What is an elliptic object?

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Dedicated to Graeme Segal on the occasion of his 60th birthday

Contents

1 Introduction

In these notes we propose an approach towards *enriched elliptic objects* over a manifold X. We hope that once made precise, these new objects will become cocycles in the generalized cohomology theory $\mathrm{tmf}^*(X)$ introduced by Hopkins and Miller [Ho], in a similar way as vector bundles over X represent elements in $K^*(X)$. We recall that one important role of $K^*(X)$ is as the home of the index of a family of Fredholm operators parametrized by X, e.g. the family of Dirac operators of a fiber bundle $E \to X$ with spin fibers. This is the parametrized version of the $\widehat{A}$-genus in the sense that the family index in $K^*(X)$ reduces to the $\widehat{A}$-genus of the fiber if X is a point. Similarly, one important role of $\mathrm{tmf}^*(X)$ is that it is the home of the parametrized version of the Witten genus in the sense that a fiber bundle $E \to X$ whose fibers are *string manifolds* (cf. section 5) gives rise to an element in $\mathrm{tmf}^*(X)$ [HBJ]. If X is a point this reduces to the Witten genus [Wi1] of the fiber (modulo torsion).

It would be very desirable to have a geometric/analytic interpretation of this parametrized Witten genus along the lines described above for the parametrized $\widehat{A}$-genus. For $X = \mathrm{pt}$, a heuristic interpretation was given by Witten who described the Witten genus of a string manifold M as the S^1-equivariant index of the 'Dirac operator on the free loop space' of M [Wi1] or as the 'partition function of the super symmetric non-linear σ-model' with target M [Wi2]; alas, neither of these things have been rigorously constructed yet. The construction of the parametrized Witten genus in $\mathrm{tmf}^*(X)$ is instead purely homotopy theoretic; the main ingredient is a Thom-isomorphism in tmf-cohomology for vector bundles with string structures. This is completely analogous to a description of the $\widehat{A}$-genus based on the Thom isomorphism in K-theory for spin vector bundles.

The cohomology theory $\mathrm{tmf}^*(X)$ derives from cohomology theories of the 'elliptic' flavor. The first such theory was constructed by Landweber and Stong [La] using Landweber's Exact Functor Theorem and the elliptic genus introduced by Ochanine [Och]. Ochanine's genus can be interpreted as coming from the formal group law associated to a particular elliptic curve; varying the elliptic curve used in the Landweber-Stong construction leads to a plethora of *elliptic cohomology theories*. The cohomology theory $\mathrm{tmf}^*(X)$ is not strictly speaking one of these, but essentially the 'inverse limit' (over the category of elliptic curves) of all these cohomology theories. (There are considerable technical difficulties with making this precise, in fact so far no complete written account is available.) Since *integral modular forms* can be defined as an inverse limit of an abelian group valued functor over the same category of elliptic curves, the elements of $\mathrm{tmf}^*(\mathrm{pt})$ are called *topological modular forms*. There is a ring homomorphism from $\mathrm{tmf}^*(\mathrm{pt})$ to the ring of integral modular forms which is rationally an isomorphism.

Unfortunately, the current *geometric* understanding of elliptic cohomology still is very much in its infancy despite the efforts of various people; see [Se1], [KS], [HK], [BDR]. The starting point of our new approach are the elliptic objects suggested by Graeme Segal in [Se1], which we call *Segal elliptic objects*. Segal's idea was to view a vector bundle $E \to X$ with connection as a *1-dimensional field theory over* X in the following sense: To each point $x \in X$, the bundle E associates a vector space E_x, and to each path in X the connection on E associates a linear map between these vector spaces. Segal suggested that a *2-dimensional conformal* field theory over X could be used as a cocycle for some elliptic cohomology theory. It would associate Hilbert spaces to loops in X, and Hilbert-Schmidt operators to conformal surfaces (with boundary) in X.

The main problem with Segal elliptic objects is that excision does not seem to hold. One of our contributions is to suggest a modification of the definition in order to get around this problem. This is where von Neumann algebras (associated to points in X) and their bimodules (associated to arcs in X) enter the picture. We will explain our modification in detail in the coming sections of this introduction. In the case $X = \mathrm{pt}$, we obtain in particular a modification of the notion of a *vertex operator algebra* (which was shown to be equivalent to a Segal elliptic object in [Hu], at least for genus zero surfaces; the super symmetric analogue appeared in [Ba]).

Another, more technical, problem in Segal's definition is that he had to introduce 'riggings' of 1- and 2-manifolds. These are certain additional structures (like parametrizations of the boundary circles) which we shall recall after Definition 4.1.1. Our first observation is that one can avoid these extra structures all together by enriching the conformal surfaces with fermions, and that these fermions give rise naturally to the degree of an elliptic object. This degree coincides for closed surfaces with the correct power of the determinant line as explained in [Se2] and in fact the space of fermions is a natural extension of the determinant line to surfaces with boundary (in the absence of parametrizations).

In Definition 4.1.3 we explain the resulting *Clifford elliptic objects of degree* n which includes a 'super symmetric' aspect. The reader should be warned that there remains an issue with how to make this super symmetric aspect precise; we formulate what we need as Hypothesis 3.3.13 and use it in the proof of Theorem 1.0.2 below.

We motivate these Clifford elliptic objects by first explaining carefully the K-theoretic analogues in Section 3. It turns out that the idea of a connection has to be modified because one needs the result of 'parallel transport' to depend on the length of the parametrizing interval. In other words, we explain how a K-cocycle is given by a super symmetric 1-dimensional *Euclidean* field theory, see Definition 3.2.2. In this simpler case, we do formulate the super symmetric aspect in detail and we discuss why it is essential for K-theory. As is well known, the best way to define K-theory in degree n, $K^n(X)$, is to introduce the finite dimensional Clifford algebras C_n. We shall explain how these algebras arise naturally when enriching intervals with fermions. We conclude in Section 3.2 the following new description of the K-theory spectrum:

Theorem 1.0.1. *For any $n \in \mathbb{Z}$, the space of super symmetric 1-dimensional Euclidean field theories of degree n has the homotopy type of K_{-n}, the $(-n)$-th space in the Ω-spectrum representing periodic K-theory.*

There is an analogous statement for periodic KO-theory, using real field theories.

Roughly speaking, Segal's idea, which we are trying to implement here, was to replace 1-dimensional by 2-dimensional field theories in the above theorem in order to obtain the spectrum of an elliptic cohomology theory. The following result is our first point of contact with modular forms and hence with $\mathrm{tmf}^n(X)$.

Theorem 1.0.2. *Given a degree n Clifford elliptic object E over X, one gets canonically a Laurent series*

$$MF(E) \in K^{-n}(X)[[q]][q^{-1}].$$

Moreover, if n is even and $X = \mathrm{pt}$, then $MF(E) \in \mathbb{Z}[[q]][q^{-1}]$ is the q-expansion of a 'weak' modular form of weight $n/2$. This means that the product of $MF(E)$ with a sufficiently large power of the discriminant Δ is a modular form.

In terms of our new definition of K-theory, the map $E \mapsto MF(E)$ is given by crossing with the standard circle S^1, and hence is totally geometric. As we shall explain, the length of an interval is very important in K-theory, and by crossing with S^1 it is turned into the conformal modulus of an annulus. The above result shows that the modularity aspects of an elliptic object are satisfied with only minor modifications of Segal's original definition. This is related to the fact that for $X = \mathrm{pt}$ the deficiency regarding excision is not present.

In Section 4 we make a major modification of Segal's elliptic objects and explain our *enriched elliptic objects* which are defined so that excision can be satisfied in the theory. Each enriched elliptic object gives in particular a Clifford elliptic object (which is closely related to a Segal elliptic object) but there are also data assigned to points and arcs in X, see Definition 1.2.1. Roughly speaking, in addition to Hilbert spaces associated to loops in X, we assign von Neumann algebras $\mathcal{A}(x)$ to points $x \in X$ and bimodules to arcs in X, in a way that Segal's Hilbert space can be decomposed as a Connes fusion of bimodules whenever the loop decomposes into arcs, see Section 1.2. One purpose of the paper is to make these statements precise. We shall not, however, give the ultimate definition of elliptic cocycles because various aspects of the theory have not been completely worked out yet.

Our main result, which to our mind justifies all definitions, is the following analogue of the tmf-orientation for string vector bundles [Ho, §6], [AHS]. As the underlying Segal elliptic object, we in particular recover in the case $E = TX$ the 'spinor bundle' over the loop space LX. Our enrichment expresses the locality (in X) of this spinor bundle. We expect that this enriched elliptic object will play the role of an elliptic Euler class and, in a relative version, of the elliptic Thom class.

Theorem 1.0.3. *Let E be an n-dimensional vector bundle over a manifold X. Assume that E comes equipped with a string structure and a string connection. Then there is a canonical degree n enriched elliptic object over X such that for all $x \in X$ the algebras $\mathcal{A}(x)$ are hyperfinite type III_1 factors. Moreover, if one varies the string connection then the resulting enriched elliptic objects are isomorphic.*

A vector bundle over X has a *string structure* if and only if the characteristic classes w_1, w_2 and $p_1/2$ vanish. In Section 5 we define a string structure on an n-dimensional spin bundle as a lift of the structure group in the following extension of topological groups:

$$1 \longrightarrow PU(A) \longrightarrow \mathrm{String}(n) \longrightarrow \mathrm{Spin}(n) \longrightarrow 1$$

Here A is an explicit hyperfinite type III_1 factor, the 'local fermions on the circle', cf. Example 4.3.2. Its unitary group is contractible (in the strong operator topology) and therefore the resulting projective unitary group $PU(A) = U(A)/\mathbb{T}$ is a $K(\mathbb{Z}, 2)$. The extension is constructed so that $\pi_3 \mathrm{String}(n) = 0$ which explains the condition on the characteristic class $p_1/2$. This interpretation of string structures is crucial for our construction of the enriched elliptic object in Theorem 1.0.3, the relation being given by a monomorphism

$$\mathrm{String}(n) \longrightarrow \mathrm{Aut}(A)$$

which arises naturally in the definition of the group extension above. It should be viewed as the 'fundamental representation' of the group $\mathrm{String}(n)$. The notion of a *string connection*, used in the above theorem, will be explained before Corollary 5.3.6.

1.1 Segal elliptic objects and Excision

A *Segal elliptic object* over X [Se1, p.199] associates to a map γ of a closed rigged 1-manifold to a target manifold X a topological vector space $H(\gamma)$, and to any conformal rigged surface Σ with map $\Gamma\colon \Sigma \to X$ a vector $\Psi(\Gamma)$ in the vector space associated to the restriction of Γ to $\partial\Sigma$ (we will define *riggings* in Definition 4.1.2 below). This is subject to the axiom

$$H(\gamma_1 \amalg \gamma_2) \cong H(\gamma_1) \otimes H(\gamma_2)$$

and further axioms for Ψ which express the fact that the gluing of surfaces (along closed submanifolds of the boundary) corresponds to the composition of linear operators. Thus an elliptic object over a point is a *conformal field theory* as axiomatized by Atiyah and Segal: it's a functor from a category $\mathcal{C}(X)$ to the category of topological vector spaces. Here the objects in $\mathcal{C}(X)$ are maps of closed rigged 1-manifolds into X, and morphisms are maps of conformal rigged surfaces into X.

Originally, the hope was that these elliptic objects would lead to a geometric description of elliptic cohomology. Unfortunately, excision for the geometric theory defined via elliptic objects didn't seem to work out. More precisely, consider the Mayer-Vietoris sequence

$$\dots \longrightarrow E^n(X) \longrightarrow E^n(U) \oplus E^n(V) \longrightarrow E^n(U \cap V) \longrightarrow \dots$$

associated to a decomposition $X = U \cup V$ of X into two open subsets $U, V \subset X$. This is an *exact sequence* for any cohomology theory $X \mapsto E^n(X)$. For K-theory the exactness of the above sequence at $E^n(U) \oplus E^n(V)$ comes down to the fact that a vector bundle $E \to X$ can be reconstructed from its restrictions to U and V.

Similarly, we expect that the proof of exactness for a cohomology theory built from Clifford elliptic objects of degree n would involve being able to reconstruct an elliptic object over $U \cup V$ from its restriction to U and V. This does not seem to be the case: suppose (H, Ψ) is an elliptic object over $U \cup V$ and consider two paths γ_1, γ_2 between the points x and y. Assume that the path γ_1 lies in U, that γ_2 lies in V, and denote by $\bar{\gamma}_2$ the path γ_2 run backwards. Then the restriction of (H, Ψ) to U (resp. V) contains not enough information on how to reconstruct the Hilbert space $H(\gamma_1 \cup \bar{\gamma}_2)$ associated to the loop $\gamma_1 \cup \bar{\gamma}_2$ in $U \cup V$.

1.2 Decomposing the Hilbert space

Our basic idea on how to overcome the difficulty with excision is to notice that in the basic geometric example coming from a vector bundle with string connection (see Theorem 1.0.3), there is the following additional structure: To a point $x \in X$ the string structure associates a graded type III_1-factor $\mathcal{A}(x)$ and to a finite number of points x_i it assigns the spatial tensor product of the $\mathcal{A}(x_i)$. Moreover, to a path γ from x to y, the string connection gives a graded right module $\mathcal{B}(\gamma)$ over $\mathcal{A}(\partial\gamma) = \mathcal{A}(x)^{\mathrm{op}} \bar{\otimes} \mathcal{A}(y)$. There are canonical isomorphisms over $\mathcal{A}(\partial\bar{\gamma}) = \mathcal{A}(y)^{\mathrm{op}} \bar{\otimes} \mathcal{A}(x)$ (using the 'conjugate' module from Section 4.3).

$$\mathcal{B}(\bar{\gamma}) \cong \overline{\mathcal{B}(\gamma)}$$

The punchline is that Hilbert spaces like $H(\gamma_1 \cup \bar{\gamma}_2)$ discussed above can be decomposed as

$$H(\gamma_1 \cup \bar{\gamma}_2) \cong \mathcal{B}(\gamma_1) \boxtimes_{\mathcal{A}(\partial\gamma_i)} \mathcal{B}(\bar{\gamma}_2).$$

where we used the *fusion product* of modules over von Neumann algebras introduced by Connes [Co1, V.B.δ]. Following Wassermann [Wa], we will refer to this operation as *Connes fusion.* Connes' definition was motivated by the fact that a homomorphism $A \to B$ of von Neumann algebras leads in a natural way to an $B - A$-bimodule such that composition of homomorphisms corresponds to his fusion operation [Co1, Prop. 17 in V.B.δ]. In [Wa], Wassermann used Connes fusion to define the correct product on the category of positive energy representations of a loop group *at a fixed level.*

We abstract the data we found in the basic geometric example from Theorem 1.0.3 by giving the following preliminary

Definition 1.2.1. (Preliminary!) A *degree n enriched elliptic object* over X is a tuple $(H, \Psi, \mathcal{A}, \mathcal{B}, \phi_{\mathcal{B}}, \phi_H)$, where

1. (H, Ψ) is a degree n Clifford elliptic object over X. In particular, it gives a Hilbert space bundle over the free loop space LX,

2. $\mathcal{A}$ is a von Neumann algebra bundle over X,

3. $\mathcal{B}$ is a module bundle over the free path space PX. Here the end point map $PX \to X \times X$ is used to pull back two copies of the algebra bundle $\mathcal{A}$ to PX, and these are the algebras acting on $\mathcal{B}$. The modules $\mathcal{B}(\gamma)$ come equipped with gluing isomorphisms (of $\mathcal{A}(x_1)^{\mathrm{op}}\bar{\otimes}\mathcal{A}(x_3)$-modules)

$$\phi_{\mathcal{B}}(\gamma,\gamma') : \mathcal{B}(\gamma \cup_{x_2} \gamma') \xrightarrow{\cong} \mathcal{B}(\gamma) \boxtimes_{\mathcal{A}(x_2)} \mathcal{B}(\gamma')$$

if γ is a path from x_1 to x_2, and γ' is a path from x_2 to x_3.

4. ϕ_H is an isomorphism of Hilbert spaces associated to each pair of paths γ_1 and γ_2 with $\partial\gamma_1 = \partial\gamma_2$:

$$\phi_H(\gamma_1,\bar{\gamma}_2)\colon H(\gamma_1 \cup_{\partial\gamma_i} \gamma_2) \xrightarrow{\cong} \mathcal{B}(\gamma_1) \boxtimes_{\mathcal{A}(\partial\gamma_i)} \mathcal{B}(\bar{\gamma}_2).$$

All algebras and modules are $\mathbb{Z}/2$-graded and there are several axioms that we require but haven't spelled out above.

Remark 1.2.2. This is only a preliminary definition for several reasons. Among others,

- we left out the conditions for surfaces glued along non-closed parts of their boundary. The vectors $\Psi(\Gamma)$ of a Clifford elliptic objects compose nicely when two surfaces are glued along *closed* submanifolds of the boundary, compare Lemma 2.3.14. Our enriched elliptic objects compose in addition nicely when two surfaces are glued along arcs in the boundary, see Proposition 4.3.10.

- we left out the super symmetric part of the story. We will explain in Section 3.2 why super symmetric data are essential even in the definition of K-theory.

- we left out the fermions from the discussions. These will be used to define the degree n of an elliptic object, and they are extra data needed so that a conformal spin surface actually gives a vector in the relevant Hilbert space. If the surface Σ is closed, then a fermion is a point in the n-th power of the Pfaffian line of Σ. Since the Pfaffian line is a square-root of the determinant line, this is consistent with the fact that a degree n elliptic object should give a modular form of weight $n/2$ when evaluated on tori, see Sections 3.3 and 4.1.

- Segal's Hilbert space associated to a circle will actually be *defined* by 4 above, rather than introducing the isomorphisms ϕ_H. So it will not play a central role in the theory, but can be reconstructed from it. At this point, we wanted to emphasize the *additional* data needed to resolve the problem with excision, namely a decomposition of Segal's Hilbert space.

Most of these deficiencies will be fixed in Section 4 by defining a degree n enriched elliptic object as a certain functor from a bicategory $\mathcal{D}_n(X)$ made from d-manifolds (with n fermions) mapping into X, $d = 0, 1, 2$, to the bicategory vN of von Neumann algebras, their bimodules and intertwiners.

We end this introduction by making a brief attempt to express the meaning of an enriched elliptic object over X in physics lingo: it is a conformal field theory with $(0,1)$ super symmetry (and target X), whose fermionic part has been quantized but whose bosonic part is classical. This comes from the fact that the Fock spaces are a well established method of fermionic quantization, whereas there is up to date no mathematical way of averaging the maps of a surface to a curved target X. Moreover, the enriched (0-dimensional) aspect of the theory is some kind of an open string theory. It would be very interesting to relate it to Cardy's boundary conformal field theories.

1.3 Disclaimer and Acknowledgments

This paper is a survey of our current understanding of the geometry of elliptic objects. Only ideas of proofs are given, and some proofs are skipped all together. We still believe that it is of service to the research community so make such work in progress accessible.

It is a pleasure to thank Dan Freed, Graeme Segal and Antony Wassermann for many discussions about conformal field theory. Graeme's deep influence is obvious, and Dan's approach to Chern-Simons theory [Fr1] motivated many of the considerations in Section 5. Antony's groundbreaking work [Wa] on Connes fusion for positive energy representations was our starting point for the central definitions in Section 4. He also proof-read the operator algebraic parts of this paper, all remaining mistakes were produced later in time.
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2 Field theories

Following Graeme Segal [Se2], we explain in this section the axiomatic approach to field theories, leading up to a definition of 'Clifford linear field theories of degree n' (cf. definitions 2.3.16 and 2.3.20) after introducing the necessary background on Fock spaces, spin structures and Dirac operators.

2.1 d-dimensional field theories

Roughly speaking, a d-dimensional field theory associates to a closed manifold Y of dimension $d-1$ a Hilbert space $E(Y)$ and to a bordism Σ from Y_1 to Y_2 a Hilbert

Schmidt operator $E(Y_1) \to F(Y_2)$ (a bounded operator T is *Hilbert-Schmidt* if the sum of the norm squares of its matrix elements is finite). The main requirement is that gluing bordisms should correspond to composing the associated operators. As is well-known, this can be made precise by defining a *d-dimensional field theory* to be a functor

$$E\colon \mathcal{B}^d \longrightarrow \mathrm{Hilb},$$

from the d-dimensional bordism category $\mathcal{B}^d$ to the category Hilb of complex Hilbert spaces which are compatible with additional structures on these categories spelled out below. The precise definition of the categories $\mathcal{B}^d$ and Hilb is the following:

- The objects of the *d-dimensional bordism category* $\mathcal{B}^d$ are closed oriented manifolds of dimension $d-1$, equipped with geometric structures which characterize the flavor of the field theory involved (see remarks below). If Y_1, Y_2 are objects of $\mathcal{B}^d$, the orientation preserving geometric diffeomorphisms from Y_1 to Y_2 are morphisms from Y_1 to Y_2 which form a subcategory of $\mathcal{B}^d$. There are other morphisms, namely oriented geometric bordisms from Y_1 to Y_2; i.e., d-dimensional oriented manifolds Σ equipped with a geometric structure, together with an orientation preserving geometric diffeomorphism $\partial\Sigma \cong \bar{Y}_1 \amalg Y_2$, where $\bar{Y}_1$ is Y_1 equipped with the opposite orientation. More precisely, two bordisms Σ and Σ' are considered the *same* morphism if they are orientation preserving geometric diffeomorphic relative boundary. Composition of bordisms is given by gluing; the composition of a bordism Σ from Y_1 to Y_2 and a diffeomorphism $Y_2 \to Y_3$ is again the bordism Σ, but with the identification $\partial\Sigma \cong \bar{Y}_1 \amalg Y_2$ modified by composition with the diffeomorphism $Y_2 \to Y_3$.

- The objects of Hilb are separable Hilbert spaces (over the complex numbers). The morphisms from H_1 to H_2 are the bounded operators $T\colon H_1 \to H_2$; the strong topology on the space of bounded operators makes Hilb a topological category.

Without additional geometric structures on the objects and the bordisms, such a field theory would be referred to as a *topological* field theory. If the geometric structures are conformal structures on bordisms and objects, the associated field theory is called *conformal* (for short CFT). If the conformal structure is a replaced by a Riemannian metric, one obtains what is usually referred to as a *Euclidean* field theory (EFT) to distinguish it from the Lorentz case. We sometimes use the term *field theory* (FT) if the geometric structures are not specified.

The main examples of field theories in these notes will have at least a conformal structure on the manifolds, and in addition all manifolds under consideration will be equipped with a spin structure (see Definition 2.3.4 for a careful explanation of spin structures on conformal manifolds). It is important to point out that every spin manifold has a canonical involution associated to it (which doesn't move the points of the manifold but flips the two sheets of the spin bundle). This has the effect that all algebraic objects associated to spin manifolds will be $\mathbb{Z}/2$-graded. This is the first step towards super symmetry and our reason for introducing spin structures in the main Definitions 2.3.16 and 2.3.20. We should point out that those definitions (where the categories of geometric manifolds are denoted by $\mathcal{CB}^2_n$ respectively $\mathcal{EB}^1_n$) introduce the spin structures (and the degree n) *for the first time.* The following warm-up discussions, in particular Definition 2.1.3, only use an orientation, not a spin structure (even though the notation $\mathcal{CB}^2$ respectively $\mathcal{EB}^1$ is very similar).

Summarizing, the reader should expect spin structures whenever there is a degree n in the discussion. Indeed, we will see that the degree makes sense only in the presence of spin structures.

Definition 2.1.1. (Additional structures on the categories $\mathcal{B}^d$ and Hilb).

- **Symmetric monoidal structures.** The disjoint union of manifolds (respectively the tensor product of Hilbert spaces) gives $\mathcal{B}^d$ (resp. Hilb) the structure of symmetric monoidal categories. The unit is given by the empty set and $\mathbb{C}$, respectively.

- **Involutions and anti-involutions.** There are involutions $\mathcal{B}^d \to \mathcal{B}^d$ and Hilb $\to$ Hilb. On the category $\mathcal{B}^d$ this involution is given by reversing the orientation on the d-manifold (objects) as well as the bordisms (morphisms); this operation will be explained in detail in Definition 2.3.1. We note that if Σ is a bordism from Y_1 to Y_2, then Σ with the opposite orientation can be interpreted as a bordism from $\bar{Y}_1$ to $\bar{Y}_2$. For an object $H \in$ Hilb, $\bar{H}$ is the space H with the opposite complex structure; for a morphism $f\colon H_1 \to H_2$, the morphism $\bar{f}\colon \bar{H}_1 \to \bar{H}_2$ is equal to f as a map of sets.

  There are also anti-involutions (i.e., contravariant functors) $*\colon \mathcal{B}^d \to \mathcal{B}^d$ and $*\colon$ Hilb $\to$ Hilb. These are the identity on objects. If $T\colon H_1 \to H_2$ is a bounded operator, then $T^*\colon H_2 \to H_1$ is its adjoint; similarly, if Σ is a bordism from Y_1 to Y_2, then Σ^* is Σ with the opposite orientation, considered as a morphism from Y_2 to Y_1. Finally, if ϕ is a diffeomorphism from Y_1 to Y_2 then $\phi^* \stackrel{\text{def}}{=} \phi^{-1}$.

- **Adjunction transformations.** There are natural transformations

  $$\mathcal{B}^d(\emptyset, Y_1 \amalg Y_2) \longrightarrow \mathcal{B}^d(\bar{Y}_1, Y_2) \qquad \mathrm{Hilb}(\mathbb{C}, H_1 \otimes H_2) \longrightarrow \mathrm{Hilb}(\bar{H}_1, H_2)$$

  On $\mathcal{B}^d$ this is given by reinterpreting a bordism Σ from $\emptyset$ to $Y_1 \amalg Y_2$ as a bordism from $\bar{Y}_1$ to Y_2. On Hilb we can identify $\mathrm{Hilb}(\mathbb{C}, H_1 \otimes H_2)$ with the space of Hilbert-Schmidt operators from H_1 to H_2 and the transformation is the inclusion from Hilbert-Schmidt operators into all bounded operators. It should be stressed that neither transformation is in general surjective: in the category $\mathcal{B}^d$, a diffeomorphism from Y_1 to Y_2 is not in the image; in Hilb, not every bounded operator is a Hilbert-Schmidt operator. For example, a diffeomorphism ϕ gives a unitary operator $E(\phi)$ (if the functor E preserves the anti-involution $*$ above). In infinite dimensions, a unitary operator is never Hilbert-Schmidt.

Remark 2.1.2. In the literature on field theory, the functor E always respects the above involution, whereas E is called a *unitary* field theory if it also respects the anti-involution. It is interesting that in our honest example in Section 4.3 there are actually 3 (anti) involutions which the field theory has to respect.

Definition 2.1.3. The main examples of field theories we will be interested in are 2-dimensional conformal field theories and 1-dimensional Euclidean field theories. We will use the following terminology: A *conformal field theory* or *CFT* is a functor

$$E\colon \mathcal{CB}^2 \to \mathrm{Hilb}$$

compatible with the additional structures on the categories detailed by Definition 2.1.1 where $\mathcal{CB}^2$ is the 'conformal' version of $\mathcal{B}^2$, i.e., the bordisms in this category are 2-dimensional and equipped with a conformal structure. A *Euclidean field theory* or *EFT* is a functor

$$E\colon \mathcal{EB}^1 \to \mathrm{Hilb}$$

compatible with the additional structures of Definition 2.1.1 where $\mathcal{EB}^1$ is the 'Euclidean' version of $\mathcal{B}^1$, i.e., the bordisms in this category are 1-dimensional and equipped with a Riemannian metric.

Example 2.1.4. Let M be a closed Riemannian manifold, let $H = L^2(M)$ be the Hilbert space of square integrable functions on M and let $\Delta\colon H \to H$ be the Laplace operator. Then we can construct a 1-dimensional EFT $E\colon \mathcal{EB}^1 \to \mathrm{Hilb}$ by defining

$$E(\mathrm{pt}) = H \qquad E(I_t) = e^{-t\Delta} \qquad E(S_t) = \mathrm{tr}(e^{-t\Delta}).$$

Here pt is the one-point object in $\mathcal{EB}^1$, I_t is the interval of length t, considered as a morphism from pt to pt, and $S_t \in \mathcal{EB}^1(\emptyset, \emptyset)$ is the circle of length t. We note that unlike the Laplace operator Δ the heat operator $e^{-t\Delta}$ is a bounded operator, even a trace class operator and hence it is meaningful to take the trace of $e^{-t\Delta}$.

It is not hard to show that the properties in Definition 2.1.1 allow us to extend E uniquely to a real EFT. More interestingly, the operator $E(\Sigma)\colon E(Y_1) \to E(Y_2)$ associated to a bordism Σ from Y_1 to Y_2 can be described in terms of a path integral over the space of maps from Σ to M. This is the Feynman-Kac formula, which for $\Sigma = I_t$ gives $e^{-t\Delta}$.

Definition 2.1.5. More generally, if X is a manifold, we may replace the category $\mathcal{B}^d$ above by the category $\mathcal{B}^d(X)$, whose objects are closed oriented $(d-1)$-manifolds equipped with a piecewise smooth map to X; similarly the morphisms of $\mathcal{B}^d(X)$ are oriented bordisms equipped with maps to X and orientation preserving diffeomorphisms compatible with the given maps to X. We note that $\mathcal{B}^d(X)$ can be identified with $\mathcal{B}^d$ if X is a point. The four structures described above on the bordism category $\mathcal{B}^d$ can be extended in an obvious way to the category $\mathcal{B}^d(X)$. We define a *d-dimensional field* *theory over* X to be a functor $E\colon \mathcal{B}^d(X) \to \text{Hilb}$ which is compatible with the monoidal structure, (anti)-involutions, and adjunction transformations mentioned above. Analogously, we can form the categories $\mathcal{CB}^2(X)$ (resp. $\mathcal{EB}^1(X)$) of 2-dimensional conformal bordisms over X (resp. 1-dimensional Euclidean bordisms over X).

Example 2.1.6. This is a 'parametrized' version of Example 2.1.4 (which in the notation below is the case $X = \text{pt}$ and $Z = M$). Suppose that $\pi\colon Z \to X$ is a Riemannian submersion. Then we can construct a 1-dimensional EFT $E\colon \mathcal{EB}^1(X) \to \text{Hilb}$ over X as follows. On objects, E associates to a map γ from a 0-manifold Y to X the Hilbert space of L^2-functions on the space of lifts $\{\tilde{\gamma}\colon Y \to Z \mid \pi \circ \tilde{\gamma} = \gamma\}$ of γ; in particular if $Y = \text{pt}$ and $\gamma(\text{pt}) = x$, then $E(Y, \gamma)$ is just the space of L^2-functions on the fiber over x. We can associate an operator $E(\Sigma, \Gamma)\colon E(Y_1, \gamma_1) \to E(Y_2, \gamma_2)$ to a bordism (Σ, Γ) from (Y_1, γ_1) to (Y_2, γ_2) by integrating over the space of maps $\tilde{\Gamma}\colon \Sigma \to Z$ which are lifts of $\Gamma\colon \Sigma \to X$. For $\Sigma = I_t$ and if Γ maps all of Σ to the point x, then the operator constructed this way is via the Feynman-Kac formula just $e^{-t\Delta_x}$, where Δ_x is the Laplace operator on the fiber over x.

