq 4$, by comparing the James spectral sequences of the fibrations $\xi(\pi, w)$ and $\xi'(\pi, w)$. Note that it is easy to generalize the James spectral sequence from vector bundles to fiber bundles with fiber $\mathbb{R}^n$ and zero-section. We only have to replace the relative fibration (disk bundle, sphere bundle) by the relative fibration (total space, total space\zero-section) to which we can also apply the relative Serre spectral sequence.

However, we also have to use topological transversality in dimension 4 (see [11]), to be able to use the Pontrjagin-Thom isomorphism

$$\Omega_4^{TOP}(\pi, w) \cong \pi_4(M\xi'(\pi, w)) .$$

It is clear that a splitting of the exact sequence (****) is the same as the choice of an element of order 2 in $\Omega_4^{TOP}(\pi, w)$ with nontrivial $\mathfrak{ks}$-invariant. If $w = 0$ then the relation (***) holds in $\Omega_4^{TOP}(\pi, w)$. This implies that a $\xi'(\pi, w)$-manifold with nontrivial $\mathfrak{ks}$-invariant has nontrivial signature and thus cannot have finite order. In particular, the exact sequence (****) does not split.

Now consider the case $w \neq 0$. The James spectral sequence for $\xi(\pi, w)$ and $\xi'(\pi, w)$ gives a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega_4^{Spin} \arrow[r] \arrow[d, "\cdot 2"] & F_{2,2}(\pi, w) \arrow[r, "p"] \arrow[d] & E^\infty_{2,2}(\pi, w) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega_4^{Spin\,TOP} \arrow[r] \arrow[d, "\mathfrak{ks}"] & F^{TOP}_{2,2}(\pi, w) \arrow[r, "p^{TOP}"] \arrow[d, "\mathfrak{ks}"] & E^\infty_{2,2}(\pi, w) \arrow[r] & 0 \\
& \mathbb{Z}/2 \arrow[r, equal] & \mathbb{Z}/2 \,. & &
\end{tikzcd}$$

Since $E^\infty_{2,2} = H_2(\pi; \mathbb{Z}/2)/\mathrm{Image}(d_i)$ is 2-torsion and the first vertical map is multiplication by 2, the middle horizontal sequence splits. This implies that the image of the signature on $F^{TOP}_{2,2}(\pi, w)$ is $8 \cdot \mathbb{Z}$ and therefore the map

$$\left(\frac{\sigma}{8}, p^{TOP}\right): F^{TOP}_{2,2}(\pi, w) \to \mathbb{Z} \times E^\infty_{2,2}(\pi, w)$$

is a well-defined isomorphism. Recall from part (3) of the Main Theorem that there exists a (differentiable) 4-manifold $[M] \in F_{2,2}(\pi, w)$ with w_2-type (π, w) and signature 8. It follows directly from the above isomorphism that

$$[M] - [|E_8|] \in F^{TOP}_{2,2}(\pi, w) \subseteq \Omega_4^{TOP}(\pi, w)$$

is an element of order 2 which clearly has nontrivial $\mathfrak{ks}$-invariant. $\square$

Corollary. *If $w \neq 0$, the image of the signature on $\Omega_4(\pi, w)$ equals the image of the signature on $\Omega_4^{TOP}(\pi, w)$.*

This corollary proves part (9) of our Main Theorem. The fact that 8 divides $\sigma^{TOP}(\pi, 0)$ is proven exactly as part (4) and thus the existence of $|E_8|$ finishes the proof of part (8).

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