2.2 Clifford algebras and Fock modules

Definition 2.2.1. (Clifford algebras). Let V be a real or complex Hilbert space equipped with an isometric involution $\alpha\colon V \to V$, $v \mapsto \bar{v} = \alpha(v)$ ($\mathbb{C}$-*anti*-linear in the complex case). This implies that

$$b(v, w) \overset{\text{def}}{=} \langle \bar{v}, w \rangle$$

is a symmetric bilinear form (here $\langle\ ,\ \rangle$ is the inner product on V, which is $\mathbb{C}$-anti-linear in the first and linear in the second slot in the complex case).

The *Clifford algebra* is the quotient of the real resp. complex tensor algebra generated by V by imposing the Clifford relations

$$v \cdot v = -b(v, v) \cdot 1 \qquad v \in V.$$

Suppressing the dependence on the involution in the notation, we'll just write $C(V)$ for this algebra. It is a $\mathbb{Z}/2$-graded algebra with grading involution $\epsilon\colon C(V) \to C(V)$ induced by $v \mapsto -v$ for $v \in V$; the inner product on V extends to an inner product on the Clifford algebra $C(V)$.

We will write $-V$ for the Hilbert space furnished with the involution $-\alpha$. We will adopt the convention that if an involution α on V has not been explicitly specified, then it is assumed to be the identity. For example,

- $C_n \overset{\text{def}}{=} C(\mathbb{R}^n)$ is the Clifford algebra generated by vectors $v \in \mathbb{R}^n$ subject to the relation $v \cdot v = -|v|^2 \cdot 1$,

- $C_{-n} \stackrel{\text{def}}{=} C(-\mathbb{R}^n)$ is the Clifford algebra generated by vectors $v \in \mathbb{R}^n$ subject to the relation $v \cdot v = |v|^2 \cdot 1$, and

- $C_{n,m} \stackrel{\text{def}}{=} C(\mathbb{R}^n \oplus -\mathbb{R}^m)$ is the Clifford algebra generated by vectors $v \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ subject to the relations $v \cdot v = -|v|^2 \cdot 1$, $w \cdot w = |w|^2 \cdot 1$, $v \cdot w + w \cdot v = 0$. We will use repeatedly that $C_{n,n}$ is of real type for all n:

$$C_{n,n} \cong M_{2^n}(\mathbb{R})$$

The reader should be warned that conventions in the literature concerning Clifford algebras vary greatly; our conventions with regards to C_n and $C_{n,m}$ agree for example with [Ka, Ch. III, 3.13] and [LM, Ch. I, §3].

Remark 2.2.2. (Properties of Clifford algebras.) Useful properties of the construction $V \mapsto C(V)$ include natural isomorphisms

$$C(V \oplus W) \cong C(V) \otimes C(W) \qquad \text{and} \qquad C(-V) \cong C(V)^{\text{op}}; \tag{2.2.3}$$

here as throughout the paper $\otimes$ stands for the *graded tensor product*; here the adjective 'graded' stipulates that the product of elements $a \otimes b, a' \otimes b' \in A \otimes B$ is defined by

$$(a \otimes b) \cdot (a' \otimes b') \stackrel{\text{def}}{=} (-1)^{|b||a'|} aa' \otimes bb',$$

where $|b|, |a'|$ are the degrees of b and a' respectively. The *opposite* B^{op} of a graded algebra B is B as graded vector space but with new a multiplication $*$ defined by $a * b \stackrel{\text{def}}{=} (-1)^{|a||b|} b \cdot a$ for homogeneous elements $a, b \in B$ of degree $|a|, |b| \in \mathbb{Z}/2$, respectively. Any graded left $A \otimes B$-module M can be interpreted as a bimodule over $A - B^{\text{op}}$ via

$$a \cdot m \cdot b \stackrel{\text{def}}{=} (-1)^{|m||b|} (a \otimes b) m$$

for homogeneous elements $a \in A$, $b \in B$, $m \in M$) and vice-versa. In particular, a left module M over $C(V \oplus -W)$ may be interpreted as a left module over $C(V) \otimes C(W)^{\text{op}}$; or equivalently, as a $C(V) - C(W)$-bimodule, and we will frequently appeal to this move. We note that this construction is compatible with 'passing to the opposite module', where we define the *opposite* of a graded $A - B$-module M, denoted $\overline{M}$, to be M with the grading involution ϵ replaced by $-\epsilon$ and the right B-action modified by the grading automorphism of B. This is consistent (by the above formula) with changing the grading and keeping *the same* $A \otimes B^{\text{op}}$-module structure.

Definition 2.2.4. (Fermionic Fock spaces). Let V be a Hilbert space with an isometric involution as in Definition 2.2.1. There is a standard construction of modules over the resulting Clifford algebra $C(V)$ (cf. [PS, Ch. 12], [A]); the input datum for this construction is a *Lagrangian* $L \subset V$. *By definition*, this means that L is closed, b

vanishes identically on L and that $V = L \oplus \bar{L}$. Note that the existence of L is a serious condition on our data, for example a Lagrangian cannot exist if the involution on V is trivial.

Given a Lagrangian L, the exterior algebra

$$\Lambda(\bar{L}) = \Lambda^{ev}(\bar{L}) \oplus \Lambda^{odd}(\bar{L}) = \bigoplus_{p \text{ even}} \Lambda^p(\bar{L}) \oplus \bigoplus_{p \text{ odd}} \Lambda^p(\bar{L})$$

is a $\mathbb{Z}/2$-graded module over the Clifford algebra $C(V)$:

- for $\bar{v} \in \bar{L} \subset V \subset C(V)$, the corresponding operator $c(\bar{v})\colon \Lambda(\bar{L}) \to \Lambda(\bar{L})$ is given by exterior multiplication by $\bar{v}$ ('creation operator'),
- for $v \in L$, the operator $c(v)$ is given by interior multiplication by v ('annihilation operator'); i.e., $c(v)$ acts as a graded derivation on $\Lambda(\bar{L})$, and for $\bar{w} \in \bar{L} = \Lambda^1(\bar{L})$ we have $c(v)\bar{w} = b(v, \bar{w}) = \langle w, v \rangle$.

We define the *fermionic Fock space* $F(L)$ to be the completion of $\Lambda(\bar{L})$ with respect to the inner product induced by the inner product on $\bar{L} \subset V$. We will refer to $F_{alg}(L) \stackrel{\text{def}}{=} \Lambda(\bar{L})$ as the *algebraic Fock space*; both of these $C(V)$-modules will play an important role for us.

We note that the adjoint $c(v)^*$ of the operator $c(v)\colon F(L) \to F(L)$ is given by $c(v)^* = -c(\bar{v})$ for any $v \in V$. It is customary to call $1 \in \Lambda^0 \bar{L} \subset F(L)$ the *vacuum vector* and to write $\Omega \in F(L)$ for it. It is easy to see that Ω is a cyclic vector and hence $F(L)$ is a graded irreducible module over $C(V)$. The classification of these modules is given by the following well-known result (cf. [A])

Theorem 2.2.5 (I. Segal-Shale equivalence criterion). *Two Fock representations $F(L)$ and $F(L')$ of $C(V, b)$ are isomorphic if and only if the composition of orthogonal inclusion and projection maps,*

$$L' \hookrightarrow V \twoheadrightarrow \bar{L}$$

is a Hilbert-Schmidt operator. Moreover, this isomorphism preserves the grading if and only if $\dim(\bar{L} \cap L')$ *is even.*

We recall that an operator $T\colon V \to W$ between Hilbert spaces is a *Hilbert-Schmidt operator* if and only if the sum $\sum_{i=1}^{\infty} ||Te_i||^2$ converges, where $\{e_i\}$ is a Hilbert space basis for V. We note that the space $\bar{L} \cap L'$ is finite dimensional if the map $L' \to \bar{L}$ is a Hilbert-Schmidt operator.

Remark 2.2.6. (Orientations and bimodules). Let V be a real inner product space of dimension $n < \infty$. Then there is a homeomorphism

$$\{\text{isometries } f\colon \mathbb{R}^n \to V\} \longrightarrow \mathcal{L} \stackrel{\text{def}}{=} \{\text{Lagrangian subspaces } L \subset V \oplus -\mathbb{R}^n\}$$

given by sending an isometry f to its graph. By passing to connected components, we obtain a bijection between orientations on V and $\pi_0\mathcal{L}$. According to the Segal-Shale Theorem (plus the fact that in finite dimensions *any* irreducible module is isomorphic to some Fock space), sending a Lagrangian L to the Fock space $F(L)$ induces a bijection between $\pi_0\mathcal{L}$ and the set of isomorphism classes of irreducible graded (left) $C(V \oplus -\mathbb{R}^n)$-modules; as explained in (2.2.3), these may in turn be interpreted as $C(V) - C_n$-bimodules. Summarizing, we can identify orientations on V with isomorphism classes of irreducible $C(V) - C_n$-bimodules $S(V)$. We observe that the opposite bimodule $\overline{S(V)}$, defined above 2.2.4, corresponds to the opposite orientation.

Remark 2.2.7. (Functorial aspects of the Fock space construction). Let V_1, V_2 be Hilbert spaces with involutions as in definition 2.2.1 and let $L_1 \subset V_2 \oplus -V_1$ be a Lagrangian. The associated algebraic Fock space $F_{alg}(L_1)$ (cf. Definition 2.2.4) is then a graded module over the Clifford algebra $C(V_2 \oplus -V_1)$; alternatively we can view it as a bimodule over $C(V_2) - C(V_1)$. We wish to discuss in which sense the constructions $V \mapsto C(V)$ and $L \mapsto F_{alg}(L)$ give a *functor* (cf. [Se2, §8]). Here the objects of the 'domain category' are Hilbert spaces V with involutions, and morphisms from V_1 to V_2 are Lagrangian subspaces of $V_2 \oplus -V_1$. Given morphisms $L_1 \subset V_2 \oplus -V_1$ and $L_2 \subset V_3 \oplus -V_2$, their composition is given by the Lagrangian $L_3 \subset V_3 \oplus -V_1$ obtained by 'symplectic' reduction from the Lagrangian $L \overset{\text{def}}{=} L_2 \oplus L_1 \subset V \overset{\text{def}}{=} V_3 \oplus -V_2 \oplus V_2 \oplus -V_1$, namely

$$L^{red} \overset{\text{def}}{=} L \cap U^{\perp_b} / L \cap U \subset V^{red} \overset{\text{def}}{=} V \cap U^{\perp_b}/U;$$

Here U is the isotropic subspace $U = \{(0, v_2, v_2, 0) \mid v_2 \in V_2\} \subset V$ and $U^{\perp_b}$ is its annihilator with respect to the bilinear form b. We note that the reduced space V^{red} can be identified with $V_3 \oplus -V_1$.

The objects of the 'range category' are graded algebras; the morphisms from A to B are pointed, graded $B - A$-bimodules; the composition of a pointed $B - A$-bimodule (M, m_0) and a pointed $C - B$-bimodule (N, n_0) is given by the $C - A$-bimodule $(N \otimes_B M, n_0 \otimes m_0)$. The following lemma shows that in the type I case composition of Lagrangians is compatible with the tensor product of pointed bimodules, i.e., the construction $V \mapsto C(V)$, $L \mapsto (F_{alg}(L), \Omega)$ is a (lax) functor. Here 'type' refers to the type of the von Neumann algebra generated by $C(V)$ in $B(F(L))$ as explained in Section 4.3. Type I is the easiest case where the von Neumann algebra is just the bounded operators on some Hilbert space. This corresponds geometrically to gluing along closed parts of the boundary. Gluing along, say, arcs in the boundary corresponds to type III for which a more difficult gluing lemma is needed: Connes fusion appears, see Proposition 4.3.10. It actually covers all types, so we restrict in the arguments below to the finite dimensional case. That is all one needs for 1-dimensional EFT's, i.e. for K-theory.

Gluing Lemma 2.2.8. *If the von Neumann algebra generated by $C(V_2)$ has type I, there is a unique isomorphism of pointed, graded $C(V_3) - C(V_1)$ bimodules*

$$(F_{alg}(L_2) \otimes_{C(V_2)} F_{alg}(L_1), \Omega_2 \otimes \Omega_1) \cong (F_{alg}(L_3), \Omega_3).$$

Here we assume that L_i intersect V_j trivially (which is satisfied in the geometric applications if there are no closed components, cf. Definition 2.3.12).

Proof. We note that $F_{alg}(L_2) \otimes_{C(V_2)} F_{alg}(L_1)$ is the quotient of $F_{alg}(L_2) \otimes F_{alg}(L_1) = F_{alg}(L_2 \oplus L_1) = F_{alg}(L)$ modulo the subspace $\bar{U} F_{alg}(L)$. Here $\bar{U} \subset V$ is the subspace obtained from U defined above by applying the involution $v \mapsto \bar{v}$; explicitly, $\bar{U} = \{(0, -v_2, v_2, 0) \mid v_2 \in V_2\}$; we observe that for $\bar{u} = (0, -v_2, v_2, 0) \in \bar{U}$, and $\psi_i \in F_{alg}(L_i)$ we have

$$c(\bar{u})(\psi_2 \otimes \psi_1) = (-1)^{|\psi_2|}(-\psi_2 c(v_2) \otimes \psi_1 + \psi_2 \otimes c(v_2)\psi_1).$$

We recall that an element $\bar{u} \in \bar{U} \subset V$, which decomposes as $\bar{u} = u_1 + \bar{u}_2 \in V = L \oplus \bar{L}$ with $u_i \in L$ acts on $F_{alg}(L) = \Lambda(\bar{L})$ as the sum $c(u_1) + c(\bar{u}_2)$ of the 'creation' operator $c(u_1)$ and the 'annihilation' operator $c(\bar{u}_2)$. We observe that by assumption the map $L^{red} \oplus \bar{U} \to L$ given by $(v, \bar{u}) \mapsto v + u_1$ is an isomorphism. In finite dimensions, a filtration argument shows that the $C(V^{red})$-linear map

$$\Lambda(\bar{L}^{red}) \longrightarrow \Lambda(\bar{L}^{red} \oplus U)/c(\bar{U})\Lambda(\bar{L}^{red} \oplus \bar{U})$$

is in fact an isomorphism. □

Definition 2.2.9. (Generalized Lagrangian). For our applications to geometry, we will need a slightly more general definition of a Lagrangian. This will also avoid the assumption in the gluing lemma above. A *generalized Lagrangian* of a Hilbert space V with involution is a homomorphism $L\colon W \to V$ with finite dimensional kernel so that the closure $L_W \subset V$ of the image of L is a Lagrangian. In the geometric situation we are interested in, W will be the space of harmonic spinors on a manifold Σ, V will be the space of all spinors on the boundary $\partial\Sigma$, and L is the restriction map. Then we define the *algebraic Fock space*

$$F_{alg}(L) \stackrel{\text{def}}{=} \Lambda^{top}(\ker L)^* \otimes \Lambda(\bar{L}_W),$$

where $\Lambda^{top}(\ker L)^* = \Lambda^{\dim(\ker L)}(\ker L)^*$ is the top exterior power of the dual space of the kernel of L. The algebraic Fock space is a module over the Clifford algebra $C(V)$ via its action on $\Lambda(\bar{L}_W)$.

Unlike the case discussed previously, this Fock space has only a canonical vacuum element $\Omega = 1 \otimes 1$ if $\ker L = 0$. Otherwise the vacuum vector is zero which is consistent with the geometric setting where it corresponds to the Pfaffian element of the Dirac operator: it vanishes, if there is a nontrivial kernel. Therefore, the gluing Lemma 2.3.14 in the following section has to be formulated more carefully than the gluing lemma above.

2.3 Clifford linear field theories

We recall that a d-dimensional field theory is a functor $E\colon \mathcal{B}^d \to \mathrm{Hilb}$; in particular on objects, it assigns to a closed oriented $(d-1)$-manifold Y a Hilbert space $E(Y)$. It is the purpose of this section to define *Clifford linear field theories* E *of degree* n (for $d = 1, 2$). Such a theory assigns to Y as above a Hilbert space $E(Y)$ which is a right module over $C(Y)^{\otimes n}$, where $C(Y)$ is a Clifford algebra associated to Y. The formal definition (see Definition 2.3.16 for $d = 2$ and Definition 2.3.20 for $d = 1$) is quite involved. The reader might find it helpful to look first at Example 2.3.3, which will be our basic example of a Clifford linear field theory (for $d = 1$) and which motivates our definition. This example is a variation of Example 2.1.4 with the Laplace operator replaced by the square of the Dirac operator.

Definition 2.3.1. (Spin structures on Riemannian vector bundles). Let V be an inner product space of dimension d. Motivated by Remark 2.2.6 we define a *spin structure* on V to be an irreducible graded $C(V) - C_d$-bimodule $S(V)$ (equipped with a compatible inner product as in the case of Fock spaces). If W is another inner product space with spin structure, a *spin isometry* from V to W is an isometry $f\colon V \to W$ together with an isomorphism $\hat{f}\colon S(V) \overset{\cong}{\to} f^*S(W)$ of graded $C(V) - C_d$-bimodules with inner products. We note that $f^*S(W)$ is isomorphic to $S(V)$ if and only if f is orientation preserving; in that case there are *two* choices for $\hat{f}$. In other words, the space of spin isometries $\mathrm{Spin}(V, W)$ is a double covering of the space $SO(V, W)$ of orientation preserving isometries. It is clear that spin isometries can be composed and so they can be regarded as the morphisms in a category of inner product spaces with spin structures.

Now we can use a 'parametrized version' of the above to define spin structures on vector bundles as follows. Let $E \to X$ be a real vector bundle of dimension d with Riemannian metric, i.e., a fiberwise positive definite inner product. Let $C(E) \to X$ be the Clifford algebra bundle, whose fiber over x is the Clifford algebra $C(E_x)$. A *spin structure* on E is a bundle $S(E) \to X$ of graded irreducible $C(E) - C_d$-bimodules. It is tempting (at least for topologists) to think of two isomorphic bimodule bundles as giving the *same* spin structure. However, it is better to think of the 'category of spin structures' (with the obvious morphisms), since below we want to consider the space of sections of $S(E)$ and that is a functor from this category to the category of vector spaces. Then the usual object topologists are interested in are the *isomorphism classes* of spin structures. The group $H^1(X; \mathbb{Z}/2)$ acts freely and transitively on the set of isomorphism classes.

To relate this to the usual definition of spin structure expressed in terms of a principal $\mathrm{Spin}(d)$-bundle $\mathrm{Spin}(E) \to X$ (cf. [LM, Ch. II, §1]), we note that we obtain a $C(E) - C_d$-bimodule bundle if we define

$$S(E) \overset{\mathrm{def}}{=} \mathrm{Spin}(E) \times_{\mathrm{Spin}(d)} C_d.$$

Moreover, we note that $S(E)$ determines an orientation of E by Remark 2.2.6. We define the *opposite spin structure* on E to be $\overline{S(E)}$ (whose fiber over $x \in X$ is the bimodule opposite to $S(E_x)$ in the sense of Remark 2.2.6); this induces the opposite orientation on E.

Remark 2.3.2. We note that there is a functor F from the category of spin-structures on $E \oplus \mathbb{R}$ to the category of spin structures on E. Given a spin structure on $E \oplus \mathbb{R}$, i.e., a $C(E \oplus \mathbb{R}) - C_{d+1}$-bimodule bundle $S \to X$ over, we define $F(S) \stackrel{\text{def}}{=} S^+(E \oplus \mathbb{R})$, the even part of $S(E \oplus \mathbb{R})$. This is a graded $C(E) - C_d$-bimodule, if we define the grading involution on $S^+(E \oplus \mathbb{R})$ by $\psi \mapsto e_1 \psi e_1$ ($e_1 \in \mathbb{R}$ is the standard unit vector), the left action of $v \in E \subset C(E)$ by $\psi \mapsto v e_1 \psi$ and the right action of $w \in \mathbb{R}^d \subset C_d$ by $\psi \mapsto \psi e_1 w$. The functor F is compatible with 'passing to the opposite spin structure' in the sense that there is an isomorphism of spin structures $\overline{F(S)} \cong F(\overline{S})$, which is natural in S.

Example 2.3.3. (EFT associated to a Riemannian spin manifold). Let M be a closed manifold of dimension n with a spin structure; i.e., a spin structure on its cotangent bundle T^*M. In other words, M comes equipped with a graded irreducible $C(T^*M) - C_n$-bimodule bundle $S \to M$. A Riemannian metric on M induces the Levi-Civita connection on the tangent bundle TM which in turn induces a connection ∇ on S. The *Dirac operator* $D = D_M$ is the composition

$$D\colon C^\infty(M;S) \xrightarrow{\nabla} C^\infty(M; T^*M \otimes S) \xrightarrow{c} C^\infty(M;S),$$

where c is Clifford multiplication (given by the left action of $T^*M \subset C(T^*M)$ on S). The Dirac operator D is an (unbounded) Fredholm operator on the real Hilbert space $L^2(M;S)$ of square integrable sections of S. As in Example 2.1.4 we can construct a 1-dimensional EFT $E\colon \mathcal{EB}^1 \to \text{Hilb}$ by defining

$$E(\text{pt}) = L^2(M;S) \otimes_{\mathbb{R}} \mathbb{C} \qquad E(I_t) = e^{-tD^2}.$$

However, there is more structure in this example: the fibers of S and hence the Hilbert space $L^2(M;S)$ is a $\mathbb{Z}/2$-*graded right module* over C_n (or equivalently by Remark 2.2.2, a left module over $C_n^{\text{op}} = C_{-n}$). Moreover, D and hence $E(I_t)$ *commute* with this action. It should be emphasized that we are working in the *graded* world; in particular, saying that the odd operator D commutes with the left C_{-n}-action means $D(c \cdot x) = (-1)^{|x|} c \cdot D(x)$ for a homogeneous element $c \in C_{-n}$ of degree $|c|$ and $x \in E(\text{pt})$.

Definition 2.3.4. (Spin structures on conformal manifolds). Let Σ be a manifold of dimension d and for $k \in \mathbb{R}$ let $L^k \to \Sigma$ be the oriented real line bundle (and hence trivializable) whose fiber over $x \in \Sigma$ consists of all maps $\rho\colon \Lambda^d(T_x\Sigma) \to \mathbb{R}$ such that $\rho(\lambda\omega) = |\lambda|^{k/d}\rho(\omega)$ for all $\lambda \in \mathbb{R}$. Sections of L^d are referred to as *densities*; they can be integrated over Σ resulting in a real number.

Now assume that Σ is equipped with a conformal structure (i.e., an equivalence class of Riemannian metrics where we identify a metric obtained by multiplication by a function with the original metric). We remark that for any $k \neq 0$ the choice of a metric in the conformal class corresponds to the choice of a positive section of L^k. Moreover, the conformal structure on Σ induces a *canonical* Riemannian metric on the *weightless cotangent bundle* $T_0^*\Sigma \stackrel{\text{def}}{=} L^{-1} \otimes T^*\Sigma$.

A *spin structure* on a conformal d-manifold Σ is by definition a spin structure on the Riemannian vector bundle $T_0^*\Sigma$. The *opposite spin structure on* Σ is the opposite spin structure on the vector bundle $T_0^*\Sigma$. We will use the notation $\bar{\Sigma}$ for Σ equipped with the opposite spin structure.

If Σ' is another conformal spin d-manifold, a *conformal spin diffeomorphism* from Σ to Σ' is a conformal diffeomorphism $f\colon \Sigma \to \Sigma'$ together with an isometry between the $C(T_0^*\Sigma) - C_d$-bimodule bundles $S(\Sigma)$ and $f^*S(\Sigma')$. We observe that every conformal spin manifold Σ has a canonical spin involution $\epsilon = \epsilon_\Sigma$, namely the identity on Σ together with the bimodule isometry $S(\Sigma) \to S(\Sigma)$ given by multiplication by -1.

Example 2.3.5. (Examples of spin structures) The manifold $\Sigma = \mathbb{R}^d$ has the following 'standard' spin structure: identifying $T_0^*\Sigma$ with the trivial bundle $\underline{\mathbb{R}}^d$, the bundle $S \stackrel{\text{def}}{=} \mathbb{R}^d \times C_d \to \mathbb{R}^d$ becomes an irreducible graded $C(T_0^*\Sigma) - C_d$-bimodule bundle. Restricting S we then obtain spin structures on codimension zero submanifolds like the disc $D^d \subset \mathbb{R}^d$ or the interval $I_t = [0, t] \subset \mathbb{R}$.

The above spin structure on $\mathbb{R}^d$ makes sense even for $d = 0$; here $\mathbb{R}^0$ consists of one point and $S = \mathbb{R}$ is a graded bimodule over $C_d = \mathbb{R}$ (i.e., a graded real line). We will write pt for the point equipped with this spin structure, and $\overline{\text{pt}}$ for the point equipped with its opposite spin stucture (the bimodule for pt is an 'even' real line, while the bimodule for $\overline{\text{pt}}$ is an 'odd' real line).

If Σ has a boundary $\partial\Sigma$, we note that the restriction $T_0^*\Sigma_{|\partial\Sigma}$ is *canonically* isometric to $T_0^*\partial\Sigma \oplus \underline{\mathbb{R}}$. It follows by Remark 2.3.2 that a spin structure on Σ, i.e., a $C(T_0^*\Sigma) - C(\mathbb{R}^d)$-bimodule bundle $S \to \Sigma$ restricts to a spin structure $S^+ \to \partial\Sigma$ on the boundary $\partial\Sigma$. In particular, the standard spin structure on D^2 restricts to a spin structure on $S^1 = \partial D^2$, which we refer to as the *zero-bordant* or *anti-periodic* spin structure; we'll use the notation S^{ap}.

Definition 2.3.6. (The C_{d-1}-Hilbert space $V(Y)$). If Y^{d-1} is a conformal spin manifold with spinor bundle $S \to Y$, we define

$$V(Y) \stackrel{\text{def}}{=} L^2(Y, L^{\frac{d-1}{2}} \otimes S),$$

the space of square-integrable sections of the real vector bundle $E = L^{\frac{d-1}{2}} \otimes S$. We note that using the fiberwise inner product of the spinor bundle S, we can pair sections φ, ψ of E to obtain a section of L^{d-1} which in turn may be integrated over Y to obtain a real valued inner product $\langle \varphi, \psi \rangle$ on the space of smooth sections of E; completion then gives the real Hilbert space $V(Y)$. We note that each fiber of E is a graded right C_{d-1}-module, which induces the same structure on $V(Y)$.

Definition 2.3.7. (The Clifford algebra $C(Y)$, $d = 1, 2$). Let Y^{d-1} be a conformal spin manifold and let $V(Y)$ be as above. In particular, for $d = 1$, $V(Y)$ is just a graded real Hilbert space; for $d = 2$, the Clifford algebra C_{d-1} is isomorphic to $\mathbb{C}$ and hence $V(Y)$ is a complex vector space on which the grading involution acts by a $\mathbb{C}$-anti-linear involution. After extending the $\mathbb{R}$-valued inner product to a $\mathbb{C}$-valued hermitian product, we can regard $V(Y)$ as a graded complex Hilbert space. So for $d = 1, 2$, $V(Y)$ has the structures needed to form the Clifford algebra $C(Y) \stackrel{\text{def}}{=} C(V(Y))$ as described in Definition 2.2.1. Here the involution α is given by the grading involution (which for $d = 2$ anticommutes with the action of $C_1 = \mathbb{C}$).

Example 2.3.8. (Examples of Clifford algebras $C(Y)$). If pt, $\overline{\text{pt}}$ are the point equipped with its standard resp. its opposite spin structure as defined in Definition 2.3.4, then $C(\text{pt}) = C_1$ and $C(\overline{\text{pt}}) = C_{-1}$.

If $Y = \emptyset$, then $V(Y)$ is zero-dimensional and consequently, $C(\emptyset) = \mathbb{R}$ (for $d = 1$) resp. $C(\emptyset) = \mathbb{C}$ (for $d = 2$).

Definition 2.3.9. (The generalized Lagrangian $L(\Sigma) : W(\Sigma) \to V(\partial\Sigma)$). Let Σ^d be a conformal spin manifold. Picking a Riemannian metric in the given conformal class determines the Levi-Civita connection on the tangent bundle of Σ, which in turn determines connections on the spinor bundle $S = S(T_0^*\Sigma)$, the line bundles L^k and hence $L^k \otimes S$ for all $k \in \mathbb{R}$. The corresponding *Dirac operator* $D = D_\Sigma$ is the composition

$$\begin{aligned} D\colon C^\infty(\Sigma; L^k \otimes S) &\xrightarrow{\nabla} C^\infty(\Sigma; T^*\Sigma \otimes L^k \otimes S) \\ &= C^\infty(\Sigma; L^{k+1} \otimes T_0^*\Sigma \otimes S) \xrightarrow{c} C^\infty(\Sigma; L^{k+1} \otimes S), \end{aligned} \tag{2.3.10}$$

where c is Clifford multiplication (given by the left action of $T_0^*\Sigma \subset C(T_0^*\Sigma)$ on S). It turns out that for $k = \frac{d-1}{2}$ the Dirac operator is in fact *independent* of the choice of the Riemannian metric.

According to Green's formula, we have

$$\langle D\psi, \phi\rangle - \langle \psi, D\phi\rangle = \langle c(\nu)\psi_|, \phi_|\rangle \qquad \psi, \phi \in C^\infty(\Sigma, L^{\frac{d-1}{2}} \otimes S),$$

where $\psi_|, \phi_|$ is the restriction of ψ resp. ϕ to $\partial\Sigma$ and ν is the unit conormal vector field (the section of $T_0^*\Sigma_{|\partial\Sigma}$ corresponding to $1 \in \mathbb{R}$ under the natural isomorphism $T_0^*\Sigma_{|\partial\Sigma} \cong T_0^*\partial\Sigma \oplus \mathbb{R}$). Replacing ψ by ψe_1 in the formula above and using the fact that multiplication by e_1 is skew-adjoint, we obtain

$$\langle D\psi e_1, \phi\rangle + \langle \psi, D\phi e_1\rangle = \langle c(\nu)\psi_| e_1, \phi_|\rangle. \tag{2.3.11}$$

Let $W(\Sigma) \stackrel{\text{def}}{=} \ker D^+$ where D^+ has domain $C^\infty(\Sigma, L^{\frac{d-1}{2}} \otimes S^+)$ and consider the restriction map to the boundary

$$L(\Sigma) : W(\Sigma) \longrightarrow L^2(\partial\Sigma, L^{\frac{d-1}{2}} \otimes S) = V(\partial\Sigma)$$

The closure L_Σ of the image of $L(\Sigma)$ is the *Hardy space* of boundary values of harmonic sections of $L^{\frac{d-1}{2}} \otimes S^+$. The kernel of $L(\Sigma)$ is the space of harmonic spinors on Σ which vanish on the boundary. If $\Sigma_0 \subseteq \Sigma$ denotes the subspace of closed components of Σ then $\ker L(\Sigma) = \ker D^+_{\Sigma_0}$ is the (finite dimensional) subspace of harmonic spinors on Σ_0.

The Green formula shows that L_Σ is isotropic with respect to the bilinear form $b(v,w) = \langle \epsilon(v), w\rangle$, where the involution ϵ is given by $\epsilon(v) = c(\nu)ve_1$. Comparison with Remark 2.3.2 shows that ϵ is precisely the grading involution on S^+ defining the spin structure on $\partial\Sigma$ and it agrees with the grading involution on $V(\partial\Sigma)$. Analytically, much more involved arguments show that L_Σ is in fact a *Lagrangian* subspace [BW]. This implies that $L(\Sigma)$ is a generalized Lagrangian in the sense of Definition 2.2.9.

Moreover, the map $L(\Sigma) : W(\Sigma) \to V(\partial\Sigma)$ is linear with respect to $C_d^{ev} = C_{d-1}$, since the Dirac operator D commutes with the right C_d-action.

We give the following definition only for dimensions $d = 1, 2$ because these are the cases where C_{d-1} is commutative and hence one has a good definition of the 'exterior algebra' over C_{d-1}. For higher dimensions, one could ignore the C_{d-1}-action, but we will not discuss this case as it's not important for our applications.

Definition 2.3.12. (The $C(\partial\Sigma)$-modules $F_{alg}(\Sigma)$ and $F(\Sigma)$). Using the generalized Lagrangian from the previous definition, we define $F_{alg}(\Sigma) \stackrel{\text{def}}{=} F_{alg}(L(\Sigma))$, the algebraic Fock module over $C(\partial\Sigma)$ from Definition 2.2.9. This is a real vector space for $d = 1$ and a complex vector space for $d = 2$. Recall that

$$F_{alg}(L(\Sigma)) = \Lambda^{top}(\ker L(\Sigma))^* \otimes \Lambda(\bar{L}_\Sigma) \tag{2.3.13}$$

and that $\bar{L}_\Sigma$ (and hence the exterior algebra) is equipped with a natural inner product. If $\Sigma_0 \subseteq \Sigma$ denotes again the subspace of closed components of Σ then $\ker L(\Sigma) = \ker D^+_{\Sigma_0}$. We note that $D^+_{\Sigma_0}$ is is skew-adjoint by equation (2.3.11) with respect to the natural hermitian pairing between the domain and range of this operator: for $\psi \in C^\infty(\Sigma; L^{(d-1)/2} \otimes S^+)$ and $\phi \in C^\infty(\Sigma; L^{(d+1)/2} \otimes S^-)$ the point-wise inner product of $\psi \cdot e_1$ and ϕ gives a section of L^2 which may be integrated over Σ to give a complex number; this allows us to identify $L^2(\Sigma, L^{(d+1)/2} \otimes S^-)$ with the dual of $L^2(\Sigma, L^{(d-1)/2} \otimes S^+)$. In particular, $\Lambda^{top}(\ker L(\Sigma))^* = \Lambda^{top}(\ker D^+_{\Sigma_0})^*$ is the *Pfaffian line* $\mathrm{Pf}(\Sigma)$ of the skew-adjoint operator $D^+_{\Sigma_0}$, which comes equipped with the Quillen metric [BF] (this is a *real* line for $d = 1$ and a *complex* line for $d = 2$). Hence both factors on the right hand side of equation 2.3.13 are equipped with natural inner products and we obtain a Hilbert space $F(\Sigma)$ as the completion of $F_{alg}(\Sigma)$, which is still a module over $C(\partial\Sigma)$. We note that the Fock space $F(\Sigma)$ can be regarded as a generalization of the Pfaffian line, since for a *closed* Σ the Fock space $F(\Sigma)$ is *equal* to $\mathrm{Pf}(\Sigma)$.
For $d = 1$ we have $F_{alg}(\Sigma) = F(\Sigma)$ because both are finite dimensional.

If Σ is a conformal spin bordism from Y_1 to Y_2, then $F(\Sigma)$ is a left module over $C(\partial\Sigma) = C(Y_1)^{\mathrm{op}} \otimes C(Y_2)$; in other words, a $C(Y_2) - C(Y_1)$-bimodule.

We need to understand how these Fock modules behave under gluing surfaces together (we shall not discuss the 1-dimensional analogue explicitly but the reader will easily fill this gap). So let Σ_i be conformal spin surfaces with decompositions

$$\partial\Sigma_1 = Y_1 \cup Y_2, \quad \partial\Sigma_2 = Y_2 \cup Y_3,$$

where $Y_i \cap Y_{i+1}$ could be nonempty (but always consists of the points ∂Y_i).Let $\Sigma_3 \stackrel{\text{def}}{=} \Sigma_1 \cup_{Y_2} \Sigma_2$. Then this geometric setting leads to the algebraic setting in Remark 2.2.7: We have $V_i \stackrel{\text{def}}{=} V(Y_i)$ and $L_i \stackrel{\text{def}}{=} L_{\Sigma_i}$ so that we can derive a gluing isomorphism. Note that there are two cases, depending on the type of the von Neumann algebra generated by $C(V_2) = C(Y_2)$: If Y is closed then we are in type I, and if Y has boundary we are in type III where a more sophisticated gluing lemma is needed. Note also that we really have generalized Lagrangians $L(\Sigma_i)$ which are used in the gluing lemma below. It follows from our algebraic gluing lemma (for type I) together with the canonical isomorphisms of Pfaffian lines for disjoint unions of closed surfaces.

Gluing Lemma 2.3.14. *If Y_2 is a closed 1-manifold, there are natural isomorphisms of graded $C(Y_3) - C(Y_1)$ bimodules*

$$F_{alg}(\Sigma_2) \otimes_{C(Y_2)} F_{alg}(\Sigma_1) \cong F_{alg}(\Sigma_3).$$

Again there is a refined version of this lemma for all types of von Neumann algebras which uses Connes fusion, see Proposition 4.3.10. It will actually imply that the above isomorphism are isometries and hence carry over to the completions $F(\Sigma_i)$.

Remark 2.3.15. A different way to see the isometry for completions is to observe that our assumption on Y_2 being closed (i.e. that the von Neumann algebra $A(Y_2)$ is of type I) implies that $A(Y_i) \cong B(H_i)$ for some Hilbert spaces H_i and also that

$$F(\Sigma_1) \cong HS(H_2, H_1), \quad F(\Sigma_2) \cong HS(H_3, H_2), \quad F(\Sigma_3) \cong HS(H_3, H_1).$$

Then the isomorphism for Lemma 2.3.14 is just given by composing these Hilbert-Schmidt operators. Note that if Y_i bound conformal spin surfaces S_i then we may choose $H_i = F(S_i)$ in which case everything becomes canonical. It is important to note that in the case relevant for string vector bundles, this last assumption will be satisfied because we will be working in a relative situation where Y_i consists of two copies of the same manifold, one with a trivial bundle, and with a nontrivial bundle over it.

After these preliminaries, we are now ready to define Clifford linear field theories of degree n. To motivate the following definition, we recall Example 2.3.3 of a 1-dimensional EFT: here the Hilbert space $E(\text{pt})$ assocítated to the point pt has additional structure: $E(\text{pt})$ is a $\mathbb{Z}/2$-graded left module over the Clifford algebra $C_{-n} = (C(\text{pt})^{\text{op}})^{\otimes n}$ (see Example 2.3.8). Roughly speaking, a *Clifford field theory of degree n* is a field theory (of dimension $d = 1$ or 2) with extra structure ensuring that the Hilbert space $E(Y)$ associated to a manifold Y of dimension $d-1$ is a graded left module over the Clifford algebra $C(Y)^{-n} \overset{\text{def}}{=} (C(Y)^{\text{op}})^{\otimes n}$. To make this precise, we define Clifford linear field theories of degree n as functors from $\mathcal{CB}^2_n$ (resp. $\mathcal{EB}^1_n$) to the category of Hilbert spaces; here $\mathcal{CB}^2_n$ (resp. $\mathcal{EB}^1_n$) are 'larger' versions of the categories $\mathcal{CB}^2$ (resp. $\mathcal{EB}^1$) such that the endomorphisms of the object given by Y contains the Clifford algebra $C(Y)^{-n}$. This implies that for such a functor E the Hilbert space $E(Y)$ is left module over $C(Y)^{-n}$ (or equivalently, a right module over $C(Y)^{\otimes n}$).

Definition 2.3.16. (CFT of degree n). A *Clifford linear conformal field theory of degree* $n \in \mathbb{Z}$ is a continuous functor

$$E\colon \mathcal{CB}^2_n \longrightarrow \text{Hilb},$$

compatible with the additional structures in Definition 2.1.1 on both categories. We recall that these are the monodial structures, involutions and anti-involutions, and adjunction transformations on both categories. In addition we require that the functor E is compatible with the linear structure on morphisms in the sense that the equations 2.3.17 below hold. For brevity's sake, we will refer to such a theory also just as *CFT of degree* n (we note that we have defined the notion of 'degree' only for these Clifford linear theories).

The objects of $\mathcal{CB}^2_n$ are closed conformal spin 1-manifolds Y. If Y_1, Y_2 are objects of $\mathcal{CB}^2_n$, there are two types of morphisms from Y_1 to Y_2, namely

- pairs (f, c) consisting of a spin diffeomorphism $f\colon Y_1 \to Y_2$ and an element $c \in C(Y_1)^{-n}$; here $C(Y_1)^k$ stands for the graded tensor product of $|k|$ copies of $C(Y_1)$ if $k \geq 0$ resp. $C(Y_1)^{\text{op}}$ if $k < 0$. In particular, there are morphisms

$$f \overset{\text{def}}{=} (f, 1 \in C(Y_1)^{-n}) \in \mathcal{CB}^2_n(Y_1, Y_2) \quad \text{and} \quad c \overset{\text{def}}{=} (1_{Y_1}, c) \in \mathcal{CB}^2_n(Y_1, Y_1)$$

- pairs (Σ, Ψ), where Σ is a conformal spin bordism from Y_1 to Y_2, and $\Psi \in F_{alg}(\Sigma)^{-n}$. Here $F = F_{alg}(\Sigma)$ is the algebraic Fock space, and F^k stands for the graded tensor product of $|k|$ copies of F if $k \geq 0$ resp. of $\bar{F}$ if $k \leq 0$. A conformal spin bordism Σ from Y_1 to Y_2 is a conformal spin manifold together with a spin diffeomorphism $\partial\Sigma \cong \bar{Y}_1 \amalg Y_2$. More precisely, we identify the morphisms (Σ, Ψ) and (Σ', Ψ') if there is a conformal spin diffeomorphism $\Sigma \to \Sigma'$ compatible with the boundary identification with $\bar{Y}_1 \amalg Y_2$ such that Ψ is sent to Ψ' under the induced isomorphism on Fock spaces. We recall from definition 2.3.12 that if Σ has no closed components, then $F_{alg}(\Sigma)$ is a Fock space which by definition 2.2.4 has a *canonical* cyclic vector Ω. Then $\Omega^{-n} \in F_{alg}(\Sigma)^{-n}$ and we will write

$$\Sigma \overset{\text{def}}{=} (\Sigma, \Omega^{-n}) \in \mathcal{CB}^2_n(Y_1, Y_2).$$

We note that every 1-manifold has a unique conformal structure; hence our definition of spin structure and the construction of the Clifford algebra $C(Y)$ applies to every oriented 1-manifold Y. Composition of morphisms is given as follows:

- If (f_1, c_1) is a morphism from Y_1 to Y_2, and (f_2, c_2) is a morphism from Y_2 to Y_3, then $(f_2, c_2) \circ (f_1, c_1) = (f_2 \circ f_1, f_1^* c_2 \cdot c_1)$. In particular, interpreting as above a spin diffeomorphism $f\colon Y_1 \to Y_2$ as a morphism from Y_1 to Y_2, and an element $c \in C(Y_1)^{-n}$ as an endomorphism of Y_1 we have $(f, c) = f \circ c$.

- If (Σ_1, Ψ_1) is a morphism from Y_1 to Y_2, and (Σ_2, Ψ_2) is a morphism from Y_2 to Y_3, their composition is given by $(\Sigma_2 \cup_{Y_2} \Sigma_1, \Psi_2 \cup_{Y_2} \Psi_1)$, where $\Sigma_3 = \Sigma_2 \cup_{Y_2} \Sigma_1$ is obtained by 'gluing' along the common boundary component Y_2, and the fermion $\Psi_3 = \Psi_2 \cup_{Y_2} \Psi_1$ on Σ_3 is obtained by 'gluing' the fermions Ψ_2 and Ψ_1, i.e., it is the image of $\Psi_2 \otimes \Psi_1$ under the ($(-n)$-th power of the) 'fermionic gluing homomorphism' from Lemma 2.3.14:
$$F_{alg}(\Sigma_2) \otimes_{C(Y_2)} F_{alg}(\Sigma_1) \longrightarrow F_{alg}(\Sigma_3).$$
In the present context where Y_2 is closed, the assumptions of Lemma 2.3.14 are indeed satisfied.

- Composing a morphism (Σ, Ψ) from Y_1 to Y_2 with a diffeomorphism $f\colon Y_2 \to Y_3$ is again (Σ, Ψ), but now regarding Σ as a bordism from Y_1 to Y_3, and $F_{alg}(\Sigma)$ as a bimodule over $C(Y_3) - C(Y_1)$ by means of f. Precomposition of (Σ, Ψ) by a diffeomorphism is defined analogously.

- For $c_i \in C(Y_i)^{-n} \subset \mathcal{CB}^2_n(Y_i, Y_i)$ we have
$$c_2 \circ (\Sigma, \Psi) = (\Sigma, c_2 \cdot \Psi) \qquad \text{and} \qquad (\Sigma, \Psi) \circ c_1 = (\Sigma, \Psi \cdot c_1).$$
We note that $F_{alg}(\Sigma)$ is a $C(Y_2) - C(Y_1)$-bimodule and hence $F_{alg}(\Sigma)^{-n}$ is a $C(Y_2)^{-n} - C(Y_1)^{-n}$-bimodule, which explains the products $c_2 \cdot \Psi$ and $\Psi \cdot c_1$.

We require that a CFT $E\colon \mathcal{CB}^2_n \to \mathrm{Hilb}$ is compatible with the linear structure on morphisms in the sense that given a spin diffeomorphism $f\colon Y_1 \to Y_2$ or a conformal spin bordism Σ from Y_1 to Y_2 the maps
$$C(Y_1)^{-n} \longrightarrow \mathrm{Hilb}(E(Y_1), E(Y_1)) \qquad F_{alg}(\Sigma)^{-n} \longrightarrow \mathrm{Hilb}(E(Y_1), E(Y_2)) \tag{2.3.17}$$
given by $c \mapsto E(f, c)$ (resp. $\Psi \mapsto E(\Sigma, \Psi)$) are linear maps.

Remark 2.3.18. (Basic properties of Clifford conformal field theories) Let Σ be a bordism from Y_1 to Y_2 with no closed components. Then Ω^{-n} is a cyclic vector in the $C(Y_2)^{-n} - C(Y_1)^{-n}$-bimodule $F_{alg}(\Sigma)^{-n}$ and hence every morphism (Σ, Ψ) can be written as $(\Sigma, c_2 \Omega^{-n} c_1) = c_2 \circ (\Sigma, \Omega^{-n}) \circ c_1$. This shows that the morphisms in the category $\mathcal{CB}^2_n$ are generated by diffeomorphisms f, Clifford elements c, and conformal bordisms $\Sigma = (\Sigma, \Omega^{-n})$ (with no closed components).

We note that the spin involution $\epsilon = \epsilon_Y$ (see Definition 2.3.4) on a conformal spin 1-manifold Y induces the grading involution on the assoicated Clifford algebra $C(Y)$. This implies that as morphisms in the category $\mathcal{CB}^2_n$, it commutes with the even elements of the Clifford algebra $C(Y)^{-n}$, while it anti-commutes with the odd elements. In particular, if $E\colon \mathcal{CB}^2_n \to \mathrm{Hilb}$ is a CFT of degree n, then the Hilbert space $E(Y)$ is a *graded* left $C(Y)^{-n}$-module (or equivalently, a right $C(Y)^n$-module). If Σ is a conformal spin bordism from Y_1 to Y_2, then the 'spin involution' ϵ_Σ restricts to ϵ_{Y_i} on the boundary and hence we have the relation

$$\epsilon_{Y_2} \circ \Sigma = \Sigma \circ \epsilon_{Y_1}$$

in $\mathcal{CB}^2_n$. In particular, the corresponding bounded operator $E(\Sigma)\colon E(Y_1) \to E(Y_2)$ is *even*.

We claim that $E(\Sigma)$ is in fact a *Hilbert-Schmidt operator* from $E(Y_1)$ to $E(Y_2)$. To see this, observe that $\Sigma \in \mathcal{CB}^2_n(Y_1, Y_2)$ is in the image of the natural transformation

$$\mathcal{CB}^2_n(\emptyset, \bar{Y}_1 \amalg Y_2) \longrightarrow \mathcal{CB}^2_n(Y_1, Y_2)$$

by regarding Σ as a bordism from $\emptyset$ to $\bar{Y}_1 \amalg Y_2$. This implies that $E(\Sigma)$ is in the image of the corresponding natural transformation in Hilb

$$\mathrm{Hilb}(\mathbb{C}, \overline{E(Y_1)} \otimes E(Y_2)) \longrightarrow \mathrm{Hilb}(E(Y_1), E(Y_2)),$$

which consists exactly of the Hilbert-Schmidt operators from $E(Y_1)$ to $E(Y_2)$.

Remark 2.3.19. We note that if Σ is a bordism from Y_1 to Y_2, and E is a Clifford linear theory of degree n then the map $F(\Sigma)^{-n} \longrightarrow \mathrm{Hom}(E(Y_1), E(Y_2))$, $\Psi \mapsto E(\Sigma, \Psi)$ in fact induces a $C(Y_2)^{-n}$-linear map

$$E(\Sigma)\colon F(\Sigma)^{-n} \otimes_{C(Y_1)^{-n}} E(Y_1) \longrightarrow E(Y_2)$$

Definition 2.3.20. (EFT of degree n). A *Clifford linear* 1*-dimensional Euclidean field theory of degree* n is a continuous functor

$$\mathcal{EB}^1_n \longrightarrow \mathrm{Hilb}$$

compatible with the additional structures in Definition 2.1.1 and the linear structure on the morphisms (equation 2.3.17). Here the 1-dimensional degree n bordism category $\mathcal{EB}^1_n$ is defined as for $\mathcal{CB}^2_n$, except that the dimension of all manifolds involved is down by one: the objects of $\mathcal{EB}^1_n$ are 0-dimensional spin manifolds Y and the bordisms Σ are 1-dimensional; furthermore the geometric structure on these bordisms are *Riemannian metrics* rather than conformal structures. We want to emphasize that now the Clifford algebras $C(Y)$ and the Fock spaces $F(\Sigma)$ are *finite dimensional real vector spaces.*

We note that we can define *real* EFT's; these are functors from $\mathcal{EB}^1_n$ to the category $\mathrm{Hilb}^{\mathbb{R}}$ of *real Hilbert spaces* with the same properties. In fact our motivating example 2.3.3 is the complexification of a real EFT.

It should be pointed out that there are no naive 'real' versions of CFT's, since e.g. the map $C(Y)^{-n} \to \mathrm{Hilb}(E(Y), E(Y))$ is required to be linear, which means *complex* linear if Y is 1-dimensional (in which case $C(Y)$ is an algebra over $\mathbb{C}$). Consequently, we can't restrict the vector spaces $E(Y)$ to be real.

Definition 2.3.21. (Clifford linear field theories over a manifold X). As in definition 2.1.5 we define Clifford linear field theories over a manifold X as follows. Let $\mathcal{CB}^2_n(X)$ resp. $\mathcal{EB}^1_n(X)$ be categories whose objects are as in the categories $\mathcal{CB}^2_n$ resp. $\mathcal{EB}^1_n$ except that all objects Y (given by manifolds of dimension 1 resp. 0) come equipped with piecewise smooth maps to X. Similarly all bordisms Σ come with piecewise smooth maps to X. The additional structures on $\mathcal{CB}^2_n$ and $\mathcal{EB}^1_n$ extend in an obvious way to $\mathcal{CB}^2_n(X)$ and $\mathcal{EB}^1_n(X)$, respectively. We define a *Clifford linear CFT of degree n over X* to be a functor $E\colon \mathcal{CB}^2_n \to \mathrm{Hilb}$ compatible with the additional structures.

Similarly a *Clifford linear EFT of degree n over X* is a functor $E\colon \mathcal{EB}^1_n(X) \to \mathrm{Hilb}$ compatible with the additional structures.

Example 2.3.22. (Basic example of a Clifford linear EFT over X). Let $\xi \to X$ be an n-dimensional spin vector bundle with metric and compatible connection over a manifold X. Let $S(\xi) \to X$ be the associated spinor bundle (a $C(\xi) - C_n$-bimodule bundle, see Definition 2.3.1). Then there is a Clifford linear EFT over X of degree n: this is a functor $E\colon \mathcal{EB}^1_n(X) \to \mathrm{Hilb}$ which maps the object of $\mathcal{EB}^1_n(X)$ given by $\mathrm{pt} \mapsto x \in X$ to the Hilbert space $S(\xi)_x$; on morphisms, $E(c)$ for $c \in C(\mathrm{pt})^{-n} = C_n^{\mathrm{op}}$ is given by the right C_n-module structure on $S(\xi)$. If $\gamma\colon I_t \to X$ is a path from x to y representing a morphism in $\mathcal{EB}^1_n$, then $E(\gamma)\colon S(\xi)_x \to S(\xi)_y$ is given by parallel translation along γ. The properties of a Clifford linear field theory then determine the functor E.

2.4 Twisted Clifford algebras and Fock modules

In this section we shall generalize all the definitions given in Section 2.3 to the twisted case, i.e. where the manifolds are equipped with vector bundles and connections. This is a straightforward step, so we shall be fairly brief. At the end of the definition of the twisted Clifford algebra (respectively twisted Fock module), we'll explain the relative version of the constructions, which involves the twisted and untwisted objects. It is these relative objects which will be used in Section 5.

Definition 2.4.1. (The C_{d-1}-Hilbert space $V(\xi)$). Let Y^{d-1} be a conformal spin manifold with spinor bundle S, and let $\xi \to Y$ be a vector bundle, equipped with a Riemannian metric. Define

$$V(\xi) \overset{\mathrm{def}}{=} L^2(Y, L^{\frac{d-1}{2}} \otimes S \otimes \xi),$$

the space of square-integrable sections of the real vector bundle $E = L^{\frac{d-1}{2}} \otimes S \otimes \xi$. Each fiber of E is a graded right C_{d-1}-module, which induces the same structure on $V(\xi)$.

Definition 2.4.2. (The Clifford algebras $\boldsymbol{C(\xi)}$ and $\boldsymbol{C(\gamma)}$). The above definition gives for $d = 1$ a graded real Hilbert space $V(\xi)$; for $d = 2$, the Clifford algebra C_{d-1} is isomorphic to $\mathbb{C}$ and hence $V(\xi)$ is a complex vector space on which the grading involution acts by a $\mathbb{C}$-anti-linear involution. As in Definition 2.3.7, $V(\xi)$ has thus the structures needed to form the Clifford algebra $C(\xi) \stackrel{\text{def}}{=} C(V(\xi))$ for $d = 1, 2$.

In case that $\xi = \gamma^* E$ is the pullback of an n-dimensional vector bundle $E \to X$ via a smooth map $\gamma : Y \to X$, we define the following *relative* Clifford algebra:

$$C(\gamma) \stackrel{\text{def}}{=} C(\gamma^* E) \otimes C(Y)^{-n}$$

For example, if $Y = \text{pt}$ and $\gamma(\text{pt}) = x \in X$ then this gives the algebra $C(x) = C(E_x) \otimes C_{-n}$. Recall that a spin structure on E_x can then be described as a graded irreducible (left) $C(x)$-module.

Definition 2.4.3. (The generalized Lagrangian $\boldsymbol{L(\xi) : W(\xi) \to V(\partial\xi)}$). Let Σ^d be a conformal spin manifold with boundary Y. Assume that the bundle ξ extends to a vector bundle with metric and connection on Σ. We denote it again by ξ and let $\partial\xi$ be its restriction to Y. Let S be the spinor bundle of Σ and recall from Definition 2.3.4 that the restriction of S^+ to Y is the spinor bundle of Y. Consider the *twisted (conformal) Dirac operator*

$$\begin{aligned} D_\xi \colon C^\infty(\Sigma; L^{\frac{d-1}{2}} \otimes S \otimes \xi) &\xrightarrow{\nabla} C^\infty(\Sigma; T^*\Sigma \otimes L^{\frac{d-1}{2}} \otimes S \otimes \xi) \\ &= C^\infty(\Sigma; L^{\frac{d+1}{2}} \otimes T_0^*\Sigma \otimes S \otimes \xi) \xrightarrow{c} C^\infty(\Sigma; L^{\frac{d+1}{2}} \otimes S \otimes \xi), \end{aligned} \quad (2.4.4)$$

where ∇ is the connection on $L^{\frac{d-1}{2}} \otimes S \otimes \xi$ determined by the connection on ξ and the Levi-Civita connection on $L^{\frac{d-1}{2}} \otimes S$ for the choice of a metric in the given conformal class. Let $W(\xi) \stackrel{\text{def}}{=} \ker D_\xi^+$ where D_ξ^+ has domain $C^\infty(\Sigma, L^{\frac{d-1}{2}} \otimes S^+ \otimes \xi)$ and consider the restriction map to the boundary

$$L(\xi) : W(\xi) \longrightarrow L^2(\partial\Sigma, L^{\frac{d-1}{2}} \otimes S \otimes \xi) = V(\partial\xi)$$

The closure L_ξ of the image image of $L(\xi)$ is the *twisted Hardy space* of boundary values of harmonic sections of $L^{\frac{d-1}{2}} \otimes S^+ \otimes \xi$. $\ker L(\xi)$ is the space of twisted harmonic spinors which vanish on the boundary. If $\Sigma_0 \subseteq \Sigma$ denotes the subspace of closed components of Σ and ξ_0 is the restriction of ξ, then $\ker L(\xi) = \ker D_{\xi_0}^+$ is the (finite dimensional) subspace of twisted harmonic spinors on Σ_0. As before, one shows that $L(\xi)$ is C_{d-1}-linear and that L_ξ is a *Lagrangian* subspace of $V(\partial\xi)$.

Definition 2.4.5. (The $C(\partial\xi)$-modules $F_{alg}(\xi)$ and $F(\xi)$). We define $F_{alg}(\xi) \stackrel{\text{def}}{=} F_{alg}(L(\xi))$, the algebraic Fock module over $C(\partial\xi)$ determined by the generalized Lagrangian $L(\xi) : W(\xi) \to V(\partial\xi)$, see Definition 2.2.9. As before, this is a real Hilbert space for $d = 1$ and a complex Hilbert space for $d = 2$. As in Definition 2.3.12, $F_{alg}(\xi)$ can be completed to the Hilbert space $F(\xi) = F(L(\xi))$.

In case that $\xi = \Gamma^*E$ is the pullback of an n-dimensional vector bundle $E \to X$ via a smooth map $\Gamma : \Sigma \to X$, we define the following *relative* Fock modules:

$$F(\Gamma) \stackrel{\text{def}}{=} F(\Gamma^*E) \otimes F(\Sigma)^{-n} \text{ and } F_{alg}(\Gamma) \stackrel{\text{def}}{=} F_{alg}(\Gamma^*E) \otimes F_{alg}(\Sigma)^{-n}$$

These are left modules over the relative Clifford algebra $C(\gamma)$ from Definition 2.4.2, where $\gamma = \Gamma|_Y$. It is important to note that the vacuum vector for Γ is by definition $\Omega_\Gamma \in F_{alg}(\Gamma^*E)$.
If Σ is closed then $\text{Pf}(\Gamma) \stackrel{\text{def}}{=} F_{alg}(\Gamma)$ is the relative Pfaffian line.

There are again gluing laws for twisted Fock spaces as in Lemma 2.3.14 and Proposition 4.3.10.

3 K-theory and 1-dimensional field theories

3.1 The space of 1-dimensional Euclidean field theories

We recall from definition 2.3.20 that an EFT of degree n is a continuous functor E from the Euclidean bordism category $\mathcal{EB}^1_n$ to the category Hilb of Hilbert spaces compatible with the symmetric monoidal structure, the (anti-) involutions $*$ and $\bar{\cdot}$, the 'adjunction transformations' (see 2.1.1) and the linear structure on morphisms (see equation 2.3.17). An important feature is that the Hilbert space $E(\text{pt})$ associated to the point is a graded left C_{-n}-module, or equivalently, a graded right C_n-module. In our basic example 2.3.3, $E(\text{pt})$ is the space of square integrable sections of the spinor bundle $S \to M$ of a spin n-manifold, where the right C_n-action is induced by the right C_n-action on S.

It might be important to repeat the reason why the algebra C_{-n} comes up: The geometric example dictates that $E(\text{pt})$ be a *right* C_n module. (This goes back to the fact that a frame for a vector space V is an isometry $\mathbb{R}^n \to V$, and hence $O(n)$ acts on the right on these frames.) However, from a functorial point of view, the endomorphisms of an object in a category act on the left. This is preserved under the covariant functor E. Since we built in C_{-n} as the endomorphisms of the object $\text{pt} \in \mathcal{EB}^1_n$, $E(\text{pt})$ becomes a *left* C_{-n}-module. Equivalently, this is a right C_n-module, exactly what we want.

In this subsection we consider the space of EFT's. We want to assume that the (right) C_n-module $E(\text{pt})$ is a submodule of some fixed graded complex Hilbert space H (equipped with a right C_n-action such that all irreducible modules occur infinitely often) in order to obtain a *set* of such functors.

Proposition 3.1.1. *There is a bijection*

$$\{EFT\text{'s of degree } n\} \xrightarrow{R} \mathrm{Hom}(\mathbb{R}_+, HS^{ev,sa}_{C_n}(H))$$

Here $\mathbb{R}_+$ is the additive semi-group of positive real numbers, and $HS^{ev,sa}_{C_n}(H)$ is the semi-group of Clifford linear, even (i.e., grading preserving), self-adjoint Hilbert-Schmidt operators with respect to composition.

Definition 3.1.2. (Construction of R). Let $\mathbb{R}$ be equipped with the standard spin structure (see Example 2.3.5). We note that the translation action of $\mathbb{R}$ on itself is by spin isometries, allowing us to identify all the spin 0-manifolds $\{t\}$ with the object pt of the bordism category $\mathcal{EB}^1_n$. We recall $C(\mathrm{pt}) = C_1$ (Example 2.3.8) and hence $C(\mathrm{pt})^{-n} = C_{-n}$.

For $t > 0$ let $I_t \in \mathcal{EB}^1_n(\mathrm{pt}, \mathrm{pt})$ be the endomorphism given by the Riemannian spin 1-manifold $[0, t] \subset \mathbb{R}$. We note that the composition $I_t \circ I_{t'}$ is represented by gluing together the spin 1-manifolds $[0, t']$ and $[0, t]$, identifying $0 \in [0, t]$ with $t' \in [0, t']$ by means of the translation $t' \in \mathbb{R}_+$. This results in the spin 1-manifold $[0, t + t']$. We note that $I_t^* = I_t$, since reflection at the midpoint of the interval I_t is a spin structure reversing isometry.

As discussed in remark 2.3.18, if $E\colon \mathcal{EB}^1_n \to \mathrm{Hilb}$ is a Clifford linear EFT of degree n, then $E(\mathrm{pt})$ is a right C_n-module and $E(I_t)\colon E(\mathrm{pt}) \to E(\mathrm{pt})$ is an even, Clifford linear Hilbert-Schmidt operator. Furthermore, due to $I_t^* = I_t$, and the required compatibility of E with the anti-involution $*$, the operator $E(I_t)$ is self-adjoint. The relation $I_t \circ I_{t'} = I_{t+t'}$ in the category $\mathcal{EB}^1_n$ implies that

$$\mathbb{R}_+ \longrightarrow HS^{ev,sa}_{C_n}(E(\mathrm{pt})) \qquad t \mapsto E(I_t) \tag{3.1.3}$$

is a semi-group homomorphism. Extending the Hilbert-Schmidt endomorphism $E(I_t)$ of $E(\mathrm{pt}) \subset H$ to all of H by setting it zero on $E(\mathrm{pt})^\perp$ defines the desired semi-group homomorphism $R(E)\colon \mathbb{R}_+ \to HS^{ev,sa}_{C_n}(H)$.

Sketch of proof of Proposition 3.1.1. Concerning the injectivity of the map R, we observe that the functor $E\colon \mathcal{EB}^1_n \to \mathrm{Hilb}$ can be recovered from $E(\mathrm{pt})$ (as graded right module over C_n) and $E(I_t)$ as follows. Every spin 0-manifold Z is a disjoint union of copies of pt and $\bar{\mathrm{pt}}$ and hence $E(Z)$ is determined by the Hilbert space $E(\mathrm{pt})$, $E(\overline{\mathrm{pt}}) = \overline{E(\mathrm{pt})}$ and the requirement that E sends disjoint unions to tensor products. Concerning the functor E on morphisms, we note that $E(c)$ for $c \in C(\mathrm{pt})^{-n} = C_{-n} \subset \mathcal{EB}^1_n(\mathrm{pt}, \mathrm{pt})$ is determined by the (left) C_{-n}-module structure on $E(\mathrm{pt}) \subset H$. Similarly, the image of the endomorphism $\epsilon \in B^n_1(\mathrm{pt}, \mathrm{pt})$ is the grading involution on $E(\mathrm{pt})$. Now the morphisms of the category $\mathcal{EB}^1_n$ are *generated* by I_t, $c \in C(\mathrm{pt})^{-n}$ and ϵ using the operations of composition, disjoint union, the involution $\bar{\cdot}$ and the adjunction transformations $\mathcal{EB}^1_n(\emptyset, Z_1 \amalg Z_2) \to \mathcal{EB}^1_n(\overline{Z}_1, Z_2)$. For example, I_t can be interpreted as an element of $\mathcal{EB}^1_n(\mathrm{pt}, \mathrm{pt})$ or $\mathcal{EB}^1_n(\emptyset, \overline{\mathrm{pt}} \amalg \mathrm{pt})$ or $\mathcal{EB}^1_n(\mathrm{pt} \amalg \overline{\mathrm{pt}}, \emptyset)$. The second and third interpretation correspond to each other via the involution $\bar{\cdot}$; the first is the image of the second under the natural transformation $\mathcal{EB}^1_n(\emptyset, \overline{\mathrm{pt}} \amalg \mathrm{pt}) \to \mathcal{EB}^1_n(\mathrm{pt}, \mathrm{pt})$. It can be shown that the composition

$$\emptyset \xrightarrow{I_t} \overline{\mathrm{pt}} \amalg \mathrm{pt} \xrightarrow{I_{t'}} \emptyset \tag{3.1.4}$$

is the circle $S^{ap}_{t+t'}$ of length $t + t'$ with the anti-periodic spin structure, while

$$\emptyset \xrightarrow{I_t} \overline{\mathrm{pt}} \amalg \mathrm{pt} \xrightarrow{\epsilon \amalg 1} \overline{\mathrm{pt}} \amalg \mathrm{pt} \xrightarrow{I_{t'}} \emptyset \tag{3.1.5}$$

is $S^{per}_{t+t'}$, the circle of length $t + t'$ with the periodic spin structure. The best way to remember this result is to embed I as the upper semicircle into the complex plane. For $x \in I$, the real line $S^+_x(I)$ can be identified with the complex numbers whose square lies in $T_xI \subset \mathbb{C}$. It follows that the spinor bundle $S^+(I)$ is a band twisted by $\pi/2$ (or a 'quarter twist'). This is consistent with the fact that $S(\partial I)$ consists of one even line, and one odd line (which are orthogonal). Gluing two such quarter twisted bands together gives a half twisted band (i.e. the anti-periodic spin structure on the corresponding circle). This also follows from the fact that this circle bounds a disk in the complex plane, and is thus spin zero-bordant. Gluing together two quarter twisted bands using the half twist ϵ gives a fully twisted band (i.e. the periodic spin structure on S^1).

The fact that the morphisms in $\mathcal{EB}^1_n$ are generated by I_t, ϵ and $c \in C(\mathrm{pt})^{-n}$ implies that the functor E is determined by the semi-group homomorphism $E(I_t)$. Hence the map $E \mapsto E(I_t)$ is injective. Surjectivity of this map is proved by similar arguments by analyzing the relations between these generators. □

Remark 3.1.6. As in our motivating Example 2.3.3 for a Clifford linear field theory, let M be a Riemannian spin manifold of dimension n and consider the semi-group of Hilbert-Schmidt operators $t \mapsto e^{-tD^2}$ acting on the Hilbert space $L^2(M; S)$. Then Proposition 3.1.1 (or rather its version for *real* EFT's) shows that there is a real Clifford linear EFT of degree n with $E(\mathrm{pt}) = L^2(M; S)$ and $E(I_t) = e^{-tD^2}$.

This EFT contains interesting information, namely the *Clifford index* of D, an element of $KO_n(\mathrm{pt})$, see [LM, §II.10]. We recall (see e.g. [LM, Ch. I, Theorem 9.29]) that $KO_n(\mathrm{pt})$ can described as $KO_n(\mathrm{pt}) = \mathfrak{M}(C_n)/i^*\mathfrak{M}(C_{n+1})$, where $\mathfrak{M}(C_n)$ is the Grothendieck group of graded right modules over the Clifford algebra C_n, and i^* is induced by the inclusion map $C_n \to C_{n+1}$. Hence the C_n-module $\ker D^2$ represents an element of $KO_n(\mathrm{pt})$. The crucial point is that $[\ker D^2] \in KO_n(\mathrm{pt})$ is *independent of the choice of Riemannian metric used in the construction of D*. The argument is this: the eigenspace E_λ of D^2 with eigenvalue λ is a C_n-module; for $\lambda > 0$ the automorphism $\lambda^{-1/2}\epsilon D$ of E_λ has square -1 and anti-commutes with right multiplication by $v \in \mathbb{R}^n \subset C_n$. In other words, the graded C_n-module structure on E_λ extends to a C_{n+1}-module structure. This shows that $[\ker D^2] = [E_{<\rho}] \in KO_n(\mathrm{pt})$, where $E_{<\rho}(D^2)$ for $\rho > 0$ is the (finite dimensional) sum of all eigenspaces E_λ with eigenvalue $\lambda < \rho$. Choosing a ρ not in the spectrum of D^2, the C_n-module $E_{<\rho}(D^2)$ can be identified with $E_{<\rho}((D')^2)$ for any sufficiently close operator D', in particular for Dirac operators corresponding to slightly deformed metrics on M.

This shows that $[E_{<\rho}(D^2)] \in KO_n(\mathrm{pt})$ is independent of the choice of $\rho > 0$ as well as the metric on M. We note that in terms of the EFT, the Clifford index can be described as $[E_{>\rho}(E(I_t))] \in KO_n(\mathrm{pt})$, where $E_{>\rho}(E(I_t))$ is the sum of all eigenspaces of $E(I_t)$ with eigenvalue $> \rho$ (a finite dimensional graded C_n-module); the argument above shows that this is independent of t and $\rho > 0$.

This example suggests that the space of 1-dimensional EFT's of degree n contains interesting 'index information' and that we should analyze its homotopy type. Unfortunately, the result is that it is contractible! To see this, use Proposition 3.1.1 to identify this space with the space of semi-groups $t \mapsto P_t$ of even, self-adjoint, C_n-linear Hilbert-Schmidt operators. We note that if P_t is such a semi-group, then so is $t \mapsto s^t P_t$ for any $s \in [0,1]$, which implies that the space of these semi-groups is contractible.

3.2 Super symmetric 1-dimensional field theories

After the 'bad news' expressed by the last remark, we'll bring the 'good news' in this section: if we replace 1-dimensional EFT's by *super symmetric* EFT's, then we obtain a space with a very interesting homotopy type. Before stating this result and explaining what a super symmetric EFT *is*, let us motivate a little better why super symmetry is to be expected to come in here.

Remark 3.2.1. Let E be a real EFT of degree n. Then motivated by Remark 3.1.6, one is tempted to define its Clifford index in $KO_n(\mathrm{pt})$ to be represented by the C_n-module $E_{>\rho}(E(I_t))$ (the sum of the eigenspace of $E(I_t)$ with eigenvalue $> \rho$). However, in general this *does* depend on t and ρ; moreover, for fixed t, ρ replacing the semi-group $E(I_t)$ by the deformed operator $s^t E(I_t)$ leads to a *trivial* module for sufficiently small s! This simply comes from the fact that this operator has no Eigenvalues $> \rho$ for sufficiently small s.

What goes wrong is this: the arguments in Remark 3.2.13 show that there is a non-negative, self-adjoint, even operator A (not bounded!) on some subspace $H' \subset H$ which is an infinitesimal generator of the semi-group $E(I_t)$ in the sense that $E(I_t) = e^{-tA} \in HS^{ev,sa}_{C_n}(H') \subset HS^{ev,sa}_{C_n}(H)$ (this inclusion is given by extending by 0 on the orthogonal complement of H' in H). However, in general A is *not* the square of an odd operator D, and so the argument in Remark 3.1.6 showing that $[E_{>\rho}(E(I_t))] \in KO_n(\mathrm{pt})$ is independent of t, ρ fails.

The argument goes through for those semi-groups $\mathbb{R}_+ \to HS^{ev,sa}_{C_n}(H)$ whose generators are squares of odd operators; we will see that these are precisely those semi-group which extend to 'super homomorphims' $\mathbb{R}^{1|1}_+ \to HS^{sa}_{\mathbb{C}_n}(H)$.

Definition 3.2.2. (Susy EFT of degree n). A *super symmetric* 1*-dimensional Euclidean field theory* (or susy EFT) of degree n is a continuous functor

$$E\colon \mathcal{SEB}^1_n \longrightarrow \mathrm{Hilb}$$

satisfying the compatibility conditions 2.1.1. Here $\mathcal{SEB}^1_n$ is the 'super' version of the 1-dimensional bordism category $\mathcal{EB}^1_n$, where 1-dimensional Riemannian manifolds (which are morphisms in $\mathcal{EB}^1_n$) are replaced by *super manifolds* of dimension $(1|1)$ with an appropriate 'super' structure corresponding to the metric.

We refer to [DW] or [Fr2] for the definition of super manifolds. To a super manifold M of dimension $(n|m)$ we can in particular associate

- its 'algebra of smooth functions' $C^\infty(M)$, which is a $\mathbb{Z}/2$-graded, graded commutative algebra;
- an ordinary manifold M^{red} of dimension n so that $C^\infty(M^{red})$ (the smooth functions on M^{red}) is the quotient of $C^\infty(M)$ by its nil radical.

One assumes that $C^\infty(M)$ is a locally free module over $C^\infty(M^{red})$. A basic example of a super manifold of dimension $(n|m)$ is $\mathbb{R}^{n|m}$ with

$$(\mathbb{R}^{n|m})^{red} = \mathbb{R}^n \quad \text{and} \quad C^\infty(\mathbb{R}^{n|m}) = C^\infty(\mathbb{R}^n) \otimes \Lambda^* \mathbb{R}^m.$$

More generally, if Σ is a manifold of dimension n and $E \to \Sigma$ is a real vector bundle of dimension m, then there is an associated super manifold M of dimension $(n|m)$ with

$$M^{red} = \Sigma \quad \text{and} \quad C^\infty(M) = C^\infty(\Sigma, \Lambda^* E^*),$$

where $C^\infty(\Sigma, \Lambda^* E^*)$ is the algebra of smooth sections of the exterior algebra bundle $\Lambda^* E^*$ generated by the dual vector bundle E^*.

In particular, if Σ is a spin bordism between 0-manifolds Y_1 and Y_2, then we can interpret Σ as a super manifold of dimension $(1|1)$ (using the even part $S^+ \to \Sigma$ of the spinor bundle) and Y_1, Y_2 as super manifolds of dimension $(0|1)$; in fact then Σ is a 'super bordisms' between Y_1 and Y_2 with Σ^{red} being the original bordism between Y_1^{red} and Y_2^{red}. The question is what is the relevant *geometric structure* on Σ, which reduces to the Riemannian metric on the underlying 1-manifold Σ^{red}? We note that a Riemannian metric on an oriented 1-manifold determines a unique 1-form which evaluates to 1 on each unit vector representing the orientation. Conversely, a nowhere vanishing 1-form determines a Riemannian metric. We generalize this point of view by defining a *metric structure* on a $(1|1)$-manifold Σ to be an even 1-form ω (see [DW, §2.6] for the theory of differential forms on super manifolds) such that

- ω and $d\omega$ are both nowhere vanishing (interpreted as sections of vector bundles over Σ^{red}) and
- the Berezin integral of ω over $(0|1)$-dimensional submanifolds is *positive*.

On $\Sigma^{red} \subset \Sigma$ such a form ω restricts to a nowhere vanishing 1-form which in turn determines a Riemannian metric on Σ^{red}.

Example 3.2.3. For example, on the $(1|1)$-dimensional super manifold $\mathbb{R}^{1|1}$ with even coordinate t and odd coordinate θ, the form $\omega = dz + \eta d\eta$ is a metric structure (we note that $d\omega = d\eta \wedge d\eta \neq 0$; the form $d\eta$ is an odd 1-form and hence *commutes* with itself according to equation (2.6.3) in [DW]). The Berezin integral of ω over $\{t\} \times \mathbb{R}^{0|1}$ gives the value 1 for every t (the form $dt - \eta d\eta$ gives the value -1 and hence is *not* a metric structure). The form ω restricts to the standard form dz on $(\mathbb{R}^{1|1})^{red} = \mathbb{R}$ by setting $\eta = 0$. In particular, the metric structure ω induces the *standard* Riemannian metric on $(\mathbb{R}^{1|1})^{red} = \mathbb{R}$.

With this terminology in place we can define $\mathcal{SEB}^1_n$. It is a category (enriched over super manifolds!) and its morphisms consist of 'super bordisms' as in $\mathcal{EB}^1_n$, except that 1-dimensional spin bordisms Σ equipped with Riemannian metrics are replaced by $(1|1)$-dimensional super bordisms equipped with a metric structure. In particular, the endomorphism spaces of each object are now super semigroups, compare Definition 3.2.7.

1-EFT's and the K-theory spectrum. We can now give a precise formulation of Theorem 1.0.1 from the introduction. It says that the space $\mathcal{EFT}_n$ of susy EFT's of degree n has the homotopy type of K_{-n}, the $(-n)$-th space in the Ω-spectrum K representing periodic complex K-theory. Here $K_0 = \Omega^\infty K$ is the 0-th space in the spectrum K, and all the spaces K_n, $n \in \mathbb{Z}$ are related to each other by $\Omega K_n \simeq K_{n-1}$. Note that this implies that the connected components of the space of susy EFT's of degree n are the homotopy groups of the spectrum:

$$\pi_0(\mathcal{EFT}_n) = \pi_0(K_{-n}) = K^{-n}(\mathrm{pt}) = K_n(\mathrm{pt}) \tag{3.2.4}$$

Remark 3.2.5. There is an $\mathbb{R}$-version of the above result (with the same proof), namely that the space $\mathcal{EFT}_n^{\mathbb{R}}$ of *real* susy EFT's of degree n is homotopy equivalent to KO_{-n}, the $(-n)$-th space in the real K-theory spectrum KO.

To convince the reader that one can really do explicit constructions in terms of these spaces of field theories, we will describe the Thom class of a spin vector bundle and the family Dirac index of bundle with spin fibers in terms of maps into these spaces (see Remarks 3.2.22 and 3.2.21).

The 'super' analog of Proposition 3.1.1 is then the following result.

Proposition 3.2.6. *There is a bijection*

$$\mathcal{EFT}_n \xrightarrow{R} \mathrm{Hom}(\mathbb{R}_+^{1|1}, HS^{sa}_{C_n}(H))$$

Here $HS^{sa}_{C_n}(H)$ is the super ($\mathbb{Z}/2$-graded) algebra of self-adjoint C_n-linear Hilbert-Schmidt operators on a Hilbert space H which is a graded right module over C_n (containing all irreducible C_n-modules infinitely often). The 'super semi group' $\mathbb{R}_+^{1|1}$ and homomorphisms between super semi groups are defined as follows.

Definition 3.2.7. (The super group $\mathbb{R}^{1|1}$). We give $\mathbb{R}^{1|1}$ the structure of a 'super group' by defining a multiplication

$$\mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \longrightarrow \mathbb{R}^{1|1} \qquad (t_1, \theta_1), (t_2, \theta_2) \mapsto (t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2);$$

here the points of $\mathbb{R}^{1|1}$ are parametrized by pairs (t, θ), where t is an even and θ is an odd variable.

What do we mean by 'odd' and 'even' variables, and how do we make sense of the above formula? A convenient way to interpret these formulas is to extend scalars by some exterior algebra Λ and form $\mathbb{R}^{1|1}(\Lambda) \stackrel{\text{def}}{=} (\mathbb{R}^{1|1} \otimes \Lambda)^{ev}$, called the Λ-*points* of $\mathbb{R}^{1|1}$. Here $\mathbb{R}^{1|1} = \mathbb{R} \oplus \mathbb{R}$ is just considered as a graded vector space, with one copy of $\mathbb{R}$ in even, the other copy of $\mathbb{R}$ of odd degree, so that $(\mathbb{R}^{1|1} \otimes \Lambda)^{ev} = \Lambda^{ev} \oplus \Lambda^{odd}$. Now considering (t, θ) as an element of $\mathbb{R}^{1|1}(\Lambda)$ the formula in Definition 3.2.7 makes sense: for $t_1, t_2 \in \Lambda^{ev}$ and $\theta_1, \theta_2 \in \Lambda^{odd}$, we have $t_1 + t_2 + \theta_1\theta_2 \in \Lambda^{ev}$ and $\theta_1 + \theta_2 \in \Lambda^{odd}$ and it is easy to check that in this fashion we have given $\mathbb{R}^{1|1}(\Lambda)$ the structure of a group. But how about a 'super group structure' on $\mathbb{R}^{1|1}$ itself? Well, in one approach to super groups putting a super group structure on $\mathbb{R}^{1|1}$ is *by definition* the same as putting a group structure on $\mathbb{R}^{1|1}(\Lambda)$ for every Λ, depending functorially on Λ. In particular, the formula in Definition 3.2.7 gives $\mathbb{R}^{1|1}$ the structure of a super group.

Hopefully, the reader can now guess what a homomorphism $A \to B$ between super groups is: it is a family of ordinary group homomorphisms $A(\Lambda) \to B(\Lambda)$ depending functorially on Λ. A particularly interesting example of a super homomorphism is given in Example 3.2.9.

Remark 3.2.8. It is well-known that the group of orientation preserving isometries of $\mathbb{R}$ equipped with its standard orientation and metric can be identified with $\mathbb{R}$ acting on itself by translations. Similarly the group of automorphisms of the super manifold $\mathbb{R}^{1|1}$ preserving the metric structure $\omega = dz + \eta d\eta$ can be identified with the super group $\mathbb{R}^{1|1}$ acting on itself by translations. Let us check that for $(t, \theta) \in \mathbb{R}^{1|1}$ the translation

$$T_{t,\theta} \colon \mathbb{R}^{1|1} \longrightarrow \mathbb{R}^{1|1} \qquad (z, \eta) \mapsto (t + z + \theta\eta, \theta + \eta)$$

preserves the form ω:

$$T^*_{t,\theta} dz = d(t + z + \theta\eta) = dz - \theta d\eta \qquad T^*_{t,\theta} d\eta = d(\theta + \eta) = d\eta$$

and hence

$$T^*_{t,\theta}\omega = (dz - \theta d\eta) + (\theta + \eta) d\eta = dz + \eta d\eta = \omega.$$

The Λ-points $\mathbb{R}^{1|1}_+(\Lambda)$ of the super space $\mathbb{R}^{1|1}_+$ consist of all $(t, \theta) \in \Lambda^{ev}_+ \oplus \Lambda^{odd}$, where the Λ^0-component of t is positive. We note that the multiplication on $\mathbb{R}^{1|1}$ restricts to a multiplication on $\mathbb{R}^{1|1}_+$, but there are no inverses; i.e., $\mathbb{R}^{1|1}_+$ is a 'super semi-group'. It is the analog of $\mathbb{R}_+$ (where 'multiplication' is given by addition): we can interpret

$\mathbb{R}^+$ as the moduli space of of intervals equipped with Riemannian metrics; similarly, $\mathbb{R}^{1|1}_+$ can be interpreted as the moduli space of 'super intervals with metric structures'. The multiplication on $\mathbb{R}_+$ (resp. $\mathbb{R}^{1|1}_+$) corresponds to the gluing of intervals (resp. super intervals).

Example 3.2.9. (A super homomorphism). Let H be the $\mathbb{Z}/2$-graded Hilbert space of L^2-sections of the spinor bundle on a compact spin manifold, let D be the Dirac operator acting on H, and let $HS^{sa}(H)$ be the space of self-adjoint Hilbert-Schmidt operators on H. Then we obtain a map of super spaces $\mathbb{R}^{1|1}_+ \to HS^{sa}_{C_{-n}}(H)$ by defining it on Λ-points in the following way:

$$\mathbb{R}^{1|1}_+(\Lambda) = \Lambda^{ev}_+ \times \Lambda^{odd} \longrightarrow HS^{sa}(H)(\Lambda) = (HS^{sa}(H) \otimes \Lambda)^{ev}$$
$$(t,\theta) \mapsto e^{-tD^2} + \theta D e^{-tD^2}.$$

Here e^{-tD^2} is defined for real-valued $t > 0$ via functional calculus; for a general t, we decompose t in the form $t = t_B + t_S$ with $t_B \in \mathbb{R}_+ = \Lambda^0_+$ and $t_S \in \bigoplus_{p=1}^{\infty} \Lambda^{2p}$ (physics terminology: t_B is the 'body' of t, while t_S is the 'soul' of t). Then we use Taylor expansion to define $e^{-tD^2} = e^{-(t_B+t_S)D^2}$ as an element of $HS^{sa}(H) \otimes \Lambda$ (we note that the Taylor expansion gives a *finite* sum since t_S is nilpotent). We note that θ and D are both odd, so that $e^{-tD^2} + \theta D e^{-tD^2}$ is indeed in the *even* part of the algebra $HS^{sa}(H) \otimes \Lambda$. We will check in the proof of Lemma 3.2.14 that the map defined above is in fact a super homomorphism.

Definition 3.2.10. (Construction of *R*). We recall from Remark 3.2.8 that for $(t,\theta) \in \mathbb{R}^{1|1}$ the translation $T_{t,\theta}\colon \mathbb{R}^{1|1} \longrightarrow \mathbb{R}^{1|1}$ preserves the metric structure given by the even 1-form $\omega = dz + \eta d\eta$ from example 3.2.3. For $t > 0$, we will write $I_{t,\theta}$ for the $(1|1)$-dimensional super manifold $[0,t] \times \mathbb{R}^{0|1} \subset \mathbb{R}^{1|1}$ equipped with the metric structure given by ω (the Λ-points of $[0,t] \times \mathbb{R}^{0|1}$ consist of all $(z,\eta) \in \mathbb{R}^{1|1}$ such that z_B, the 'body' of z is in the interval $[0,t]$). We consider $I_{t,\theta}$ as a super bordism between $\mathrm{pt} \stackrel{\text{def}}{=} 0 \times \mathbb{R}^{0|1}$ and itself by identifying $0 \times \mathbb{R}^{0|1}$ with $\{t\} \times \mathbb{R}^{0|1}$ by means of the translation $T_{(t,\theta)}$. The composition of I_{t_1,θ_1} and I_{t_2,θ_2} in the category $\mathcal{SEB}^1_n$ (given by gluing of these 'super bordisms') is then given by

$$I_{t_2,\theta_2} \circ I_{t_1,\theta_1} = I_{t_1+t_2+\theta_1\theta_2,\theta_1+\theta_2},$$

since we use the translation T_{t_1,θ_1} to identify $[0,t_2] \times \mathbb{R}^{0|1}$ with $[t_1, t_1+t_2] \times \mathbb{R}^{0|1}$. This then 'fits' together with $[0,t_1] \times \mathbb{R}^{0|1}$ to form the bigger domain $[0, t_1+t_2] \times \mathbb{R}^{0|1}$, the relevant identification between $\{0\} \times \mathbb{R}^{0|1}$ and the right hand boundary $\{t_1+t_2\} \times \mathbb{R}^{0|1}$ of this domain is given by $T_{t_2,\theta_2} \circ T_{t_1,\theta_1} = T_{t_1+t_2+\theta_1\theta_2,\theta_1+\theta_2}$.

This implies that

$$R(E)\colon \mathbb{R}^{1|1}_+ \longrightarrow HS(E(\mathrm{pt})) \subset HS(H) \qquad (t,\theta) \mapsto E(I_{t,\theta})$$

is a super homomorphism. As in Definition 3.1.2 it follows that $E(I_{t,\theta})$ is a self-adjoint operator. However, in general it won't be *even*; unlike the situation in the category $\mathcal{EB}^1_n$ in the super bordism category $\mathcal{SEB}^1_n$ the spin involution $\epsilon = \epsilon_{\mathrm{pt}}$ (see definition 2.3.4) does *not* commute with $I_{t,\theta}$; rather we have

$$\epsilon \circ I_{t,\theta} \circ \epsilon = I_{t,-\theta}. \tag{3.2.11}$$

To see this, we recall that the spin involution ϵ_Σ of a conformal spin manifold Σ is the identity on Σ and multiplication by -1 on the fibers of the spinor bundle $S(\Sigma) \to \Sigma$. If Σ is d-dimensional, this is an involution on the $(d|2^d)$-dimensional super manifold $S(\Sigma)$. In particular for $\Sigma = \{0\} \subset \mathbb{R}$, $S(\Sigma)$ can be identified with $\{0\} \times \mathbb{R}^{0|1} \subset \mathbb{R}^{1|1}$ on which ϵ acts by $(z, \eta) \mapsto (z, -\eta)$. It is easy to check that the translation $T_{t,\theta}$ and the involution ϵ (considered as automorphism of $\mathbb{R}^{1|1}$) satisfy the relation $\epsilon \circ T_{t,\theta} \circ \epsilon = T_{t,-\theta}$. This in turn implies the relation (3.2.11) between endomorphism of $\mathrm{pt} = \{0\} \times \mathbb{R}^{0|1}$

The equation (3.2.11) shows that restricting a super symmetric EFT $E\colon \mathcal{SEB}^1_n \to \mathrm{Hilb}$ to the semi-group of morphisms $I_{t,\theta} \in \mathcal{SEB}^1_n(\mathrm{pt}, \mathrm{pt})$ gives a $\mathbb{Z}/2$-equivariant semi-group homomorphism $R(E)$ as desired.

The proof of Proposition 3.2.6 is analogous to the proof of Proposition 3.1.1, so we skip it. The proof of Theorem 1.0.1 is based on a description of K-theory in terms of homomorphisms of C^*-algebras. We recall that a C^*-algebra A is a subalgebra of the algebra of bounded operators on some Hilbert space which is closed under the operation $a \mapsto a^*$ of taking adjoints and which is a closed subset of all bounded operators with respect to the operator norm. Equivalently, A is an algebra (over $\mathbb{R}$ or $\mathbb{C}$) equipped with a norm and an anti-involution $*$ satisfying some natural axioms [BR],

[Co1], [HR].

The examples of C^*-algebras relevant to us are:

- The C^*-algebra $C_0(\mathbb{R})$ of continuous real valued functions on $\mathbb{R}$ which vanish at ∞ with the supremum norm and trivial $*$-operation. This is a $\mathbb{Z}/2$-graded algebra with grading involution $\epsilon\colon C_0(\mathbb{R}) \to C_0(\mathbb{R})$ induced by $t \mapsto -t$ for $t \in \mathbb{R}$.

- The C^*-algebra $\mathcal{K}(H)$ of compact operators on a Hilbert space H; if H is graded, $\mathcal{K}(H)$ is a graded C^*-algebra. More generally, if H is a graded module over the Clifford algebra C_n, then the algebra $\mathcal{K}_{C_n}(H)$ of C_n-linear compact operators is a graded C^*-algebra.

If A, B are graded C^*-algebras, let $C^*(A, B)$ be the space of grading preserving $*$-homomorphisms $f\colon A \to B$ (i.e., $f(a^*) = f(a)^*$; such maps are automatically continuous) equipped with the topology of pointwise convergence, i.e., a sequence f_n converges to f if and only if for all $a \in A$ the sequence $f_n(a)$ converges to $f(a)$.

Theorem 3.2.12 (Higson-Guentner [HG]). *Let H be a real Hilbert space which is a graded right module over the Clifford algebra C_n (containing all irreducible C_n-modules infinitely often). Then the space $C^*(C_0(\mathbb{R}), \mathcal{K}_{C_n}(H))$ is homotopy equivalent to the $(-n)$-th space in (the Ω-spectrum equivalent to) the real K-theory spectrum KO.*

Remark 3.2.13. This picture of K-theory is derived from Kasparov's KK-theory, see e.g. [HR]. It is also closely related to a geometric picture of KO-homology due to Graeme Segal [Se3]. We note that if $\varphi\colon C_0(\mathbb{R}) \to \mathcal{K}(H)$ is a $*$-homomorphism (not necessarily grading preserving), then $\varphi(f)$ for $f \in \mathbb{C}_0(\mathbb{R})$ is a family of commuting self-adjoint compact operators. By the spectral theorem, there is a decomposition of H into mutually perpendicular simultaneous eigenspaces of this family. On a particular eigenspace the corresponding eigenvalue $\lambda(f)$ of $\varphi(f)$ determines a real number $t \in \mathbb{R}\cup\infty$ such $\lambda(f) = f(t)$ (any algebra homomorphism $C_0(\mathbb{R}) \to \mathbb{C}$ is given by evaluation at some point $t \in \mathbb{R} \cup \infty$). The eigenspaces are necessarily finite dimensional (except possibly for $t = \infty$), and the only accumulation point of points $t \in \mathbb{R}$ corresponding to a non-trivial eigenspace is ∞. Hence a C^*-homomorphism φ determines a configuration of points on the real line with labels which are mutually perpendicular subspaces of H (given by the corresponding eigenspaces); conversely, such a configuration determines a C^*-homomorphism φ. The conjugation involution on the space of all $*$-homomorphisms $C_0(\mathbb{R}) \to \mathcal{K}(H)$ (whose fixed point set is $C^*(C_0(\mathbb{R}), \mathcal{K}(H))$) corresponds to the involution on the configuration space induced by $t \mapsto -t$ and the grading involution on H. We observe that this implies that every grading preserving $*$-homomorphism $\varphi\colon C_0(\mathbb{R}) \to \mathcal{K}(H)$ is of the form $f \mapsto f(D) \in \mathcal{K}(H') \subset \mathcal{K}(H)$, where $H' \subset H$ is the subspace given by the direct sum of all subspaces E_t of H which occur as 'labels' of points $t \in \mathbb{R}$ of the configuration corresponding to φ. The operator $D\colon H' \to H'$ has E_t as its eigenspace with eigenvalue t; the equivariance condition implies that D is an *odd* operator.

A geometric model for $C^*(C_0(\mathbb{R}), \mathcal{K}_{C_n}(H))$ is obtained by requiring that the points $t \in \mathbb{R}$ are labeled by C_n-linear subspaces of H.

Theorem 1.0.1 follows from Theorem 3.2.12 and the following result.

Lemma 3.2.14. *The inclusion map $HS^{sa}_{C_{-n}}(H) \to \mathcal{K}^{sa}_{C_{-n}}(H)$ induces a homotopy equivalence of the corresponding spaces of homomorphisms of super groups from $\mathbb{R}^{1|1}_+$ to $HS^{sa}_{C_{-n}}(H)$ resp. $\mathcal{K}^{sa}_{C_{-n}}(H)$. Moreover, there is a homeomorphism*

$$C^*(C_0(\mathbb{R}), \mathcal{K}_{C_n}(H)) \xrightarrow{\approx} \mathrm{Hom}(\mathbb{R}^{1|1}_+, \mathcal{K}^{sa}_{C_n}(H))$$

Sketch of proof. Let us outline the proof of the second part. The homeomorphism is given by sending a grading preserving $*$-homomorphism $\varphi\colon C_0(\mathbb{R}) \to \mathcal{K}_{C_n}(H)$ (which may be considered a super homomorphism!) to the composition $\mathbb{R}^{1|1}_+ \xrightarrow{\chi} C_0(\mathbb{R}) \xrightarrow{\varphi} \mathcal{K}_{C_n}(H)$, where χ is the map of super spaces given on Λ-points by

$$\begin{aligned} \mathbb{R}^{1|1}_+(\Lambda) = \Lambda^{ev} \times \Lambda^{odd} \quad &\xrightarrow{\chi(\Lambda)} \quad C_0(\mathbb{R})(\Lambda) = (C_0(\mathbb{R}, \Lambda))^{ev} \\ (t, \theta) &\mapsto e^{-tx^2} + \theta x e^{-tx^2}. \end{aligned}$$

Here the expression $e^{-tx^2} + \theta x e^{-tx^2}$ is interpreted the same way as in example 3.2.9: for $t \in \mathbb{R}_+ \subset \Lambda_+^{ev}$, e^{-tx^2}, xe^{-tx^2} are obviously elements of $C_0(\mathbb{R})$; for general $t = t_B + t_S$ we use Taylor expansion around t_B.

Let us check that $\chi(\Lambda)$ is in fact a homomorphism:

$$\begin{aligned}
&(e^{-t_1x^2} + \theta_1 x e^{-t_1x^2})(e^{-t_2x^2} + \theta_2 x e^{-t_1x^2}) \\
&= e^{-(t_1+t_2)x^2} - \theta_1\theta_2 x^2 e^{-(t_1+t_2)x^2} + (\theta_1+\theta_2)xe^{-(t_1+t_2)x^2} \\
&= e^{-(t_1+t_2+\theta_1\theta_2)x^2} + (\theta_1+\theta_2)xe^{-(t_1+t_2)x^2} \\
&= e^{-(t_1+t_2+\theta_1\theta_2)x^2} + (\theta_1+\theta_2)xe^{-(t_1+t_2+\theta_1\theta_2)x^2}
\end{aligned}$$

Here the minus sign in the second line comes from permuting the odd element x past the odd element θ_2; the second equality follows by taking the Taylor expansion of $e^{-(t_1+t_2+\theta_1\theta_2)x^2}$ around the point $t_1 + t_2$; the third equality follows from the observation that the higher terms of that expansion are annihilated by multiplication by $\theta_1 + \theta_2$.

To finish the proof, one needs to show that the above map χ induces an isomorphism of graded C^*-algebras

$$C^*(\mathbb{R}_+^{1|1}) \cong C_0(\mathbb{R})$$

where the left hand side is the C^*-algebra generated by the super semigroup $\mathbb{R}_+^{1|1}$. □

Theorem 1.0.1 and its real analog identify in particular the components of the space $\mathcal{EFT}_n$ (resp. $\mathcal{EFT}_n^{\mathbb{R}}$) of complex (resp. real) super symmetric 1-field theories of degree n; there is a commutative diagram

$$\begin{array}{ccc}
\pi_0\mathcal{EFT}_n^{\mathbb{R}} & \underset{\cong}{\xrightarrow{\Theta}} & KO_n(\mathrm{pt}) \overset{\mathrm{def}}{=} \mathfrak{M}(C_n)/i^*\mathfrak{M}(C_{n+1}) \\
\downarrow & & \downarrow \\
\pi_0\mathcal{EFT}_n & \underset{\cong}{\xrightarrow{\Theta}} & K_n(\mathrm{pt}) \overset{\mathrm{def}}{=} \mathfrak{M}^{\mathbb{C}}(C_n)/i^*\mathfrak{M}^{\mathbb{C}}(C_{n+1})
\end{array},$$

where the horizontal isomorphisms are given by the theorem and the vertical maps come from complexification; $\mathfrak{M}(C_n)$ (resp. $\mathfrak{M}^{\mathbb{C}}(C_n)$) is the Grothendieck group of real (resp. complex) graded modules over the Clifford algebra C_n, and i^* is induced by the inclusion map $C_n \to C_{n+1}$ (this way of relating K-theory and Clifford algebras is well-known; see for example [LM, Ch. I, Theorem 9.29]). Explicitly, the map Θ is given by associating to a field theory E (real or complex) the graded C_n-module $E_{>\rho}(E(I_t))$ (the finite dimensional sum of the eigenspaces of $E(I_t)$ with eigenvalue $> \rho$); the argument in Remark 3.1.6 shows that its class in K-theory is independent of t, ρ and depends only on the path component of E in $\mathcal{EFT}_n$.

By Bott-periodicity, $K_{2k+1}(\mathrm{pt}) = 0$ and $K_{2k} \cong \mathbb{Z}$. This isomorphism can be described explicitly as follows.

Lemma 3.2.15. *For n even the map*

$$\mathfrak{M}^{\mathbb{C}}(C_n)/i^*\mathfrak{M}^{\mathbb{C}}(C_{n+1}) \longrightarrow \mathbb{Z} \qquad [M] \mapsto \mathrm{str}(\gamma^{\otimes n}\colon M \to M)$$

is an isomorphism. Here $\gamma \stackrel{def}{=} 2^{-1/2} i^{1/2} e_1 \in C_1 \otimes \mathbb{C}$ *and* $\gamma^{\otimes n} = \gamma \otimes \cdots \otimes \gamma \in \mathbb{C}\ell_n = \mathbb{C}\ell_1 \otimes \cdots \otimes \mathbb{C}\ell_1$*, where* $\mathbb{C}\ell_n = C_n \otimes_{\mathbb{R}} \mathbb{C}$ *is the complexified Clifford algebra.*

Proof. The complex Clifford algebra $\mathbb{C}\ell_2$ is isomorphic to the algebra of complex 2×2-matrices. Let $\Delta = \mathbb{C}^2$ be the irreducible module over $\mathbb{C}\ell_2$; make Δ a graded module by declaring the grading involution ϵ to be multiplication by the 'complex volume element' $\omega_{\mathbb{C}} = ie_1e_2 \in \mathbb{C}\ell_2$ [LM, p. 34]. It is well-known that $\Delta^{\otimes k}$ (the graded tensor product of k copies of Δ) represents a generator of $\mathfrak{M}^{\mathbb{C}}(C_{2k})/i^*\mathfrak{M}^{\mathbb{C}}(C_{2k+1})$ [LM, Ch. I, Remark 9.28]. We have

$$\gamma^{\otimes 2} = \gamma \otimes \gamma = \frac{i}{2} e_1 e_2 = \frac{1}{2}\omega_{\mathbb{C}} \in \mathbb{C}\ell_1 \otimes \mathbb{C}\ell_1 = \mathbb{C}\ell_2.$$

We note that for any homomorphism $f\colon M \to M$ on a graded vector space with grading involution ϵ we have $\mathrm{str}(f) = \mathrm{tr}(\epsilon f)$. In particular, for $M = \Delta$ with grading involution $\epsilon = \omega_{\mathbb{C}}$ we obtain

$$\mathrm{str}(\omega_{\mathbb{C}}\colon \Delta \to \Delta) = \mathrm{tr}(\omega_{\mathbb{C}}^2) = \mathrm{tr}(1_\Delta) = \dim \Delta = 2.$$

This implies $\mathrm{str}(\gamma^{\otimes 2}\colon \Delta \to \Delta) = 1$ and hence $\mathrm{str}(\gamma^{\otimes 2k}\colon \Delta^{\otimes k} \to \Delta^{\otimes k}) = 1$. □

The above lemma motivates the following

Definition 3.2.16. If M is a finite dimensional graded C_n module we define its *Clifford super dimension* as

$$\mathrm{sdim}_{C_n}(M) \stackrel{\text{def}}{=} \mathrm{str}(\gamma^{\otimes n}\colon M \to M)$$

More generally, if $f\colon M \to M$ is a C_n-linear map then we define its *Clifford super trace* as

$$\mathrm{str}_{C_n}(f) \stackrel{\text{def}}{=} \mathrm{str}(\gamma^{\otimes n} f\colon M \to M).$$

We note that the definition of the Clifford super trace continues to make sense for not necessarily finite dimensional modules M, provided f is of trace class, i.e., f is the composition of two Hilbert-Schmidt operators (this guarantees that the infinite sums giving the super trace above converge).

The simplest invariants associated to a field theory E are obtained by considering a closed d-manifold Σ equipped with a fermion $\Psi \in F_{alg}(\Sigma)$, and to regard (Σ, Ψ) as an endomorphism of the object $\emptyset$; then $E(\Sigma, \Psi) \in \mathrm{Hilb}(E(\emptyset), E(\emptyset)) = \mathrm{Hilb}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$. For $d = 1$ we obtain the following result:

Lemma 3.2.17. *Let I_t the interval of length t, let S_t^{per} (resp. S_t^{ap} resp. S_t) be the circle of length t with the periodic (resp. anti-periodic resp. unspecified) spin structure. Let $\mu\colon F_{alg}(I_t) \to F_{alg}(S_t)$ be the fermionic gluing map and let $\Psi \in F_{alg}(I_t)^{-n}$. If E is a EFT, then*

$$E(S_t^{per}, \mu(\Psi)) = \mathrm{str}(E(I_t, \Psi)) \qquad E(S_t^{ap}, \mu(\Psi)) = \mathrm{tr}(E(I_t, \Psi)).$$

This lemma follows from decomposing $S_{t+t'}^{per}$ resp. $S_{t+t'}^{ap}$ as in equation 3.1.4 (resp. 3.1.5) and noting that the algebraic analog of this chain of morphisms is just the trace (resp. super trace) of $E(I_{t_1} \circ I_{t_2}) = E(I_{t_1+t_2})$.

Remark 3.2.18. We remark that unlike $\mathrm{tr}(E(I_t, \Psi))$ the function $\mathrm{str}(E(I_t, \Psi))$ is *independent of* t: super symmetry implies that the generator A of the semi-group $E(I_t) = e^{-tA}\colon E(\mathrm{pt}) \to E(\mathrm{pt})$ is the square of an odd operator. This implies that for any $c \in C(\mathrm{pt})^{-n} = C_{-n}$ the contributions to the super trace of $E(I_t, c\Omega^{-n}) = E(c)E(I_t) = E(c)e^{-tA}$ coming from eigenspaces of A with non-zero eigenvalues vanish and hence $\mathrm{str}(E(c)e^{-tA}) = \mathrm{str}(E(c)\colon \ker A \to \ker A)$ is independent of t (cf. Remark 3.2.1).

Definition 3.2.19. If E is a EFT of degree n, we will call the function

$$Z_E(t) \overset{\text{def}}{=} \mathrm{str}_{C_n}(E(I_t)) = \mathrm{str}(\gamma^{\otimes n} E(I_t)) = E(S_t^{per}, \mu(\gamma\Omega)^{\otimes n})$$

the *partition function* of E. The previous remark shows that this function is constant if E is super symmetric. This terminology is motivated by the fact that physicists refer to the analogous function for higher dimensional field theories as *partition function* (see Definition 3.3.5 for the case of 2-dimensional conformal field theories).

Putting Theorem 1.0.1, Lemma 3.2.15 and Lemma 3.2.17 together, we obtain:

Corollary 3.2.20. *There is a bijection*

$$\pi_0 \mathcal{EFT}_{2k} \longrightarrow \mathbb{Z}$$

which sends the EFT E to its (constant) partition function $Z_E(t) \in \mathbb{Z}$.

It is desirable to describe certain important K-theory classes (like the *families index* or the *Thom class*) as maps to the space $\mathcal{EFT}_n$ of field theories of degree n. Below we do something a little less: we describe maps to the space $\mathrm{Hom}(\mathbb{R}_+^{1|1}, HS_{C_n}^{sa}(H))$, which is homeomorphic to $\mathcal{EFT}_n$ by Proposition 3.2.6; in other words, we describe the associated EFT only on the standard super interval $I_{t,\theta}$.

Remark 3.2.21. (The index of a family of spin manifolds). Let $\pi\colon Z \to X$ be a fiber bundle with fibers of dimension n. Assume that the tangent bundle τ along the fibers has a spin structure; this implies that π induces a map $\pi_*\colon KO(Z) \to KO^{-n}(X)$

called 'Umkehr map' or 'integration over the fiber'. If $\xi \to Z$ is a real vector bundle, the element

$$\pi_*(\xi) \in KO^{-n}(X) = [X, KO_{-n}] = [X, \mathcal{EFT}_n^{\mathbb{R}}]$$

can be described as follows. Let $S \to Z$ be the $C(\tau) - C_n$-bimodule bundle representing the spin structure on τ. Then we obtain a C_n-bundle over X whose fiber over $x \in X$ is $L^2(Z_x, (S \otimes \xi)_{|Z_x})$, the Hilbert space H_x of L^2-sections of $S \otimes \xi$ restricted to the fiber Z_x. The Dirac operator $D_x \otimes \xi$ on Z_x 'twisted by ξ' acts on H_x and commutes with the (right) C_n-action on H_x induced by the action on S. Since the space of C_n-linear grading preserving isometries is contractible, we may identify H_x with a *fixed* real Hilbert space $H_{\mathbb{R}}$ with C_n-action. Then the map

$$X \longrightarrow \mathrm{Hom}(\mathbb{R}_+^{1|1}, HS^{sa}_{C_n}(H_{\mathbb{R}})) \cong \mathcal{EFT}_n^{\mathbb{R}} \qquad x \mapsto ((t,\theta) \mapsto f_{t,\theta}(D_x \otimes \xi))$$

represents the element $\pi_*(\xi)$; here $f_{t,\theta} = e^{-tx^2+\theta x} = e^{-tx^2} + \theta x e^{-tx^2} \in C_0(\mathbb{R})$ for $(t,\theta) \in \mathbb{R}^{1|1}$; functional calculus can then be applied to the self-adjoint operator $D_x \otimes \xi$ to produce the super semi-group $f_{t,\theta}(D_x \otimes \xi)$ of even self-adjoint Hilbert-Schmidt operators.

Remark 3.2.22. (The *KO*-theory Thom class). Let $\pi\colon \xi \to X$ be an n-dimensional vector bundle with spin structure given by the $C(\xi) - C_n$-bimodule bundle $S \to X$ (see Remark 2.3.1). We may assume that there is a real Hilbert space $H_{\mathbb{R}}$ which is a graded C_n-module, and that S is a C_n-linear subbundle of the trivial bundle $X \times H_{\mathbb{R}}$. We note that for $v \in \xi$ the Clifford multiplication operator $c(v)\colon S_x \to S_x$ is skew-adjoint, and $\epsilon c(v)$ is selfadjoint (ϵ is the grading involution); moreover, $\epsilon c(v)$ *commutes* with the right action of the Clifford algebra $C_{-n} = C(-\mathbb{R}^n)$ if we let $w \in \mathbb{R}^n$ act via $\epsilon c(w)$. Then the map

$$\begin{aligned} \xi &\longrightarrow \mathrm{Hom}(\mathbb{R}_+^{1|1}, HS^{sa}_{C_{-n}}(H_{\mathbb{R}})) \cong \mathcal{EFT}_{-n}^{\mathbb{R}} \\ v &\mapsto ((t,\theta) \mapsto f_{t,\theta}(\epsilon c(v)) \in HS^{sa}_{C_{-n}}(S_{\pi(v)}) \subset HS^{sa}_{C_{-n}}(H_{\mathbb{R}}) \end{aligned}$$

extends to the Thom space X^ξ and represents the KO-theory Thom class of ξ in $KO^n(X^\xi) = [X^\xi, \mathcal{EFT}_{-n}^{\mathbb{R}}]$.

3.3 Conformal field theories and modular forms

In this section we will show that CFT's of degree n (see Definition 2.3.16) are closely related to modular forms of weight $n/2$. For a precise statement see Theorem 3.3.4 below. Let us first recall the definition of modular forms (cf. [HBJ, Appendix]).

Definition 3.3.1. A *modular form of weight k* is a function $f\colon \mathfrak{h} \to \mathbb{C}$ which is holomorphic (also at $i\infty$) and which has the following transformation property:

$$f(\tfrac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau) \qquad \text{for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}). \tag{3.3.2}$$

Let us recall what 'holomorphic at $i\infty$' means. The transformation property for the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ implies the translation invariance

$$f(\tau+1)=f(\tau).$$

It follows that $f\colon \mathfrak{h}\to\mathbb{C}$ factors through $\mathfrak{h}/\mathbb{Z}$, which is conformally equivalent to the punctured open unit disc D^2_0 by means of the map

$$\mathfrak{h}\longrightarrow D^2_0 \qquad \tau\mapsto q=e^{2\pi i\tau}.$$

Then $f(\tau)$ is *holomorphic at* $i\infty$ if the resulting function $f(q)$ on D^2_0 extends over the origin (note that $\tau\to i\infty$ corresponds to $q\to 0$). Equivalently, in the expansion

$$f(\tau)=\sum_{n\in\mathbb{Z}}a_nq^n \tag{3.3.3}$$

of $f(q)$ as a Laurent series around 0 (this is called the *q-expansion of f*), we require that $a_n=0$ for $n<0$.

Theorem 3.3.4. *If E is a CFT of degree n, then its partition function $Z_E\colon\mathfrak{H}\to\mathbb{C}$ (see definition 3.3.5 below) has the transformation property 3.3.2 of a modular form of weight $n/2$.*

Definition 3.3.5. (Partition function). We recall from Definition 2.3.12 that for a *closed* conformal spin surface Σ the fermionic Fock space $F_{alg}(\Sigma)$ is the Pfaffian line $\mathrm{Pf}(\Sigma)$. Given a CFT E of degree n and an element $\Psi\in\mathrm{Pf}^{-n}(\Sigma)\overset{\text{def}}{=}F_{alg}(\Sigma)^{-n}$, the pair (Σ,Ψ) represents a morphisms in the category $\mathcal{CB}^2_n$ from $\emptyset$ to $\emptyset$; hence we can apply the functor E to (Σ,Ψ) to obtain an element $E(\Sigma,\Psi)\in\mathrm{Hilb}(E(\emptyset),E(\emptyset))=\mathrm{Hilb}(\mathbb{C},\mathbb{C})=\mathbb{C}$. Since $E(\Sigma,\Psi)$ depends linearly on Ψ, we obtain an element

$$Z_E(\Sigma)\in\mathrm{Pf}^n(\Sigma)=\mathrm{Hom}(\mathrm{Pf}^{-n}(\Sigma),\mathbb{C}) \qquad \text{given by} \qquad \Psi\mapsto E(\Sigma,\Psi).$$

We recall that if $g\colon\Sigma\to\Sigma'$ is a conformal spin diffeomorphism and $g\colon\mathrm{Pf}^{-n}(\Sigma)\to\mathrm{Pf}^{-n}(\Sigma')$ is the induced isomorphism of Pfaffian lines, then for any $\Psi\in\mathrm{Pf}^{-n}(\Sigma)$ the pairs (Σ,Ψ) and $(\Sigma',g\Psi)$ represent the *same* morphism in $\mathcal{CB}^2_n(\emptyset,\emptyset)$. In particular we have

$$E(\Sigma,\Psi)=E(\Sigma',g\Psi)\in\mathbb{C} \qquad \text{and} \qquad g(Z_E(\Sigma))=Z_E(\Sigma')\in\mathrm{Pf}^n(\Sigma'). \tag{3.3.6}$$

This property can be used to interpret Z_E as a section of a complex line bundle Pf^n over the Teichmüller spaces of conformal spin surfaces which is equivariant under the action of the mapping class groups. The *partition function* of E is obtained by restricting attention to conformal surfaces of genus one with the non-bounding spin structure. This spin structure is preserved up to isomorphism by any orientation preserving diffeomorphism, i.e. by $SL_2(\mathbb{Z})$, whereas the other 3 spin structure are permuted. This will ultimately have the effect of obtaining a modular form for the full modular group $SL_2(\mathbb{Z})$ rather than for an (index 3) subgroup.

Let us describe the Teichmüller space of this spin torus explicitly. Given a point τ in the upper half plane $\mathfrak{h} \subset \mathbb{C}$, let $\Sigma_\tau \overset{\text{def}}{=} \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be the *conformal torus* obtained as the quotient of the complex plane (with its standard conformal structure) by the free action of the group $\mathbb{Z}+\mathbb{Z}\tau \subset \mathbb{C}$ acting by translations. Let A_τ be the *conformal annulus* obtained as a quotient of the strip $\{z \in \mathbb{C} \mid 0 \le \operatorname{im}(z) \le \operatorname{im}(\tau)\}$ by the translation group $\mathbb{Z}$. The annulus is a bordism from $S^1 = \mathbb{R}/\mathbb{Z}$ to itself if we identify $\mathbb{R}$ with the horizontal line $\{z \in \mathbb{C} \mid \operatorname{im}(z) = \operatorname{im}(\tau)\}$ via $s \mapsto s + \tau$. So while A_τ as a manifold depends only on the imaginary part of τ, the identification between ∂A_τ and the disjoint union of S^{per} (circle with periodic spin structure) and $\bar{S}^{per}$ depends on the real part of τ. Note that if we equip $\mathbb{C}$ with the standard spin structure given by the bimodule bundle $S = \mathbb{C} \times C_2$, then the translation action of $\mathbb{C}$ on itself lifts to an action on S (trivial on the second factor). This implies that the spin structure on $\mathbb{C}$ induces a spin structure on Σ_τ (which is the non-bounding spin structure) and A_τ.

For $Y = S^{per}$ the space Hilbert $V(Y)$ from Definition 2.3.6 can be identified with complex valued functions on the circle. In particular, the constant real functions give us an isometric embedding $\mathbb{R} \subset V(S^{per})$ and hence an embedding of Clifford algebras $C_1 = C(\mathbb{R}) \to C(S^{per})$. Let $\Omega \in F_{alg}(A_\tau)$ be the vacuum vector, let $\gamma \in C_1 \otimes \mathbb{C} \subset C(S^{per}) \otimes \mathbb{C}$ be the element constructed in Lemma 3.2.15, and let $\xi_\tau \in F_{alg}(\Sigma_\tau)$ be the image of $\gamma\Omega$ under the fermionic gluing map $\mu\colon F_{alg}(A_\tau) \to F_{alg}(\Sigma_\tau)$. If E is a CFT of degree n (not necessarily super symmetric), we define its *partition function* to be the function

$$Z_E\colon \mathfrak{h} \longrightarrow \mathbb{C} \qquad \tau \mapsto E(\Sigma_\tau, \xi_\tau^{-n}).$$

We note that for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $\tau \in \mathfrak{h}$ the map $\mathbb{C} \to \mathbb{C}$, $z \mapsto (c\tau + d)^{-1} z$ sends the lattice $\mathbb{Z}+\mathbb{Z}\tau$ to $\mathbb{Z}+\mathbb{Z}g\tau$, $g\tau = \frac{a\tau+b}{c\tau+d}$. This conformal diffeomorphism lifts to a conformal spin diffeomorphism $g\colon \Sigma_\tau \to \Sigma_{g\tau}$, which induces an isomorphism of Pfaffian lines $g_*\colon \operatorname{Pf}(\Sigma_\tau) \to \operatorname{Pf}(\Sigma_{g\tau})$.

Lemma 3.3.7. *The induced map* $\operatorname{Pf}(\Sigma_\tau)^{\otimes 2} \to \operatorname{Pf}(\Sigma_{g\tau})^{\otimes 2}$ *sends* $\xi_\tau^{\otimes 2}$ *to* $(c\tau + d)\xi_{g\tau}^{\otimes 2}$*.*

Proof of Theorem 3.3.4. We note that the previous lemma and equation 3.3.6 implies that if E is a CFT of degree $2k$, then

$$\begin{aligned} Z_E(\tau) = E(\Sigma_\tau, \xi_\tau^{-2k}) = E(\Sigma_{g\tau}, g_*(\xi_\tau^{-2k})) &= E(\Sigma_{g\tau}, (c\tau + d)^{-k}\xi_{g\tau}^{-2k})) \\ &= (c\tau + d)^{-k} E(\Sigma_{g\tau}, \xi_{g\tau}^{-2k}) = (c\tau + d)^{-k} Z_E(g\tau) \end{aligned} \quad (3.3.8)$$

This shows that the partition function of E has the transformation property (3.3.2) of a modular form of weight k as claimed by Theorem 3.3.4. ∎

Now we would like to discuss whether the partition function $Z_E(\tau)$ of a conformal field theory E of degree $2k$ *is* a modular form of weight k; in other words, whether

$Z_E(\tau)$ is holomorphic and holomorphic at $i\infty$. The key to this discussion is the following result whose proof is analogous to that of the corresponding result Lemma 3.2.17 for the partition function of a 1-dimensional EFT.

Lemma 3.3.9. $Z_E(\tau) = \mathrm{str}(E(A_\tau, (\gamma\Omega)^{-n}))$.

We note that $E(S^{per})$ is a graded module over the Clifford algebra $C(S^{per})^{-n}$ by letting $c \in C(S^{per})^{-n}$ act on $E(S^{per})$ via the operator $E(c)$. We note that the operator $E(A_\tau)$ does not commute with the action of the algebra $C(S^{per})^{-n}$, but it *does* commute with the subalgebra $C_{-n} = C_1^{-n} \subset C(S^{per})^{-n}$ generated by constant functions. Expressing $(A_\tau, (\gamma\Omega)^{-n})) \in \mathcal{CB}^2_n(S^{per}, S^{per})$ as the composition $\gamma^{-n} \circ A_\tau$ we obtain the following alternative expression for the partition function (cf. Definition 3.2.16):

$$Z_E(\tau) = \mathrm{str}(E(\gamma^{\otimes n})E(A_\tau)) = \mathrm{str}_{C_n}(E(A_\tau)). \qquad (3.3.10)$$

The compatibility of $E\colon \mathcal{CB}^2_n \to \mathrm{Hilb}$ with the involution $*$ on both categories implies that the homomorphism

$$\mathfrak{h} \to HS^{ev}_{C_n}(E(S^{per})) \qquad \tau \mapsto E(A_\tau) \qquad (3.3.11)$$

is $\mathbb{Z}/2$-equivariant, where $\mathbb{Z}/2$ acts on $\mathfrak{h}$ by $\tau \mapsto -\bar{\tau}$ and on $HS_{C_n}(H)$ by taking adjoints. *Any* homomorphism $\rho\colon \mathfrak{h} \to HS^{ev}_{C_n}(H)$ has the form

$$\rho(\tau) = q^{L_0}\bar{q}^{\bar{L}_0} \qquad q = e^{2\pi i\tau}$$

where $L_0, \bar{L}_0$ are two commuting, even, C_n-linear operators (in general unbounded), such that the eigenvalues of $L_0 - \bar{L}_0$ are integral. Moreover, the homomorphism is $\mathbb{Z}/2$-equivariant if and only if the operators L_0, $\bar{L}_0$ are self-adjoint.

We want to emphasize that the homomorphism (3.3.11) is completely analogous to the homomorphism

$$\mathbb{R}_+ \longrightarrow HS^{ev,sa}_{C_n}(E(\mathrm{pt})) \qquad t \mapsto E(I_t)$$

associated to a 1-dimensional EFT (see Equation (3.1.3)). We've shown that $E(I_t)$ is always of the form $E(I_t) = e^{-tA}$ for an (unbounded) self-adjoint, even operator A. Moreover, if E is the restriction of a *super symmetric* EFT, i.e., if E extends from the bordism category $\mathcal{EB}^1_n$ to the 'super bordism category' $\mathcal{SEB}^1_n$, then A is the square of an odd operator. This had the wonderful consequence that for any $c \in C_n$ the super trace $\mathrm{str}(cE(I_t))$ is in fact *independent* of t.

Similarly, we would like to argue that if the Clifford linear CFT E is the restriction of a 'super conformal' field theory of degree n, then the infinitesimal generator $\bar{L}_0$ of

$$E(A_\tau) = q^{L_0}\bar{q}^{\bar{L}_0} \qquad (3.3.12)$$

is the square of an odd self-adjoint operator $\bar{G}_0$. Here by 'super conformal' field theory of degree n we mean a functor $E\colon \mathcal{SCB}^2_n \to \mathrm{Hilb}$ (satisfying the usual requirements), where

$\mathcal{SCB}^2_n$ is the 'super version' of the category $\mathcal{CB}^2_n$, in which the conformal spin bordisms are replaced by super manifolds equipped with an appropriate 'geometric super structure', which induces a conformal structure on the underlying 2-dimensional spin manifold. Unfortunately our ignorance about super geometry has kept us from identifying the correct version of this 'geometric super structure', but we are confident that this can be done (or has been done already). In this situation it seems reasonable to proceed assuming this. In other words, from now on the results in this section all will be subject to the following

Hypothesis 3.3.13. *There is an appropriate notion of 'super conformal structure' with the following properties:*

1. *on the underlying* 2*-dimensional spin manifold it amounts to a conformal structure;*
2. *if* $E\colon \mathcal{CB}^2_n \to \mathrm{Hilb}$ *is a CFT of degree* n *which extends to a (yet undefined 'super symmetric CFT of degree* n*' then* $\bar{L}_0$ *is the square of an odd operator* $\bar{G}_0$ *(where* $\bar{L}_0$ *is as in equation* (3.3.12)*).*

We want to emphasize that the usual notion of 'super conformal structure' is *not* what is needed here; we will comment further in Remark 3.3.18.

Theorem 3.3.14. *Assuming Hypothesis 3.3.13, the partition function* Z_E *of a susy CFT of degree* n *is a weak integral modular form of weight* $\frac{n}{2}$*.*

Definition 3.3.15. (Weak integral modular forms). A *weak modular form* is a holomorphic function $f\colon \mathfrak{h} \to \mathbb{C}$ with the transformation property (3.3.2), whose q-expansion (3.3.3) has only finitely many terms with negative powers of q; equivalently, the function $f(q)$ on the disc has a pole, not an essential singularity at $q = 0$.

A (weak) modular form is *integral* if all coefficients a_n in its q-expansion are integers (this low-brow definition is equivalent to more sophisticated definitions). An example of an integral modular form is the *discriminant* Δ whose q-expansion has the form

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

Other examples of integral modular forms are the the *Eisenstein series*

$$c_4 = 1 + 240\sum_{k>0}\sigma_3(k)q^k \qquad c_4 = 1 - 504\sum_{k>0}\sigma_5(k)q^k,$$

(modular forms of weight 4 respectively 6) where $\sigma_r(k) = \Sigma_{d|k}d^r$. The ring of integral modular forms is equal to the quotient of the polynomial ring $\mathbb{Z}[c_4, c_6, \Delta]$ by the ideal generated by $c_4^3 - c_6^2 - (12)^3\Delta$.

Let us denote by MF_* the graded ring of weak integral modular forms; it is graded by the *degree of a modular form* to be *twice* its weight; the motivation here being that with this definition the degree of the partition function of a susy CFT E agrees with the degree of E. Recall that the discriminant Δ has a simple zero at $q = 0$. It follows that if f is a weak modular form, then $f\Delta^N$ is a modular form for N sufficiently large. As a consequence,

$$MF_* = \mathbb{Z}[c_4, c_6, \Delta, \Delta^{-1}]/(c_4^3 - c_6^2 - (12)^3\Delta).$$

Proof of Theorem 3.3.14. By equations (3.3.10) and (3.3.12) we have

$$Z_E(\tau) = \mathrm{str}_{C_n}(q^{L_0}\bar{q}^{\bar{L}_0}).$$

We recall that the eigenvalues of $L_0 - \bar{L}_0$ are integral; let $H_k \subset E(S^{per})$ be the subspace corresponding to the eigenvalue $k \in \mathbb{Z}$. According to our hypothesis 3.3.13 we have $\bar{L}_0 = \bar{G}_0^2$. This allows us to calculate the partition function $Z_E(\tau)$ as follows:

$$\begin{aligned} Z_E(\tau) &= \mathrm{str}_{C_n}(q^{L_0}\bar{q}^{\bar{L}_0}) = \mathrm{str}_{C_n}(q^{L_0}_{|\ker \bar{L}_0}) && (3.3.16)\\ &= \sum_{k\in\mathbb{Z}} \mathrm{str}_{C_n}(q^{L_0}_{|\ker \bar{L}_0\cap H_k}) = \sum_{k\in\mathbb{Z}} q^k \,\mathrm{sdim}_{C_n}(\ker \bar{L}_0 \cap H_k) && (3.3.17)\end{aligned}$$

Here the second equality follows from the fact that the eigenspace of $\bar{L}_0$ with non-zero eigenvalue don't contribute to the supertrace (see remark 3.2.1); the last equality follows from the fact that restricted to the kernel of $\bar{L}_0$ the operator $L_0 = L_0 - \bar{L}_0$ which in turn is just multiplication by k on H_k. This implies that $Z_E(\tau)$ is a holomorphic function with integral coefficients in its q-expansion.

To see that all but finitely many coefficients a_k with negative k must be zero, we note that if $\ker \bar{L}_0 \cap H_k$ were $\neq 0$ for an infinite sequence of negative values of k, we would run into a contradiction with the fact that $q^{L_0}\bar{q}^{\bar{L}_0}$ is a Hilbert-Schmidt operator. □

The above proof suggests to associate to a susy CFT E of degree n the following homomorphism of super groups

$$\psi_k \colon \mathbb{R}^{1|1} \longrightarrow HS_{C_n}(H_k) \qquad (t,\theta) \mapsto e^{2\pi i(it\bar{L}_0 + i^{1/2}\theta\bar{G}_0)}.$$

By the result of the previous section, the super homomorphism ψ_k represents an element $\Psi_k(E) \in \pi_0(\mathcal{EFT}_n) \cong K_n(\mathrm{pt})$. Let us calculate the image of this element under the isomorphism $K_n(\mathrm{pt}) \cong \mathbb{Z}$ (for n even), which is given by associating to $\Psi_k(E)$ its partition function. By the arguments leading to corollary 3.2.20, it is given by

$$\mathrm{str}_{C_n}(e^{2\pi i(it\bar{L}_0)} \text{ acting on } H_k) = \mathrm{sdim}_{C_n}(\ker \bar{L}_0 \cap H_k)$$

which is the coefficient a_k in the q-expansion of $Z_E(\tau)$ by comparison with the proof of theorem 3.3.4.

Sketch of proof of Theorem 1.0.2. This is a 'parametrized' version of the above argument: if C is a Clifford elliptic object over X, then given a point $x \in X$, we obtain a susy Clifford linear CFT as the composition $E_x \colon \mathcal{SCB}^2_n \to \mathcal{SCB}^2_n(X) \xrightarrow{C} \text{Hilb}$; here the first map is given by using the constant map to $x \in X$. From E_x we manufacture a collection of homomorphisms $\psi_k(x) \colon \mathbb{R}^{1|1} \to HS_{C_n}(H_k)$ as above; these depend continuously on x. By the results of the last subsection, the map ψ_k from X to the space of homomorphisms represents an element of $K^{-n}(X)$, giving the coefficient of q^k in the Laurent series $MF(C) \in K^{-n}(X)[[q]][q^{-1}]$. □

Remark 3.3.18. We would like to conclude this section with some comments on our Hypothesis 3.3.13. The function $\rho(\tau) = q^{L_0}\bar{q}^{\bar{L}_0}$ can be rewritten in the form

$$\rho(\tau) = q^{L_0}\bar{q}^{\bar{L}_0} = e^{2\pi i[uL_0 + v\bar{L}_0]},$$

where $u = \tau$ and $v = -\bar{\tau}$. Geometrically, the new coordinates u, v can be interpreted as follows: If we think of $\tau = x + iy$ with coordinates x, y of $\mathfrak{h} \subset \mathbb{R}^2$ and with Euclidean metric $ds^2 = dx^2 + dy^2$, then the *Wick rotation* $y \mapsto t = iy$ gives a new coordinate t with respect to which the metric becomes the Minkowski metric $dx^2 - dt^2$. We note that u, v are the light cone coordinates with respect to the Minkowski metric (i.e., the light cone consists of the points with $uv = 0$). In other words, if we write $\mathbb{R}^2_E$ for $\mathbb{R}^2$ with coordinates x, y and the usual Euclidean metric and $\mathbb{R}^2_M$ for $\mathbb{R}^2$ with x, t coordinates and the Minkowski metric, then the Wick rotation gives us an identification $\mathbb{R}^2_E \otimes \mathbb{C} = \mathbb{R}^2_M \otimes \mathbb{C}$ between the complexifications. Let $\rho_* \colon \text{Lie}(\mathbb{R}^2_E) \subset \text{Lie}(\mathbb{R}^2_E)_{\mathbb{C}} \to \text{End}_{C_n}(H)$ be the Lie algebra homomorphism induced by ρ, extended to the complexification $\text{Lie}(\mathbb{R}^2_E)_{\mathbb{C}}$. Interpreting the translation invariant vectorfields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ on $\mathbb{R}^2_E$ as elements of $\text{Lie}(\mathbb{R}^2_E)$ and $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial v}$ as elements of $\text{Lie}(\mathbb{R}^2_M)_{\mathbb{C}}$, the above equation shows that

$$\rho_*(\frac{\partial}{\partial u}) = 2\pi i L_0 \qquad \rho_*(\frac{\partial}{\partial v}) = 2\pi i \bar{L}_0.$$

Let $\mathbb{R}^{2|1}_M$ be the super Minkowski space (super space time) of dimension $(2|1)$ with even coordinates u, v and one odd coordinate θ (see [Wi2, §2.8] or [Fr2, Lecture 3]). This comes equipped with a natural geometric 'super structure' extending the Minkowski metric on the underlying $\mathbb{R}^2_M$. It has a group structure such that the translation action on itself preserves that geometric structure. The corresponding super Lie algebra is given by the space of invariant vector fields; a basis is provided by the two even vector fields $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ and the odd vector field $Q = \partial_\theta + \theta\frac{\partial}{\partial v}$. The odd element Q commutes (in the graded sense) with the even elements; the crucial relation in this super Lie algebra $\text{Lie}(\mathbb{R}^{2|1}_M)$ is (see [Wi2, p. 498]):

$$\frac{1}{2}[Q, Q] = Q^2 = \frac{\partial}{\partial v}.$$

This shows that if the representation $\rho\colon \mathbb{R}^2_M \to \mathrm{End}_{C_n}(H)$ extends to a representation of the super Poincaré group $\mathbb{R}^{2|1}_M$, then the operator $\bar{L}_0$ isthe *square of an odd operator*.

We note that the usual 'super conformal structure' on super manifolds of dimension (2|2) [CR], [Ba] is *not* the geometric structure we are looking for. This is *too much* super symmetry in the sense that if a Clifford linear conformal field theory $E\colon \mathcal{CB}^2_n \to \mathrm{Hilb}$ would extend over these super manifolds, then *both generators* L_0 and $\bar{L}_0$ of the semi-group $E(A_\tau)$ would be squares of odd operators, thus making the partition function $Z_E(\tau)$ *constant*.

4 Elliptic objects

In this section we describe various types of elliptic objects (over a manifold X). We start by recalling Segal's original definition, then we modify it by introducing fermions. After adding a super symmetric aspect (as in Section 3.2) we arrive at so called *Clifford elliptic objects*. These are still not good enough for the purposes of excision, as explained in the introduction. Therefore, we add data associated to points, rather than just circles and conformal surfaces. These are our *enriched elliptic objects*, defined as certain functors from a geometric bicategory $\mathcal{D}_n(X)$ to the bicategory of von Neumann algebras vN.

4.1 Segal and Clifford elliptic objects

We first remind the reader of a definition due to Segal [Se1, p.199].

Definition 4.1.1. A *Segal elliptic object* over X is a projective functor $\mathcal{C}(X) \to \mathcal{V}$ satisfying certain axioms. Here $\mathcal{V}$ is the category of topological vector spaces and trace class operators; the objects of $\mathcal{C}(X)$ are closed oriented 1-manifolds equipped with maps to X and the morphisms are 2-dimensional oriented bordisms equipped with a conformal structure and a map to X. In other words, $\mathcal{C}(X)$ is the subcategory of the bordism category $\mathcal{CB}^2(X)$ (see definition 2.1.5) with the same objects, but excluding those morphisms which are given by *diffeomorphisms*.

The adjective 'projective' basically means that the vector space (resp. operator) associated to map of a closed 1-manifold (resp. a conformal 2-manifold) to X is only defined up to a scalar. As explained by Segal in §4 of his paper [Se2] (after Definition 4.4), a projective functor from $\mathcal{C}(X)$ to $\mathcal{V}$ can equivalently be described as a functor

$$\widehat{E}\colon \mathcal{C}_n(X) \longrightarrow \mathcal{V},$$

where $n \in \mathbb{Z}$ is the *central charge* of the elliptic object. Here $\mathcal{C}_n(X)$ is some 'extension' of the category $\mathcal{C}(X)$, whose objects and morphisms are like those of $\mathcal{C}(X)$, but the 1-manifolds and 2-manifolds (giving the objects resp. morphisms) are equipped with an extra structure that we will refer to as *n-riggings*. The functor $\widehat{E}$ is required to satisfy a linearity condition explained below. We will use the notation $\widehat{E}$ for Segal elliptic objects and E for Clifford elliptic objects (Definition 4.1.3).

The following definition of n-riggings is not in Segal's papers, but it is an obvious adaptation of Segal's definitions if we work with manifolds with spin structures as Segal proposes to do at the end of §6 in [Se1].

Definition 4.1.2. (Riggings) Let Y be a closed spin 1-manifold which is zero bordant. We recall that associated to Y is a Clifford algebra $C(Y)$ (see Definitions 2.3.6 and 2.3.7), and that a conformal spin bordism Σ' from Y to the empty set determines an irreducible (right) $C(Y)$-module $F(\Sigma')$ (the 'Fock space' of Σ'; see Definition 2.3.12). The isomorphism type of $F(\Sigma')$ is *independent* of Σ'.

Given an integer n, we define an *n-rigging* of Y to be a right $C(Y)^{-n}$-module R isomorphic to $F(\Sigma')^{-n}$ for some Σ'. In particular, such a conformal spin bordism Σ' from Y to $\emptyset$ determines an n-rigging for Y, namely $R = F(\Sigma')^{-n}$. This applies in particular to the case Segal originally considered: if Y is parametrized by a disjoint union of circles then the same number of disks can be used as Σ'.

Let Σ be a conformal spin bordism from Y_1 to Y_2 and assume that Y_i is equipped with an n-rigging R_i. An *n-rigging* for Σ is an element λ in the complex line

$$\mathrm{Pf}^n(\Sigma, R_1, R_2) \overset{\text{def}}{=} \mathrm{Hom}_{C(Y_1)^{\otimes n}}(R_1, R_2 \boxtimes_{C(Y_2)^{-n}} F(\Sigma)^{-n})$$

which is well defined since by Definition 2.3.12 $F(\Sigma)$ is a left module over $C(Y_1)^{\mathrm{op}} \otimes C(Y_2)$. If Σ is closed, this is just the $(-n)$-th power of the Pfaffian line $\mathrm{Pf}(\Sigma) = F(\Sigma)$. More generally, if the riggings R_i come from conformal spin bordisms Σ_i' from Y_i to $\emptyset$, then the line $\mathrm{Pf}^n(\Sigma, R_1, R_2)$ can be identified with the $(-n)$-th power of the Pfaffian line of the closed conformal spin surface $\Sigma_2' \cup_{Y_2} \Sigma \cup_{Y_1} \overline{\Sigma}_1'$.

We want to point out that our definition of a 1-rigging for a closed spin 1-manifold Y is basically the 'spin version' of Segal's definition of a rigging (as defined in section 4 of [Se2], after Definition 4.4). As Segal mentions in a footnote in §6 of [Se2] a rigging (in his sense) is the datum needed on the boundary of a conformal surface Σ in order to define the determinant line $\mathrm{Det}(\Sigma)$ (which is the dual of the top exterior power of the space of holomophic 1-forms on Σ for a closed Σ). Similarly, an n-rigging on the boundary of a conformal spin surface Σ makes it possible to construct the n-th power of the Pfaffian line of Σ.

Our notion of rigging for a conformal surface, however, is different from Segal's (see Definition 5.10 in [Se2]); his is designed to give the datum needed to define non-integral powers of the determinant line $\mathrm{Det}(\Sigma)$ (corresponding to a non-integral central charge). This is then used to resolve the phase indeterminancy and get a non-projective functor. In our setting, only integral powers of the Pfaffian line arise, so that our definition has the same effect.

A Segal elliptic object $\widehat{E}\colon \mathcal{C}_n(X) \to \mathcal{V}$ is required to be linear on morphisms in the sense that the operator $\widehat{E}(\Sigma, \Gamma, \lambda)$ associated to a bordism Σ equipped with an n-rigging λ depends complex linearly on λ.

If Σ is a torus, $\mathrm{Pf}^{-2}(\Sigma)$ is canonically isomorphic to the *determinant line* $\mathrm{Det}(\Sigma)$. In particular, an n-rigging on a closed conformal spin torus amounts to the choice of an element $\lambda \in \mathrm{Det}^{n/2}(\Sigma)$. Evaluating a Segal elliptic object $\widehat{E}$ over $X = \mathrm{pt}$ of central charge n on the family of tori $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ parametrized by points $\tau \in \mathfrak{h}$ of the upper half plane (equipped with the non-bounding spin structure), we obtain a section of the complex line bundle $\mathrm{Pf}^n \to \mathfrak{h}$; it is given by $\tau \mapsto (\lambda \mapsto \widehat{E}(\Sigma_\tau, \lambda)) \in \mathrm{Hom}(\mathrm{Pf}^{-n}(\Sigma_\tau), \mathbb{C})$ (compare Definition 3.3.5). By construction, this section is $SL_2(\mathbb{Z})$-equivariant; it is holomorphic by Segal's requirement that the operator $\widehat{E}(\Sigma, \lambda)$ associated to a conformal spin bordism Σ equipped with a rigging λ depends *holomorphically* on the conformal structure (the Teichmüller space of conformal structures on Σ is a complex manifold); such CFT's are referred to as *chiral*. Moreover, the section is holomorphic at infinity thanks to the 'contraction condition' on $\widehat{E}$ (see [Se1, §6]). In other words, this construction associates a *modular form of weight* $n/2$ to a Segal elliptic object over $X = \mathrm{pt}$.

Chiral CFT's are 'rigid' in a certain sense so that they are not general enough to obtain an interesting space of such objects (this is *not* to say that elliptic cohomology couldn't be described in terms of chiral CFT's; in fact it might well be possible to obtain the elliptic cohomology spectrum from a suitable symmetric monoidal category of chiral CFT's the same way that the K-theory spectrum is obtained from the symmetric monoidal category of finite dimensional vector spaces). We propose to study 'super symmetric' CFT's which are non-chiral, but whose partition functions are holomorphic as a consequence of the built-in super symmetry, see the proof of Theorem 1.0.2 in section 3.3.

Definition 4.1.3. A *Clifford elliptic object* over X is a Clifford linear 2-dimensional CFT in the sense of Definition 2.3.21, together with a super symmetric refinement. The latter is given just like in the K-theoretic context described in Section 3.2 by replacing conformal surfaces by their (complex) $(1|1)$-dimensional partners. This is explained in more detail in Section 3.3.

Roughly speaking, the relationship between a Segal elliptic object of central charge n and a Clifford elliptic object of degree n is analogous to the relationship between the *complex* spinor bundle $S_{\mathbb{C}}(\xi)$ and the *Clifford linear* spinor bundle $S(\xi)$ associated to a spin vector bundle ξ of dimension $n = 2k$ over a manifold X (see [LM, II.5]). Given a point $x \in X$ the fiber $S_{\mathbb{C}}(\xi)_x$ is a vector space, while $S(\xi)_x$ is a graded right module over C_n, or equivalently, a graded left module over $C_{-n} = C(\mathrm{pt})^{-n}$; we can recover $S_{\mathbb{C}}(\xi)$ from $S(\xi)$ as $S_{\mathbb{C}}(\xi) = \Delta^{\otimes k} \otimes_{C_{-n}} S(\xi)$.

Similarly, given a Clifford elliptic object E over X of degree n, we can produce an associated functor

$$\widehat{E}\colon \mathcal{C}_n(X) \longrightarrow \mathcal{V}$$

as follows:

- given an object of $\mathcal{C}_n(X)$, i.e. a closed spin manifold Y with a rigging R and a map $\Gamma\colon Y \to X$, we define $\widehat{E}(Y, R, \Gamma)$ to be the Hilbert space $R \boxtimes_{C(Y)^{-n}} E(\gamma)$;
- Given a morphism $(\Sigma, \lambda, \Gamma)$ in $\mathcal{C}_n(X)$ from (Y_1, γ_1, R_1) to (Y_2, γ_2, R_2), i.e. a conformal spin bordism Σ from Y_1 to Y_2 equipped with an n-rigging $\lambda \in \mathrm{Pf}^n(\Sigma, R_1, R_2)$, we define the operator $E(\Sigma, \Gamma, \lambda)$ to be the composition

$$\widehat{E}(\gamma_1) = R_1 \boxtimes_{C(Y_1)^{-n}} E(\gamma_1) \xrightarrow{\lambda \boxtimes 1} R_2 \boxtimes_{C(Y_2)^{-n}} F(\Sigma)^{-n} \boxtimes_{C(Y_1)^{-n}} E(\gamma_1) \xrightarrow{1 \boxtimes E(\Sigma)} R_2 \boxtimes_{C(Y_2)^{-n}} E(\gamma_2) = \widehat{E}(\gamma_2); \quad (4.1.4)$$

  see Remark 2.3.19 for the meaning of $E(\Sigma)$.

It should be emphazised that the resulting functor $\widehat{E}$ is *not* a Segal elliptic object: in general the operators $\widehat{E}(\Sigma, \lambda)$ won't depend holomorphically on the complex structure on Σ, since there is no such requirement for $E(\Sigma)$ in the definition of Clifford elliptic objects. However, as mentioned above, the built in super symmetry for E will imply that the partition function (for $X = \mathrm{pt}$) of $\widehat{E}$ is holomorphic and so we obtain a (weak) modular form of weight $n/2$.

4.2 The bicategory $\mathcal{D}_n(X)$ of conformal 0-,1-, and 2-manifolds

A *bicategory* $\mathcal{D}$ consists of objects (represented by points), 1-morphisms (*horizontal* arrows) and 2-morphisms (*vertical* double arrows). There are composition maps of 1-morphisms which are associative only up to a natural transformation between functors, and an identity 1-morphism exists (but it is only an identity up to natural transformations). There are also compositions of 2-morphisms (which are strictly associative) and strict identity objects. In particular, given objects a, b there is a category $\mathcal{D}(a, b)$ whose objects are the 1-morphisms from a to b and whose morphisms are the 2-morphisms between two such 1-morphisms; the composition in $\mathcal{D}(a, b)$ is given by vertical composition of 2-morphisms in $\mathcal{D}$. Given another object c, horizontal composition gives a functor

$$\mathcal{D}(b, c) \times \mathcal{D}(a, b) \longrightarrow \mathcal{D}(a, c),$$

which is associative only up to a natural transformation. We refer to [Be] for more details.

We will first describe the geometric bicategory $\mathcal{D}_n(X)$. The objects, morphisms and 2-morphisms will be manifolds of dimension 0, 1 and 2, respectively, equipped with conformal and spin structures, and maps to X as well as the fermions from Definition 2.3.16. Note that the conformal structure is only relevant for surfaces.

Following is a list of data necessary to define a bicategory. We only spell out the case $X = \mathrm{pt}$, in the general case one just has to add piecewise smooth maps to X, for all the 0-,1-, and 2-manifolds below. So it will be easy for the reader to fill in those definitions.

objects: The objects of $\mathcal{D}_n = \mathcal{D}_n(\mathrm{pt})$ are 0-dimensional spin manifolds Z, i.e. a finite number of points with a graded real line attached to each of them.

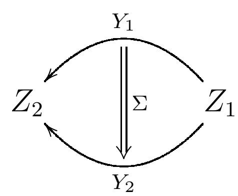
morphisms: A morphism in $\mathcal{D}_n(Z_1, Z_2)$ is either a spin diffeomorphism $Z_1 \to Z_2$, or a 1-dimensional spin manifold Y, together with a spin diffeomorphism $\partial Y \to \bar{Z}_1 \amalg Z_2$.

Composition of morphisms: For two diffeomorphisms, one uses the usual composition, and for 2 bordisms, composition is given by gluing 1-manifolds, and pictorially by

$$Z_3 \xleftarrow{Y_2} Z_2 \xleftarrow{Y_1} Z_1 \qquad = \qquad Z_3 \xleftarrow{Y_2 \cup_{Z_2} Y_1} Z_1\,,$$

The composition of a diffeomorphism and a bordism is given as for the category $\mathcal{B}_d$, namely by using the diffeomorphism to change the parametrization of one of the boundary pieces of the bordism.

2-morphisms: Given two bordism type morphisms Y_1, Y_2 from the object Z_1 to the object Z_2, then a 2-morphism in $\mathcal{D}_n(Y_1, Y_2)$ consists either of a spin diffeomorphism $Y_1 \to Y_2$ (rel. boundary), or it is given by a conformal spin surface Σ together with a diffeomorphism $\partial\Sigma \cong Y_1 \cup_{Z_1 \cup Z_2} Y_2$; this is schematically represented by the following picture:



As in the category $\mathcal{CB}_2^n$, we need in addition the following datum for a 2-morphism from Y_1 to Y_2: In the case of a diffeomorphism, we have an element $c \in C(Y_1)^{\otimes n}$; in the case of a bordism, we need a fermion Ψ in the n-th power of the algebraic Fock space $F_{alg}(\Sigma)$ from Definition 2.3.12. Moreover, we define two such pairs (Σ, Ψ) and (Σ', Ψ') to give *the same* 2-morphism from Y_1 to Y_2, if there is a conformal spin diffeomorphism $\alpha : \Sigma \to \Sigma'$ sending Ψ to Ψ'.

Given one spin diffeomorphism $\phi : Z_1 \to Z_2$ and one bordism Y, then one can form a closed spin 1-manifold Y_ϕ by gluing the ends of Y together along ϕ. Then a 2-morphism from ϕ to Y is a conformal spin surface Σ together with a fermion in $F_{alg}(\Sigma)$ and a diffeomorphism $\partial\Sigma \cong Y_\phi$. Again, two such 2-morphisms are considered equal if they are related by a conformal spin diffeomorphism.

Composition of 2-morphisms: Depending on the case at hand, (horizontal and vertical) composition in the 2-category $\mathcal{D}_n$ is either given by gluing surfaces or composing diffeomorphisms. This is very similar to the category $\mathcal{CB}_2^n$, so details are omitted.

As for K-cocycles, it will be important that our enriched elliptic objects preserve a symmetric monoidal structure, certain involutions on the bicategories, as well as certain adjunction transformations. The monoidal structure on $\mathcal{D}_n$ is simply given by disjoint union, which has a unit given by the empty set.

The involutions in $\mathcal{D}_n$. In the case of $\mathcal{B}_d$ we mentioned two involutions, called $\bar{\cdot}$ and $*$. The first reversed the spin structure on $(d-1)$-manifolds, the second on d-manifolds. So in the case of $\mathcal{D}_n$ it is natural to have 3 involutions in the game, each of which reverses the spin structure of manifolds exactly in dimension $d = 0, 1$ respectively 2. We call these involutions (in that order), *op*, $\bar{\cdot}$ and $*$ even though it might seem funny to distinguish these names. Note however, that there'll be analogous involutions on the von Neumann bicategory vN and we wish to be able to say which involutions are taken to which by our enriched elliptic object.

The adjunction transformation in $\mathcal{D}_n$. Given objects Z_1, Z_2 of $\mathcal{D}_n$, there is a functor

$$\mathcal{D}_n(\emptyset, Z_1 \amalg Z_2) \longrightarrow \mathcal{D}_n(Z_1^{\mathrm{op}}, Z_2). \tag{4.2.1}$$

On objects, it reinterprets a bordism Y from $\emptyset$ to $Z_1 \amalg Z_2$ as a bordism from Z_1^{op} to Z_2. Similarly, if Y_1, Y_2 are two such bordisms, and Σ is a morphism from Y_1 to Y_2 in the category $\mathcal{D}_n(\emptyset, Z_1 \amalg Z_2)$, then it can be reinterpreted as a morphism between Y_1 and Y_2 considered as morphisms in $\mathcal{D}_n(Z_1^{\mathrm{op}}, Z_2)$. This is natural in Z_1, Z_2; expressed in technical terms, it is a natural transformation between the two functors from $\mathcal{D}_n \times \mathcal{D}_n$ to the category of topological categories given by the domain resp. range of the functor 4.2.1. It is clear that the functor 4.2.1 is not surjective on objects or morphisms, since no diffeomorphisms can lie in the image.

4.3 Von Neumann algebras and their bimodules

References for this section are [vN], [Co1], [BR] and [Ta] for the general theory of von Neumann algebras. For the fusion aspects we recommend in addition [J2], [J4] and [Wa]. We thank Antony Wassermann for his help in writing this survey.

General facts on von Neumann algebras. A *von Neumann algebra* A is a unital $*$-subalgebra of the bounded operators $B(H)$, closed in the weak (or equivalently strong) operator topology. We assume here that H is a complex separable Hilbert space. For example, if S is any $*$-closed subset of $B(H)$, then the *commutant* (or symmetry algebra)

$$S' \stackrel{\mathrm{def}}{=} \{a \in B(H) \mid as = sa \quad \forall s \in S\}$$

is a von Neumann algebra. By von Neumann's double commutant theorem, any von Neumann algebra arises in this way. In fact, the double commutant S'' is exactly the von Neumann algebra *generated by* S. For example, given two von Neumann algebras $A_i \subseteq B(H_i)$ one defines the *spatial tensor product* $A_1 \bar{\otimes} A_2 \subseteq B(H_1 \otimes H_2)$ to be the von Neumann algebra generated by A_1 and A_2.

Just like a commutative C^*-algebra is nothing but the continuous functions on a topological space, one can show that a commutative von Neumann algebra is isomorphic to the algebra of bounded measurable functions on a measure space. The corresponding Hilbert space consists of the L^2-functions which are acted upon by multiplication.

On the opposite side of the story, one needs to understand *factors* which are von Neumann algebras with center $\mathbb{C}$. By a direct integral construction (which reduces to a direct sum if the measure space corresponding to the center is discrete), one can then combine the commutative theory with the theory of factors to understand all von Neumann algebras.

The factors come in 3 types, depending on the range of the Murray-von Neumann dimension function $d(p)$ on projections $p \in A$. This function actually characterizes equivalence classes of projections p, or equivalently, isomorphism classes of A'-modules pH. Type I factors are those von Neumann algebras isomorphic to $B(H)$ where the range of the dimension is just $\{0, 1, 2, \ldots, \dim_{\mathbb{C}}(H)\}$ (where $\dim_{\mathbb{C}}(H) = \infty$ is not excluded). For type II_1 factors, $d(p)$ can take any real value in $[0, 1]$ and for type II_∞ any value in $[0, \infty]$ ('continuous dimension'). Finally, there are type III factors for which the dimension function can only take the values 0 and ∞. Thus all nontrivial projections are equivalent. It is an empirical fact that most von Neumann algebras arising in quantum field theory are of this type.

Example 4.3.1. (Group von Neumann algebras). For a discrete countable group Γ one defines the *group von Neumann algebra* as the weak operator closure of the group ring $\mathbb{C}\Gamma$ in the bounded operators on $\ell^2(\Gamma)$. It is always of type II_1 and a factor if and only if each conjugacy class (of a nontrivial group element) is infinite. There are many deep connections between such factors and topology described for example in [Lu]. An application to knot concordance is given in [COT].

Example 4.3.2. (Local Fermions). Consider the Fock space $H = F(\Sigma)$ of a conformal spin surface Σ as in Definition 2.3.12. If Y is a compact submanifold of the boundary of Σ we can consider the Clifford algebra $C(Y)$ inside $B(H)$. The weak operator closure is a factor $A(Y)$ which is of type I if Y itself has no boundary. Otherwise $A(Y)$ is a type III factor known as the *local fermions* [Wa].

We remark that by taking an increasing union of finite dimensional subspaces of the Hilbert space of spinors $V(Y)$, it follows that $A(Y)$ is *hyperfinite*, i.e. it is (the weak operator closure of) an increasing union of finite dimensional von Neumann algebras. It is a much deeper fact that a group von Neumann algebra as in Example 4.3.1 is hyperfinite if and only if the group is amenable.

There is a classification of all hyperfinite factors due to Connes [Co1, p.45] (and Haagerup [H] in the III_1 case). The complete list is very short:

I_n: $A = B(H)$ where $n = \dim_{\mathbb{C}}(H)$ is finite or countably infinite.

II_1: Group von Neumann algebras (of amenable groups with infinite conjugacy classes). All of these turn out to be isomorphic!

II_∞: The tensor product of types I_∞ and II_1.

III_0: The Krieger factor associated to a non-transitive ergodic flow.

III_λ: The Powers factors, where $\lambda \in (0,1)$ is a real parameter coming from the 'flow of weights'.

III_1: The local fermions explained in Example 4.3.2. Again, these are all isomorphic.

This classification is obtained via the modular theory to which we turn in the next section. For example, a factor is of type III_1 if and only if there is a vacuum vector Ω for which the modular flow Δ^{it} on the vacuum representation only fixes multiples of Ω.

Tomita-Takesaki theory. We start with a factor $A \subseteq B(H_0)$ and assume that there is a cyclic and separating vector $\Omega \in H_0$. (Recall that this just means that $A\Omega$ and $A'\Omega$ are both dense in H_0). Then H_0 is called a *vacuum representation* (or standard form) for A, and the vector Ω is the *vacuum vector*. It has the following extra structure: Consider the (unbounded) operator $a\Omega \mapsto a^*\Omega$ and let S be its closure. Then S has a polar decomposition $S = J\Delta^{1/2}$, where J is a conjugate linear isometry with $J^2 = \mathrm{id}$ and Δ is a positive operator (usually unbounded). By functional calculus one gets a unitary flow Δ^{it}, referred to as the *modular flow* corresponding to Ω, and the main fact about this theory is that

$$JAJ = A' \quad \text{and} \quad \Delta^{it} A \Delta^{-it} = A.$$

Note that in particular H_0 becomes a bimodule over A by defining a right action of A on H_0 by $\pi^0(a) \stackrel{\text{def}}{=} J\pi(a)^*J$ in terms of the original left action $\pi(a)$. This structure encodes the 'flow of weights' which classifies all hyperfinite factors as explained in the previous section.

It turns out that up to unitary isomorphism, there is a unique pair (H_0, J) consisting of a (left) A-module H_0 and a conjugate linear isometry $J\colon H_0 \to H_0$ with $JAJ = A'$; such a pair is refered to as *vacuum representation of* A. For a given von Neumann algebra A there is a more sophisticated construction of such a pair, even in the absence of a cyclic and separating vector. In this invariant definition (see [Co1, p.527]), the vacuum representation is denoted by $L^2(A)$, in analogy to the commutative case where $A = L^\infty(X)$ and $L^2(A) = L^2(X)$ for some measure space X. Similarly, if A is a group von Neumann algebra corresponding to Γ then $L^2(A) = \ell^2(\Gamma)$. If one chooses Ω to be the δ-function concentrated at the unit element of Γ, then $J(\sum_i a_i g_i) = \sum_i \bar{a}_i g_i^{-1}$ and $\Delta = \mathrm{id}$.

Remark 4.3.3. A vacuum vector Ω defines a faithful normal state on A via

$$\varphi_\Omega(a) \stackrel{\text{def}}{=} \langle a\Omega, \Omega\rangle_{H_0}$$

Defining $\sigma(a) \stackrel{\text{def}}{=} \Delta^{1/2} a \Delta^{-1/2} \in A$ for *entire* elements $a \in A$ (this is a dense subset of A for which σ is defined, see [BR, I 2.5.3]) one can then verify the relation [BR, I p.96]

$$\varphi_\Omega(ba) = \varphi_\Omega(\sigma^{-1}(a)\sigma(b))$$

for all entire elements a, b in A. It follows that φ_Ω is a trace if and only if $\Delta = \mathrm{id}$. Such vacuum vectors can be found for types I and II.

Remark 4.3.4. The independence from Ω implies that the image of the modular flow (given by conjugation with Δ^{it} on A) defines a canonical central subgroup of $\mathrm{Out}(A) \stackrel{\text{def}}{=} \mathrm{Aut}(A)/\,\mathrm{Inn}(A)$. As discussed in the previous remark, this quotient flow is nontrivial exactly for type III. Alain Connes sometimes refers to it as an 'intrinsic time', defined only in the most noncommutative setting of the theory.

Example 4.3.5. Consider the example of local fermions in the special case that $\Sigma = D^2$ and $\partial\Sigma = Y \cup Y_c$ is the decomposition into the upper and lower semicircle. It was shown in [Wa] that in this case the operators J and Δ^{it} can be described geometrically: J acts on the Fock space $F(\Sigma)$ by reflection in the real axis, which clearly is of order two and interchanges $A(Y)$ and $A(Y_c) = A(Y)'$. Moreover, the modular flow Δ^{it} on $A(Y)$ is induced by the Möbius flow on D^2 (which fixes $\pm 1 = \partial Y$). This implies in particular that the Fock space is a vacuum representation (with Ω_Σ as the vacuum vector):

$$F(\Sigma) \cong L^2(A(Y))$$

The last statement is actually true for any surface Σ and any $Y \subset \partial\Sigma$ which is not the full boundary. In the latter case, $A(\partial\Sigma) = B(F(\Sigma))$, so $F(\Sigma)$ is not the vacuum representation of $A(\partial\Sigma))$. In fact, the vacuum representation $L^2(B(H))$ of $B(H)$ is given by the ideal of all Hilbert-Schmidt operators on H (with the operator J given by taking adjoints, and $\Delta = \mathrm{id}$). This is a good example of a construction of the vacuum representation without any canonical vacuum vector in sight.

Bimodules and Connes fusion. Given two von Neumann algebras A_i, an $A_2 - A_1$-*bimodule* is a Hilbert space F together with two normal (i.e. weak operator continuous) $*$-homomorphisms $A_2 \to \mathcal{B}(F)$ and $A_1^{\mathrm{op}} \to \mathcal{B}(F)$ with commuting images. Here A^{op} denotes the opposite von Neumann algebra which is the same underlying vector space (and same $*$ operator) as A but with the order of the multiplication reversed. One can imagine A_2 acting on the left on F and A_1 acting on the right.

Given an $A_3 - A_2$ bimodule F_2 and an $A_2 - A_1$ bimodule F_1, on can construct an $A_3 - A_1$ bimodule $F_2 \boxtimes_{A_2} F_1$ known as the *Connes fusion* of F_2 and F_1 over A_2. This construction is *not* the algebraic tensor product but it introduces a certain twist (by the modular operator Δ) in order to stay in the category of Hilbert spaces.

Definition 4.3.6. (Connes fusion). It is the completion of the pre-Hilbert space given by the algebraic tensor product $\mathfrak{F}_2 \odot F_1$, where

$$\mathfrak{F}_2 \stackrel{\text{def}}{=} B_{A_2^{\text{op}}}(H_0, F_2)$$

are the bounded intertwiners from the vacuum $H_0 = L^2(A_2)$ to F_2. An inner product is obtained by the formula

$$\langle x \otimes \xi, y \otimes \eta\rangle \stackrel{\text{def}}{=} \langle \xi, (x,y)\cdot \eta\rangle_{F_1} \quad \xi, \eta \in F_1,\ x, y \in \mathfrak{F}_2$$

where we have used the following A_2-valued inner product on $\mathfrak{F}_2$:

$$(x,y) \stackrel{\text{def}}{=} x^* y \in B_{A_2^{op}}(H_0, H_0) = A_2.$$

Note that this makes $\mathfrak{F}_2$ into a right Hilbert module over A_2 and that the Connes fusion is nothing but the Hilbert module tensor product with F_1 and its A_2-action. Since A_1^{op} and A_3 still act in the obvious way, it follows that the Hilbert space $F = F_2 \boxtimes_{A_2} F_1$ is an $A_3 - A_1$-bimodule.

This definition looks tantalizingly simple, for example one can easily check that the relations

$$xa \otimes \xi - x \otimes a\xi = 0, \quad a \in A_2, \xi \in F_1, x \in \mathfrak{F}_2$$

are satisfied in $F_2 \boxtimes_{A_2} F_1$ (because this vector is perpendicular to all elements of $\mathfrak{F}_2 \odot F_1$ with respect to the above inner product). This assertion is true using the obvious A_2^{op}-action on $\mathfrak{F}_2$ for which $xa(v) = x(av),\ v \in H_0$. However, if one wants to write elements in the Connes fusion in terms of *vectors* in the original bimodules F_1 and F_2 (rather than using the intertwiner space $\mathfrak{F}_2$), then it turns out that this action has to be twisted by Δ. This can be seen more precisely as follows: First of all, one has to pick a vacuum vector $\Omega \in H_0$ (or at least a normal, faithful semifinite weight) and the construction below will depend on this choice. There is an obvious embedding

$$i_\Omega : \mathfrak{F}_2 \hookrightarrow F_2, \quad x \mapsto x(\Omega)$$

and the crucial point is that this map is *not* A_2^{op}-linear. To do this calculation carefully, write $\pi(a)$ for the left A_2 action on H_0 and $\pi^0(a)$ for the right action. Recall from the previous section that $\pi^0(a) = J\pi(a)^* J$ which implies the following formulas, using $J\Omega = \Omega = \Delta\Omega$ and that $S = J\Delta^{1/2}$ has the defining property $S(\pi(a)\Omega) = \pi(a)^*\Omega$.

$$\begin{aligned} x(\Omega)\cdot a &= x(\pi^0(a)\Omega) = x(J\pi(a)^* J\Omega) \\ &= x(J\Delta^{1/2}\Delta^{-1/2}\pi(a)^*\Delta^{1/2}\Omega) \\ &= x(S(\Delta^{1/2}\pi(a)\Delta^{-1/2})^*\Omega) \\ &= x(\Delta^{1/2}\pi(a)\Delta^{-1/2})\Omega) \end{aligned}$$

Recall from Remark 4.3.3 that $\sigma(a) = \Delta^{1/2} a \Delta^{-1/2} \in A_2$ is defined for entire elements $a \in A_2$. Then we see that i_Ω has the intertwining property

$$i_\Omega(x\sigma(a)) = i_\Omega(x)a \quad \text{for all entire } a \in A_2. \tag{4.3.7}$$

This explains the connection between the Connes fusion defined above and the one given in [Co1, p.533] as follows. Consider the A_2^{op}-invariant subset $\mathfrak{F}_2\Omega = \mathrm{im}(i_\Omega)$ of the bimodule F_2. These are exactly the 'ν-bounded vectors' in [Co1, Prop.6, p.531] where in our case the weight ν is simply given by $\nu(a) = \langle a\Omega, \Omega\rangle_{H_0}$. One can then start with algebraic tensors

$$\xi_2 \otimes \xi_1 \text{ with } \xi_1 \in F_1, \xi_2 \in \mathfrak{F}_2\Omega \subset F_2$$

instead of the space $F_1 \odot \mathfrak{F}_2$ used above. This is perfectly equivalent except that the σ-twisting of the map i_Ω translates the usual algebraic tensor product relations into the following 'Connes' relations which hold for all entire $a \in A_2$:

$$\xi_2 a \otimes \xi_1 = \xi_2 \otimes \sigma(a)\xi_1 = \xi_2 \otimes \Delta^{1/2} a \Delta^{-1/2} \xi_1, \quad \xi_1 \in F_1, \xi_2 \in \mathfrak{F}_2\Omega \subset F_2$$

Remark 4.3.8. (Symmetric form of Connes fusion) There is the following more symmetric way of defining the Connes fusion which was introduced in [Wa] in order to actually *calculate* the fusion ring of positive energy representations of the loop group of $SU(n)$. One starts with the algebraic tensor product $\mathfrak{F}_2 \odot \mathfrak{F}_1$ and defines the inner product by

$$\langle x_2 \otimes x_1, y_2 \otimes y_1\rangle \overset{\text{def}}{=} \langle x_1^* y_1 \Omega, x_2^* y_2 \Omega\rangle_{H_0} = \langle y_2^* x_2 x_1^* y_1 \Omega, \Omega\rangle_{H_0} \qquad \text{for } x_i, y_i \in \mathfrak{F}_i$$

One can translate this '4-point formula' to the definition given above by substituting $\xi = x_1(\Omega), \eta = y_1(\Omega)$. It uses again that $x_i^* y_i \in A_2$ and also the choice of a vacuum vector. After translating this definition into the subspaces $\mathfrak{F}_i\Omega$ of F_i, one can also write the Connes relations (for entire a in A_2) in the following symmetric form:

$$\xi_2 \Delta^{-1/4} a \Delta^{1/4} \otimes \xi_1 = \xi_2 \otimes \Delta^{1/4} a \Delta^{-1/4} \xi_1, \quad \xi_i \in \mathfrak{F}_i\Omega \subset F_i$$

Remark 4.3.9. (Subfactors). We should mention that the fusion of bimodules has had a tremendous impact on low dimensional topology through the work of Jones, Witten and many others, see [J3] for a survey. In the context of the Jones polynomial for knots, only the hyperfinite II_1 factor was needed, so the subtlety in the Connes fusion disappears (because $\Delta = \mathrm{id}$ if one uses the trace to define the vacuum). However, the interesting data came from *subfactors* $A \subset B$, i.e. inclusions of one factor into another. They give rise to the $A - B$ bimodule $L^2(B)$. Iterated fusion leads to very interesting bicategories and tensor categories, compare Remark 4.3.13.

The main reason Connes fusion arises in our context, is that we want to glue two conformal spin surfaces along *parts* of their boundary. As explained in Section 2.2, the surfaces lead naturally to Fock modules over Clifford algebras. Using the notation from the Gluing Lemma 2.3.14, the question arises how to express $F(\Sigma_3)$ as a $C(Y_3) - C(Y_1)$ bimodule in terms of $F(\Sigma_2)$ and $F(\Sigma_1)$. In Lemma 2.3.14 we explained the case of type I factors, where the modular operator $\Delta = \mathrm{id}$ so there is no difference between the algebraic tensor product and Connes fusion. This case includes the finite dimensional setting (and hence K-theory) as well as the gluing formulas for Segal and Clifford elliptic objects (as explained in Example 4.3.2, type I corresponds exactly to the case where the manifold Y along which one glues is a *closed* 1-manifold).

After all the preparation, the following answer might not come as a surprise. We only formulate it in the absence of closed components in Σ_i, otherwise the vacuum vectors might be zero. There is a simply modification in the general case which uses isomorphisms of Pfaffian lines of disjoint unions of closed surfaces. Similarly, there is a twisted version of this result which we leave to the reader. We note that in the following the definition of fusion has to be adjusted to take the grading on the Fock modules into account. This can be done by the usual trick of Klein transformations.

Proposition 4.3.10. *There is a unique unitary isometry of* $C(Y_3) - C(Y_1)$ *bimodules*

$$F(\Sigma_2) \boxtimes_{A(Y_2)} F(\Sigma_1) \xrightarrow{\cong} F(\Sigma_3)$$

sending $\Omega_2 \otimes \Omega_1$ *to* Ω_3.

Recall that $A(Y_2)$ is the von Neumann algebra generated by $C(Y_2)$ in the bounded operators on $F(\Sigma_2)$. One knows that $F(\Sigma_2)$ is a vacuum representation for $A(Y_2)$ with vacuum vector Ω_2 [Wa]. Therefore, the above expression $\Omega_2 \otimes \Omega_1$ is well defined in the Connes fusion. The uniqueness of the isomorphism follows from the fact that both sides are irreducible $C(Y_3) - C(Y_1)$ bimodules.

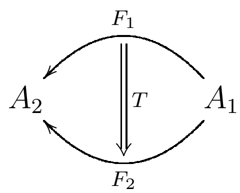
The bicategory vN **of von Neumann algebras.** The objects of vN are von Neumann algebras and a morphism from an object A_1 to an object A_2 is a an $A_2 - A_1$ bimodule.

Composition of morphisms: Is given by Connes fusion which will be denoted pictorially by

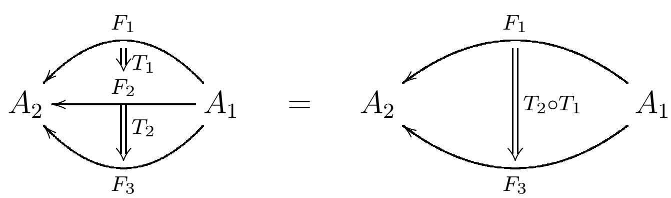
$$A_3 \xleftarrow{F_2} A_2 \xleftarrow{F_1} A_1 \qquad = \qquad A_3 \xleftarrow{F_2 \boxtimes_{A_2} F_1} A_1,$$

Recall that this operation is associative up to higher coherence (which is fine in a bicategory). Moreover, the identity morphism from A to A is the vacuum representation $H_0 = L^2(A)$ which therefore plays the role of the 'trivial' bimodule. This is in analogyto the trivial 1-dimensional representation of a group.

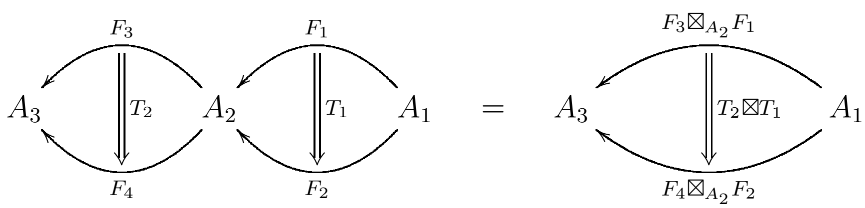
2-morphisms: Given two morphisms F_1, F_2 from the object A_1 to the object A_2, then a 2-morphism from F_1 to F_2 is a bounded intertwining operator $T \in \mathcal{B}_{A_2-A_1}(F_1, F_2)$, i.e. a bounded operator which commutes with the actions of A_1 and A_2. Pictorially,



vertical composition: Let F_i, $i = 1, 2, 3$ be three morphisms from A_1 to A_2, let T_1 be a 2-morphism from F_1 to F_2 and let T_2 be a 2-morphism from F_2 to F_3. Then their vertical composition is the 2-morphism $T_2 \circ T_1$, which is just the composition of the bounded operators T_1 and T_2. Pictorially,



horizontal composition: The following picture should be self explaining.



Additional structures on vN. The bicategory vN has a symmetric monoidal structure given on objects by the spatial tensor product of von Neumann algebras. There are also monodial structures on the categories of bimodules by considering the Hilbert tensor product of the underlying Hilbert spaces.

Involutions on vN. There are also 3 involutions

$$A \mapsto A^{\mathrm{op}}, \qquad F \mapsto \bar{F}, \qquad T \mapsto T^*$$

on the bicategory vN, where the first was explained above and the third is the usual adjoint map. The *conjugate* $A_1 - A_2$ bimodule $\bar{F}$ (for a $A_2 - A_1$ bimodule F) is given by the formula

$$a_1 \cdot \bar{v} \cdot a_2 \stackrel{\mathrm{def}}{=} \overline{(a_2^* \cdot v \cdot a_1^*)}, \quad v \in F.$$

We leave it to the reader to extend the above definitions so that they really define involutions on the bicategory vN. This should be done so that the functoriality agrees with the 3 involutions in the bicategory $\mathcal{D}_n(X)$ because our enriched elliptic object will have to preserve these involutions.

Adjunction transformations on vN. Just like in $\mathcal{D}_n(X)$, we are looking for adjunction transformations of 1- respectively 2-morphisms

$$\mathrm{vN}(\mathbb{C}, A_1 \bar{\otimes} A_2) \longrightarrow \mathrm{vN}(A_1^{\mathrm{op}}, A_2) \quad \text{and} \quad \mathrm{vN}(\mathbb{C}, F_2 \boxtimes_A F_1) \longrightarrow \mathrm{vN}(\bar{F}_2, F_1).$$

where the left hand side is defined by considering the inclusion of the algebraic tensor product in $A_1 \bar{\otimes} A_2$. The resulting bimodule is still referred to as a $A_1 \bar{\otimes} A_2$-module because the bimodule structure is in some sense boring.

To address the right hand side, let $A = A_1 \bar{\otimes} A_2$ and consider an A-module F_1 and an A^{op}-module F_2 (both thought of as lying in the image of the left hand side transformation). Then we may form the Connes fusion $F_2 \boxtimes_A F_1$ as the completion of $\mathfrak{F}_2 \odot F_1$, see Definition 4.3.6. There is a natural map

$$\Theta : \mathfrak{F}_2 \odot F_1 \longrightarrow B_A(\bar{\mathfrak{F}}_2, F_1), \quad x \otimes \eta \mapsto \theta_{x,\eta} \ \text{ where } \theta_{x,\eta}(\bar{y}) \stackrel{\mathrm{def}}{=} (y, x)\eta.$$

Here we have used again the A-valued inner product $(y, x) = y^* x$ on $\mathfrak{F}_2$, as wellas the linear isometry

$$\mathfrak{F}_2 = B_{A^{\mathrm{op}}}(H_0, F_2) \longrightarrow \bar{\mathfrak{F}}_2 \stackrel{\mathrm{def}}{=} B_A(H_0, \bar{F}_2) \quad x \mapsto \bar{x} \stackrel{\mathrm{def}}{=} xJ.$$

Recall that $\bar{F}_2$ is an A-module and so is $\bar{\mathfrak{F}}_2$.

Lemma 4.3.11. *In the above setting, the mapping $\theta_{x,\eta}$ is indeed A-linear.*

Proof. We use the careful notation used to derive equation 4.3.7, where $\pi(a)$ denotes the A-action on H_0 and $\pi^0(a) = J\pi(a)^* J$ the A^{op}-action. Then we get that for $a \in A$ and $y \in \mathfrak{F}_2$

$$a\bar{y} = \bar{y}\pi^0(a) = yJ(J\pi(a)^* J) = (y\pi(a)^*)J = \overline{ya^*}$$

This implies

$$\begin{aligned} \theta_{x,\eta}(a\bar{y}) &= (ya^*, x)\eta = (ya^*)^* x\eta \\ &= ay^* x\eta = a(y, x)\eta \\ &= a\theta_{x,\eta}(\bar{y}) \end{aligned}$$

which is exactly the statement of our lemma. ☐

Note that Θ takes values in the Banach space of A-intertwiners with the operator norm. It actually turns out that it is an isometry with respect to the fusion inner product. To check this statement, we assume for simplicity that A is of type III. Then there is a unitary A-intertwiner $U : H_0 = L^2(A) \to F_2$ and hence $y^*x = (y^*U)(U^*x)$ is a product of two elements in A and $||y^*U|| = ||\bar{y}||$. Now let $f = \sum_i x_i \otimes \eta_i$ be and arbitrary element in $\mathfrak{F}_2 \odot F_1$. Then the norm squared of $\Theta(f)$ is calculated as follows:

$$\begin{aligned} ||\Theta(f)||^2 &= \sup_{0 \neq \bar{y} \in \bar{\mathfrak{F}}_2} \frac{||\sum_i (y, x_i)\eta_i||^2}{||\bar{y}||^2} \\ &= \sup_{0 \neq \bar{y} \in \bar{\mathfrak{F}}_2} \frac{||\sum_i (y^*U)(U^*x_i)\eta_i||^2}{||\bar{y}||^2} \\ &= ||\sum_i (U^*x_i)\eta_i||^2 = \sum_{i,j} \langle \eta_i, x_i^* U U^* x_j \eta_j \rangle_{F_1} \\ &= \sum_{i,j} \langle \eta_i, (x_i, x_j)\eta_j \rangle_{F_1} = \langle \sum_i x_i \otimes \eta_i, \sum_j x_j \otimes \eta_j \rangle_{\mathfrak{F}_2 \odot F_1} \\ &= ||f||^2_{\mathfrak{F}_2 \odot F_1} \end{aligned}$$

This implies the following result because we have a functorial isometry which for $F_2 = L^2(A)$ clearly is an isomorphism. Note that the same result holds for bimodules, if there are two algebras acting on the left of F_2 respectively the right of F_1.

Proposition 4.3.12. *The above map Θ extends to an isometry*

$$\Theta : F_2 \boxtimes_A F_1 \xrightarrow{\cong} B_A(\bar{\mathfrak{F}}_2, F_1)$$

In order to define our adjunction transformation announced above, we now have to compare the right hand side of the isometry to $\mathrm{vN}(\bar{F}_2, F_1)$. If one thinks of the latter as all A-intertwiners then there is a serious problem in relating the two, because of the twisting property 4.3.7 of the inclusion $i_\Omega : \mathfrak{F}_2 \hookrightarrow F_2$. This is where the modular operator Δ has to come in. At this moment in time, we don't quite know how to resolve the issue, but it seems very likely that one has to change the definition of $\mathrm{vN}(\bar{F}_2, F_1)$ slightly. Note that one can't use the right hand side of the above isometry because these intertwiners cannot be composed, certainly not in an obvious way. This problem is related to the fact that in the example of a string vector bundle, we can only associate vectors in the fusion product to conformal spin surfaces.

Remark 4.3.13. It is interesting to point out the following special subcategories of the bicategory above. In the Jones example for a subfactor $A \subset B$, there are two objects (namely A and B) and the morphisms are all bimodules obtained by iterated fusion from ${}_AL^2(B)_B$. The crucial finite index property of Jones guarantees that for all irreducible bimodules ${}_AF_B$ that arise the vacuum representation H_0 is contained exactly once in

$F \boxtimes_B \bar{F}$ and $\bar{F} \boxtimes_A F$. This condition expresses the fact that F has *finite* 'quantum dimension'. In [Oc], these bicategories were further developed and applied to obtain 3-manifold invariants.

If one fixes a single von Neumann algebra A, then one can consider the bicategory 'restricted to A'. This means that one has only $A - A$ bimodules and their intertwiners, together with the fusion operation. This is an example of a *tensor category.* Borrowing some notation from Section 5.4 we get the following interesting subcategory: Fix a compact simply connected Lie group G and a level $\ell \in H^4(BG)$. Then there is a canonical III_1-factor A and an embedding

$$\Phi : G \hookrightarrow \mathrm{Out}(A) = \mathrm{Aut}(A)/\,\mathrm{Inn}(A).$$

We can thus consider those bimodules which are obtained from twisting $L^2(A)$ by an element in $\mathrm{Aut}(A)$ which projects to $\Phi(g)$ for some $g \in G$. We believe that a certain 'quantization' of this tensor category gives the category of positive energy representation of the loop group LG at level ℓ.

4.4 Enriched elliptic objects and the elliptic Euler class

Definition 4.4.1. An *enriched elliptic object of degree* n over X is a continuous functor $\mathcal{D}_n(X) \to \mathrm{vN}$ to the bicategory of von Neumann algebras. It is assumed to preserve the monodial structures (disjoint union gets taken to tensor product), the 3 involutions *op*, $\bar{\cdot}$ and $*$, as well as the adjunction transformations explained above. Finally, it has to be $\mathbb{C}$-linear in an obvious sense on Clifford algebra elements and fermions.

Again, this is only a preliminary definition because some of the categorical notions have not been defined yet, and it does not contain super symmetry. The main example of an enriched elliptic object comes from a string vector bundle, hopefully leading to an Euler class and a Thom class in elliptic cohomology. In fact, we hope that it will ultimately lead to a map of spectra

$$M\,\mathrm{String} \longrightarrow \mathrm{tmf}$$

We explain the construction of the Euler class momentarily class but we shall use several notions which are only developed in the coming sections. Thus the following outline can be thought of as a motivation for the reader to read on.

We next outline the construction of a degree n enriched elliptic object corresponding to an n-dimensional vector bundle $E \to X$ with string connection. This is our proposed 'elliptic Euler class' of E and it is the main example that guided many of our definitions. In Remark 5.0.7 we explain briefly how the analogous K-theory Euler class is defined for a vector bundle with spin connection. As usual, this will be our guiding principle.

Recall from Definition 4.4.1 that an enriched elliptic object of degree n in the manifold X is a certain functor between bicategories

$$\mathcal{E}_E : \mathcal{D}_n(X) \longrightarrow \mathrm{vN}$$

So we have to explain the values of $\mathcal{E}_E$ in dimensions $d = 0, 1, 2$. Let $\mathcal{S}$ be the string connection on E as explained in Definition 5.0.9. This definition is crucial for the understandingof our functor $\mathcal{E}_E$ and we expect the reader to come back to this section once she is familiar with the notion of a string connection.

The functor $\boldsymbol{\mathcal{E}_E}$ in dimension 0. This is the easiest case because we can just set $\mathcal{E}_E(\boldsymbol{x}) = \mathcal{S}(\boldsymbol{x})$ for a map $\boldsymbol{x} : Z \to X$ of a 0-dimensional spin manifold Z. Recall that $\mathcal{S}(\boldsymbol{x})$ is a von Neumann algebra which is completely determined by the string structure on E (no connection is needed). By construction, the monoidal structures on the objects of our bicategories are preserved.

The functor $\boldsymbol{\mathcal{E}_E}$ in dimension 1. For each piecewise smooth map $\gamma : Y \to X$ of a spin 1-manifold Y, the string connection $\mathcal{S}(\gamma)$ is a graded irreducible $C(\gamma) - \mathcal{S}(\partial\gamma)$ bimodule. Here $C(\gamma)$ is the relative Clifford algebra from Definition 2.4.2. To define $\mathcal{E}_E(\gamma)$ we use the same Hilbert space but considered only as a $C(Y)^{-n} - \mathcal{S}(\partial\gamma)$-bimodule. Note that this means that the module is far from being irreducible. If one takes orientations into account, one gets a bimodule over the incoming-outgoing parts of $\partial\gamma$. The gluing law of the string connection $\mathcal{S}$ translates exactly into the fact that our functor $\mathcal{E}_E$ preserves composition of 1-morphisms, i.e.it preserves Connes fusion.

The functor $\boldsymbol{\mathcal{E}_E}$ in dimension 2. Consider a conformal spin surface Σ and a piecewise smooth map $\Gamma : \Sigma \to X$, and let $Y = \partial\Sigma$ and $\gamma = \Gamma|_Y$. Then the string connection on Γ is a unitary isometry of left $C(\gamma)$-modules

$$\mathcal{S}(\Gamma) : F(\Gamma) \cong \mathcal{S}(\gamma)$$

Here we used the relative Fock module $F(\Gamma) = F(\Gamma^* E) \otimes F(\Sigma)^{-n}$ from Definition 2.4.5. Recall from the same definition that the vacuum vector Ω_Γ of Γ lies in $F(\Gamma^* E)$. Given a fermion $\Psi \in F_{alg}(\Sigma)^{-n}$, we may thus define

$$\mathcal{E}_E(\Gamma, \Psi) \stackrel{\text{def}}{=} \mathcal{S}(\Gamma)(\Omega_\Gamma \otimes \Psi) \in \mathcal{S}(\gamma) = \mathcal{E}_E(\gamma).$$

This is exactly the datum we need on 2-morphisms (Γ, Ψ) in $\mathcal{D}(X)_n$. The behavior of the string connection with respect to a conformal spin diffeomorphism $\phi : \Sigma \to \Sigma'$ implies the following important condition on an enriched elliptic object. Assuming that ϕ restricts to the identity on the boundary and noting that conformality implies $\Omega_{\Gamma'} = F(\phi)(\Omega_\Gamma)$, we may conclude that

$$\mathcal{E}_E(\Gamma', F(\phi)(\Psi)) = \mathcal{S}(\Gamma')(F(\phi)(\Omega_\Gamma \otimes \Psi) = \mathcal{S}(\Gamma)(\Omega_\Gamma \otimes \Psi) = \mathcal{E}_E(\Gamma, \Psi).$$

Finally, $\mathcal{E}_E$ preserves horizontal and vertical composition by the gluing laws of the string connection $\mathcal{S}$ as well as those of the vacuum vectors.

5 String structures and connections

Given an n-dimensional vector bundle $E \to X$, we want to introduce a topological notion of a *string structure* and then the geometric notion of a *string connection* on E. As usual we start with the analogy of a spin structure. It is the choice of a principal Spin(n)-bundle $P \to X$ together with an isomorphism of the underlying principal $GL(n)$-bundle with the frame bundle of E. In particular, one gets an inner product and an orientation on E because one can use the sequence of group homomorphisms

$$\mathrm{Spin}(n) \xrightarrow{2} \mathrm{SO}(n) \leq O(n) \leq GL(n).$$

Recall that the last inclusion is a homotopy equivalence and that, for $n > 8$, the first few homotopy groups of the orthogonal groups $O(n)$ are given by the following table:

k	0	1	2	3	4	5	6	7
$\pi_k O(n)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

It is well known that there are topological groups and homomorphisms

$$S(n) \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to O(n)$$

which kill exactly the first few homotopy groups. More precisely, SO(n) is connected, Spin(n) is 2-connected, $S(n)$ is 6-connected, and the above maps induce isomorphisms on all higher homotopy groups. This homotopy theoretical description of k-connected covers actually works for any topological group in place of $O(n)$ but it only determines the groups up to homotopy equivalence. For the 0-th and 1-st homotopy groups, it is also well known how to construct the groups explicitly, giving the smallest possible models: one just takes the connected component of the identity, and then the universal covering. In our case this gives SO(n), an index 2 subgroup of $O(n)$, and Spin(n), the universal double covering of SO(n). In particular, both of these groups are Lie groups. However, a group $S(n)$ cannot have the homotopy type of a Lie group since π_3 vanishes. To our best knowledge, there has yet not been found a canonical construction for $S(n)$ which has reasonable 'size' and a geometric interpretation.

The groups String(n). In Section 5.4 we construct such a concrete model for $S(n)$ as a subgroup of the automorphism group of 'local fermions' on the circle. These are certain very explicit von Neumann algebras, the easiest examples of hyperfinite type III_1 factors. We denote by String(n) our particular models of the groups of homotopy type $S(n)$, and we hope that the choice of this name will become apparent in the coming sections. In fact, Section 5.4 deals with the case of compact Lie groups rather than just Spin(n), and we thank Antony Wassermann for pointing out to us this generalization. It is also his result that the unitary group $U(A)$ of a hyperfinite III_1-factor is contractible (see Theorem 5.3.3). This is essential for the theorem below because it implies that the corresponding projective unitary group $PU(A) \stackrel{\mathrm{def}}{=} U(A)/\mathbb{T}$ is a $K(\mathbb{Z}, 2)$.

Theorem 5.0.2. *Consider a compact, simply connected Lie group G and a level $\ell \in H^4(BG)$. Then one can associate to it a canonical von Neumann algebra $A_{G,\ell}$ which is a hyperfinite factor of type III_1. There is an extension of topological groups*

$$1 \longrightarrow PU(A_{G,\ell}) \xrightarrow{i} G_\ell \longrightarrow G \longrightarrow 1$$

such that the boundary map $\pi_3 G \to \pi_2 PU(A_{G,\ell}) \cong \mathbb{Z}$ is given by $\ell \in H^4(BG) \cong \operatorname{Hom}(\pi_3 G, \mathbb{Z})$. Moreover, there is a monomorphism

$$\Phi : G_\ell \hookrightarrow \operatorname{Aut}(A_{G,\ell})$$

such that the composition $\Phi \circ i$ is given by the inclusion of inner automorphisms into all of $\operatorname{Aut}(A_{G,\ell})$.

Applied to $G = \operatorname{Spin}(n)$ and $\ell = p_1/2 \in H^4(B\operatorname{Spin}(n))$ (or 'level 1') this gives type III_1 factors A_n ($\cong A_1^{\bar{\otimes} n}$) and groups $\operatorname{String}(n)$ as discussed above.

Definition 5.0.3. A G_ℓ-structure on a principal G-bundle is a lift of the structure group through the above extension. In particular, a *string structure* on a vector bundle is a lift of the structure group from $SO(n)$ to $\operatorname{String}(n)$ using the homomorphisms explained above.

Corollary 5.0.4. *A G_ℓ-structure on a principal G-bundle $E \to X$ gives a bundle of von Neumann algebras over X.*

This bundle is simply induced by the monomorphism Φ above, and hence over each $x \in X$ the fiber $A(x)$ comes equipped with a G-equivariant map

$$\alpha_x : \operatorname{Iso}_G(G, E_x) \longrightarrow \operatorname{Out}(A_{G,\ell}, A(x))$$

where $\operatorname{Out}(A, B) \stackrel{\text{def}}{=} \operatorname{Iso}(A, B)/\operatorname{Inn}(A)$ are the outer isomorphisms. G-equivariance is defined using the homomorphism $\widetilde{\Phi} : G \to \operatorname{Out}(A)$. It is not hard to see that the pair $(A(x), \alpha_x)$ contains exactly the same information as the string structure on E_x. We shall use this observation in Definition 5.3.4 and hence introduce the following notation (abstracting the case $V = E_x$ above):

Definition 5.0.5. Given (G, ℓ) and a G-torsor V, define a $G_\ell - V$*-pointed factor* to be a factor A together with a G-equivariant map

$$\alpha : \operatorname{Iso}_G(G, V) \longrightarrow \operatorname{Out}(A_{G,\ell}, A)$$

where G-equivariance is defined using the homomorphism $\widetilde{\Phi} : G \to \operatorname{Out}(A)$. The choice of (A, α) is a G_ℓ*-structure* on V.

For the purposes of our application, it is actually important that all the von Neumann algebras are graded. It is possible to improve the construction for $G = \operatorname{Spin}(n)$ so that the resulting algebra is indeed graded by using local fermions rather than local loops, see Section 5.4. The above algebra A_n is then just the even part of this graded algebra.

Characteristic Classes. The homotopy theoretical description given at the beginning of Section 5 implies the following facts about existence and uniqueness of additional structures on a vector bundle E in terms of characteristic classes. We point out that we are more careful about spin (and string) structure as is customary in topology: A spin structure is really the choice of a principal $\mathrm{Spin}(n)$-bundle, and not only up to isomorphism. In our language, we obtain the usual notion of a spin structure by taking isomorphism classes in the category of spin structures. Similar remarks apply to string structures. The purpose of this refinement can be seen quite clearly in Proposition 5.0.6.

Let E be a vector bundle over X. Then

- E is *orientable* if and only if the Stiefel-Whitney class $w_1 E \in H^1(X;\mathbb{Z}/2)$ vanishes. Orientations of E are in 1-1 correspondence with $H^0(X;\mathbb{Z}/2)$.
- In addition, E has a *spin structure* if and only if the Stiefel-Whitney class $w_2 E \in H^1(X;\mathbb{Z}/2)$ vanishes. Isomorphism classes of spin structures on E are in 1-1 correspondence with $H^1(X;\mathbb{Z}/2)$.
- In addition, E has a *string structure* if and only if the characteristic class $p_1/2(E) \in H^4(X;\mathbb{Z})$ vanishes. Isomorphism classes of string structures on E are in 1-1 correspondence with $H^3(X;\mathbb{Z})$.

More generally, a principal G-bundle E (classified by $c : X \to BG$) has a G_ℓ-structure if and only if the characteristic class $c^*(\ell) \in H^4(X;\mathbb{Z})$ vanishes. Isomorphism classes of G_ℓ-structures on E are in 1-1 correspondence with $H^3(X;\mathbb{Z})$.

In the next two sections we will enhance these topological data by geometric ones, namely with the notion of a *string connection.* These are needed to construct our enriched elliptic object for a string vector bundle, just like a spin connection was needed to define the K-cocycles in Section 3.

Since $\mathrm{String}(n)$ is not a Lie group, it is necessary to come up with a new notion of a connection on a principal $\mathrm{String}(n)$-bundle. We first present such a new notion in the spin case, assuming the presence of a metric connection on the bundle.

Spin connections. By Definition 2.3.1, a spin structure on an n-dimensional vector bundle $E \to X$ with Riemannian metric is a graded irreducible bimodule bundle $S(E)$ over the Clifford algebra bundle $C(E) - C_n$. For a point $x \in X$, we denote the resulting bimodule $S(E_x)$ by $\mathcal{S}(x)$. It is a left module over the algebra $C(x) = C(E_x) \otimes C_{-n}$ from Definition 2.4.2. We now assume in addition that X is a manifold and that E is equipped with a metric connection.

Proposition 5.0.6. *A* spin connection $\mathcal{S}$ *on* E *gives for each piecewise smooth path* γ *from* x_1 *to* x_2*, an isomorphism between the following two* $C(\partial\gamma) \overset{def}{=} C(x_1)^{\mathrm{op}} \otimes C(x_2)$ *(left) modules:*

$$\mathcal{S}(\gamma) : F(\gamma) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{R}}(\mathcal{S}(x_1), \mathcal{S}(x_2)),$$

where $F(\gamma)$ is the relative Fock module from Definition 2.4.5 (defined using the connection on E). We assume that $\mathcal{S}$ varies continuously with γ and is independent of the parametrization of I. Moreover, $\mathcal{S}$ satisfies the following gluing condition: Given another path γ' from x_2 to x_3, there is a commutative diagram

$$\begin{array}{ccc} F(\gamma' \cup_{x_2} \gamma) & \xrightarrow{\mathcal{S}(\gamma' \cup_{x_2} \gamma)} & \mathrm{Hom}(\mathcal{S}(x_1), \mathcal{S}(x_3)) \\ \downarrow \cong & & \circ \uparrow \cong \\ F(\gamma') \otimes_{C(x_2)} F(\gamma) & \xrightarrow[\cong]{\mathcal{S}(\gamma') \otimes \mathcal{S}(\gamma)} & \mathrm{Hom}(\mathcal{S}(x_2), \mathcal{S}(x_3)) \otimes_{C(x_2)} \mathrm{Hom}(\mathcal{S}(x_1), \mathcal{S}(x_2)) \end{array}$$

where the left vertical isomorphism is the gluing isomorphisms from Lemma 2.3.14.

Remark 5.0.7. The vacuum vectors in the Fock modules $\Omega_\gamma \in F(\gamma^* E)$ define a parallel transport in $S(E)$ as follows: Recall that $F(\gamma) = F(\gamma^* E) \otimes F(I)^{-n}$ and that $F(I) = C_1$. Thus the vector $\Omega_\gamma \in F(\gamma^* E)$ together with the identity $\mathrm{id} \in C_{-n} = F(I)^{-n}$ gives a homomorphism from $\mathcal{S}(x_1)$ to $\mathcal{S}(x_2)$ via $\mathcal{S}(\gamma)(\Omega_\gamma \otimes \mathrm{id})$. One checks that this homomorphism is in fact C_n-linear and coincides with the usual parallel transport in the spinor bundle $S(E)$.

It is interesting to observe that these vacuum vectors Ω_γ exist for any vector bundle E with metric and connection but it is the spin connection in the sense above which makes it possible to view them as a parallel transport.

Remark 5.0.8. There is a *unique* spin connection in the setting of the above proposition. In the usual language, this is well known and follows from the fact that the fiber of the projection $\mathrm{Spin}(n) \to SO(n)$ is discrete. For our definitions, existence and uniqueness follows from the fact that all the bimodules are irreducible (and of real type) and hence the isometries $\mathcal{S}(\gamma)$ are determined up to sign. Since they vary continuously and satisfy the gluing condition above, it is possible to see this indeterminacy in the limit where γ is the constant map with image $x \in X$. Then the right hand side contains a canonical element, namely $\mathrm{id}_{\mathcal{S}(x)}$ and our gluing condition implies that it is the image under $\mathcal{S}(\gamma)$ of $\Omega_\gamma \otimes \mathrm{id}$. Hence the indeterminacy disappears.

These are the data a spin structure associates to points in X and a spin connection associates to paths in X. It is easy to extend the spin connection to give data associated to arbitrary 0- and 1-dimensional spin manifolds mapping to X, just like in Proposition 3.1.1.

In the next section, and in particular Lemma 5.1.4, we shall explain how all these data are really derived from 'trivializing' a 2-dimensional field theory (called Stiefel-Whitney theory in this paper). This derivation is necessary to motivate our definition of a string connection as a 'trivialization' of the Chern-Simons (3-dimensional) field theory. Because of the shift of dimension from 2 to 3, a string connection will necessarily have 0-, 1- and 2-dimensional data. As above, it is enough to formulate the top-dimensional data for manifolds with boundary (intervals in the case of spin, conformal surfaces in the case of string), since the usual gluing formulas determines the data on closed manifolds. Also as above, the 0-dimensional data are purely topological, and in the case of a string structure are given by the von Neumann algebra bundle from Corollary 5.0.4.

String connections. We recall from Definition 2.4.2 that there is a (relative) complex Clifford algebra $C(\gamma)$ defined for every piecewise smooth map $\gamma : Y \to X$, where Y is a spin 1-manifold and X comes equipped with a metric vector bundle E. If Y is closed then a connection on $\gamma^* E$ gives a preferred isomorphism class of graded irreducible (left) $C(\gamma)$-modules as follows: Consider the conformal spin surface $Y \times I$ and extend the bundle

$$\gamma^* E \cup \underline{\mathbb{R}}^{\dim(E)} \stackrel{\text{def}}{=} (\gamma^* E \times 0) \amalg (Y \times 1 \times \mathbb{R}^{\dim(E)})$$

over $Y \times \{0, 1\}$ to a bundle E' (with connection) on $Y \times I$ (this uses the fact that E is orientable, and hence trivial over 1-manifolds). In Definition 2.4.5 we explained how to construct a Fock module $F(E')$ from boundary values of harmonic sections on $Y \times I$. It is a graded irreducible $C(\gamma)$ module and the isomorphism class of $F(E')$ is independent of the extension of the bundle with connection. It will be denoted by $[F(\gamma)]$. If $\Gamma : \Sigma \to X$ is a piecewise smooth map of a conformal spin surface with boundary $\gamma : Y \to X$, then a connection on $\Gamma^* E$ gives a particular representative $F(\Gamma)$ in this isomorphism class as explained in Definition 2.4.5.

Definition 5.0.9. Let $E \to X$ be an n-dimensional vector bundle with spin connection. Assume further that a string structure on E has been chosen and denote by $A(x)$ the fiber of the corresponding von Neumann algebra bundle. A *string connection* $\mathcal{S}$ on E consists of the following data.

dim 0: For each map $\boldsymbol{x} : Z \to X$ of a 0-dimensional spin manifold Z, $\mathcal{S}(\boldsymbol{x})$ is a von Neumann algebra given by the von Neumann tensor product $A(x_1) \bar{\otimes} \ldots \bar{\otimes} A(x_n)$ if $\boldsymbol{x}(Z)$ consists of the spin points $x_1, \ldots, x_n$. By definition, $A(\bar{x}) = A(x)^{op}$ and $\mathcal{S}(\emptyset) = \mathbb{C}$. All these data are completely determined by the string structure alone.

dim 1: For each piecewise smooth map $\gamma : Y \to X$ of a spin 1-manifold Y, $\mathcal{S}(\gamma)$ is a graded irreducible $C(\gamma) - \mathcal{S}(\partial\gamma)$ bimodule. These fit together to bimodule bundles over $\mathrm{Maps}(Y, X)$ and we assume that on these bundles there are lifted actions $\mathcal{S}(\phi)$ of the spin diffeomorphisms $\phi \in \mathrm{Diff}(Y, Y')$ which are the identity on the boundary. It is clear that these are bimodule maps only if one takes the action of ϕ on $C(\gamma)$ into account, as well as the action of $\phi|_{\partial\gamma}$ on $\mathcal{S}(\partial\gamma)$.

Given another such $\gamma' : Y' \to X$ with 0-dimensional intersection on the boundary $\boldsymbol{x} \stackrel{\text{def}}{=} \partial\gamma \cap \partial\gamma' = \partial_{in}\gamma = \partial_{out}\gamma'$, there are $C(\gamma \cup_{\boldsymbol{x}} \gamma') - \mathcal{S}(\partial(\gamma \cup_{\boldsymbol{x}} \gamma'))$ bimodule isomorphisms

$$\mathcal{S}(\gamma, \gamma') : \mathcal{S}(\gamma \cup_{\boldsymbol{x}} \gamma') \xrightarrow{\cong} \mathcal{S}(\gamma) \boxtimes_{\mathcal{S}(\boldsymbol{x})} \mathcal{S}(\gamma')$$

where we used Connes fusion of bimodules on the right hand side, and also the identifications

$$C(\gamma \cup_{\boldsymbol{x}} \gamma') \cong C(\gamma) \otimes C(\gamma') \text{ and } \mathcal{S}(\partial(\gamma \cup_{\boldsymbol{x}} \gamma')) \subset \mathcal{S}(\partial\gamma)\bar{\otimes}\mathcal{S}(\partial\gamma')$$

The isomorphisms $\mathcal{S}(\gamma, \gamma')$ must satisfy the obvious associativity constraints. Note that for closed Y, we just get an irreducible $C(\gamma)$-module $\mathcal{S}(\gamma)$, multiplicative under disjoint union. We assume that $\mathcal{S}(\gamma)$ is a (left) module in the preferred isomorphism class $[F(\gamma)]$ explained above.

dim 2: Consider a conformal spin surface Σ and a piecewise smooth map $\Gamma : \Sigma \to X$, and let $Y = \partial\Sigma$ and $\gamma = \Gamma|_Y$. Then there are two irreducible (left) $C(\gamma)$-modules in the same isomorphism class, namely $F(\Gamma)$ and $\mathcal{S}(\gamma)$. The string connection on Γ is a unitary isometry of left $C(\gamma)$-modules

$$\mathcal{S}(\Gamma) : F(\Gamma) \cong \mathcal{S}(\gamma)$$

such that for each conformal spin diffeomorphism $\phi : (\Sigma, \Gamma) \to (\Sigma', \Gamma')$ the following diagram commutes:

$$\begin{array}{ccc} F(\Gamma) & \xrightarrow[\cong]{\mathcal{S}(\Gamma)} & \mathcal{S}(\gamma) \\ {\scriptstyle F(\phi)}\downarrow & & \downarrow{\scriptstyle \mathcal{S}(\phi|_{\partial\gamma})} \\ F(\Gamma') & \xrightarrow[\cong]{\mathcal{S}(\Gamma')} & \mathcal{S}(\gamma') \end{array}$$

The module maps $\mathcal{S}(\Gamma)$ fit together to continuous sections of the resulting bundles over the relevant moduli spaces. The irreducibility of the modules in question implies that there is only a circle worth of possibilities for each $\mathcal{S}(\Gamma)$. This is the *conformal anomaly*.

Finally, there are gluing laws for surfaces which meet along a part Y of their boundary. If Y is closed this can be expressed as the composition of Hilbert-Schmidt operators,. If Y has itself boundary one uses Connes fusion, see Proposition 4.3.10.

Note that the irreducible $C(\gamma)$-module $\mathcal{S}(\gamma)$ for $\gamma \in LM$ plays the role of the spinor bundle on loop space LM. We explain in Section 4.4 how the vacuum vectors for conformal surfaces lead to a 'conformal connection' of this spinor bundle. All of Section 5.2 is devoted to discuss the motivation behind our above definition of a string connection. This definition can also be given in the language of gerbes with 1- and 2-connection, see e.g. [Bry]. But the gerbe in question needs to be defined on the total space of the principal Spin(n)-bundle, restricting to the Chern-Simons gerbe on each fiber. We feel that such a definition is at least as complicated as ours, and it lacks the beautiful connection to von Neumann algebras and Connes fusion.

5.1 Spin connections and Stiefel-Whitney theory

We first explain a 2-dimensional field theory based on the second Stiefel-Whitney class. We claim no originality and thus skip most proofs. Stiefel-Whitney theory is defined on manifolds with the geometric structure (or classical field) given by an oriented vector bundle with inner product, and hence is a functor

$$\mathrm{SW} : \mathcal{B}_2^{SO} \longrightarrow \mathrm{Hilb}_{\mathbb{R}}$$

where $\mathcal{B}_2^{SO}$ is the category explained in Section 2.1, where the geometric structure is an oriented vector bundle with inner product. In the following definitions we could used $\mathbb{Z}/2$ instead of $\mathbb{R}$ as the values, but it will be convenient for further use to stay in the language of (real) Hilbert spaces. We use the embedding $\mathbb{Z}/2 = \{\pm 1\} \subset \mathbb{R}$ and note that these are the numbers of unit length. Note also that a $\mathbb{Z}/2$-torsor is the same thing as a 1-dimensional real Hilbert space, also called a *real line* below.

Definition of Stiefel-Whitney theory. Stiefel-Whitney theory SW associates to a closed geometric 2-manifold $E \to \Sigma$ the second Stiefel-Whitney number

$$\mathrm{SW}(E \to \Sigma) \stackrel{\mathrm{def}}{=} \langle w_2(E), [\Sigma] \rangle \in \mathbb{Z}/2 = \{\pm 1\} \subset \mathbb{R}$$

To a closed geometric 1-manifold $E \to Y$ it associates the real line

$$\mathrm{SW}(E \to Y) \stackrel{\mathrm{def}}{=} \{(F, r) \mid r \in \mathbb{R}, F \to Y \times I, F|_{Y \times \{0,1\}} = E \cup \underline{\mathbb{R}}^{\dim(E)}\}/\sim$$

where $\underline{\mathbb{R}}^n$ denotes the trivial bundle and $(F_1, r_1) \sim (F_2, r_2)$ if and only if $w_2(F_1 \cup F_2 \to Y \times S^1) \cdot r_1 = r_2$. If $\partial\Sigma = Y$ and $E' \to \Sigma$ extends $E \to Y$, then the equivalence class of $(F, w_2(E \cup F \cup \underline{\mathbb{R}}^{\dim(E)}))$ is a well defined element

$$\mathrm{SW}(E' \to \Sigma) \in \mathrm{SW}(E \to \partial\Sigma).$$

It is independent of the choice of the bundle F by additivity of w_2. This theory by itself is not very interesting but we shall make several variations, and ultimately generalize it to Chern-Simons theory. The first observation is that one can also define the value $\mathrm{SW}(E \to Z)$ for a 0-manifold Z. According to the usual field theory formalism we expect that this is a category whose morphism spaces are real lines (which can then be used to calculated the value of the field theory on 1-manifolds). In the spirit of the above definition, we start with vector bundles $F \to Z \times I$ which extend the bundle $E \cup \mathbb{R}^{\dim(E)}$ on $Z \times \{0, 1\}$. These are the objects in a category $\mathrm{SW}(E \to Z)$ with morphisms defined by

$$\mathrm{Mor}(F_1, F_2) \stackrel{\mathrm{def}}{=} \mathrm{SW}(F_1 \cup F_2 \to Z \times S^1) = \mathrm{Mor}(F_2, F_1)$$

To complete the description of the theory, we need to associate something to a bundle $E' \to Y$ over a 1-manifold with boundary $Z = \partial Y$ (with restricted bundle $E = E'|_Z$). It should be an 'element' in the category $\mathrm{SW}(E \to Z)$ which can then be used to formulate the appropriate gluing laws of the theory. There are various possible interpretations of such an 'element' but in the best case, it would mean an object a in the category. In order to find such an object, we slightly enlarge the above category, allowing as objects not just vector bundles over $Z \times I$ but more generally, vector bundles over Y with $\partial Y = Z \times \{0, 1\}$. The reader will easily see that this has the desired effect.

Definition 5.1.1. Stiefel-Whitney theory is the *extended* 2-dimensional field theory described above, where the geometric structure on Y is given by an oriented vector bundle. Here the word *extended* refers to the fact that SW also assigns a small category to 0-manifolds, and objects of this category to 1-manifolds with boundary.

Relative, real Dirac theory. There is an interesting reformulation of the theory which uses that fact that our domain manifolds Σ are equipped with a spin structure and that the bundle E comes a with connection. Enhance for a moment the geometrical structure on Σ by a conformal structure. Then we have the Dirac operator D_Σ, as well as the twisted Dirac operator D_E. If Σ^2 is a closed, we get an index in $KO_2 \cong \mathbb{Z}/2$. For a closed conformal 1-manifold Y the Dirac operator is just covariant differentiation in the spinor bundle from Definition 2.3.1. Hence it comes equipped with a real Pfaffian line $\mathrm{Pf}(D_Y)$, see Definition 2.3.12. If $Y = \partial\Sigma$ then the relative index of Σ is an element of unit length in $\mathrm{Pf}(D_Y)$. The same holds for the twisted case. Finally, for a bundle with metric over a 0-manifold, we define the following *relative, real Dirac category*: The objects are Lagrangian subspaces L in $V \perp -\mathbb{R}^n$, where V is again the orthogonal sum of the fibers and n is the dimension of V. These Lagrangians should be thought of as boundary conditions for the Dirac operator on a bundle on $Z \times I$ which restricts to $V \cup \mathbb{R}^n$ on the boundary. In particular, the boundary values of harmonic sections of a bundle E over 1-manifold Y define an object in the category for $E|_{\partial Y} = V_0 \cup V_1$ by rewriting the spaces in question as follows

$$-(V_0 \perp -\mathbb{R}^n) \perp (V_1 \perp -\mathbb{R}^n) = (-V_0 \perp V_1) \perp (\mathbb{R}^n \perp -\mathbb{R}^n)$$

This is in total analogy to the above rewriting of the isometry groups. The morphisms in the category are given by the real lines

$$\mathrm{Mor}(L_1, L_2) \stackrel{\text{def}}{=} \mathrm{Hom}_{C(V)-C_n}(F(L_1), F(L_2)),$$

where $F(L_i)$ are the Fock spaces from Definition 2.2.4. They are irreducible graded bimodules over the Clifford algebras $C(V) - C_n$. Recall from Remark 2.2.6 that the orientation of V specifies a connected component of such Lagrangians L and we only work in this component.

Lemma 5.1.2. *There is a canonical isomorphism between the two extended 2-dim. field theories, Stiefel-Whitney theory and relative, real Dirac theory. For $n = \dim(E)$ this means the following statements in the various dimensions*

dim 2: $\mathrm{SW}(E \to \Sigma) = \mathrm{index}(D_\Sigma \otimes E) - n \cdot \mathrm{index}(D_\Sigma) \in \mathbb{Z}/2$,

dim 1: $\mathrm{SW}(E \to Y) \cong \mathrm{Pf}(D_Y \otimes E) \otimes \mathrm{Pf}^{-n}(D_Y)$, *such that for* $Y = \partial\Sigma$ *the element* $\mathrm{SW}(E' \to \Sigma)$ *is mapped to the relative index.*

dim 0: *For an inner product space* V*, the category* $\mathrm{SW}(V)$ *is equivalent to the above relative Dirac category, in a way that the objects defined by* 1*-manifolds with boundary correspond to each other.*

The extra geometric structure of bundles with connection is needed to define the right hand side theory, as well as for the isomorphisms above.

Proof. The 2-dimensional statement follows from index theory, and for the 1-dimensional statement one uses the relative index on $Y \times I$. In dimension zero, recall from Remark 2.2.6 that a Lagrangian subspace L in $V \perp -\mathbb{R}^n$ is given by the graph of a unique isometry $V \to \mathbb{R}^n$. Moreover, parallel transport along a connection gives exactly the Lagrangian of boundary values of harmonic spinors along an interval. □

Spin structures as trivializations of Stiefel-Whitney theory. Fix a manifold X and an n-dimensional oriented vector bundle $E \to X$ with metric connection. One may restrict the Stiefel-Whitney theory to those bundles (with connection) that are pull-backs of E via a piecewise smooth map $Y \to X$. Thus geometric structures on Y make up the set $\mathrm{Maps}(Y, X)$, and we call the resulting theory SW_E.

Lemma 5.1.3. *A spin structure on* $E \to X$ *gives a trivialization of the Stiefel-Whitney theory* SW_E *in the following sense:*

dim 2: $\mathrm{SW}_E(\Sigma \to X) = 0$ *if* Σ *is a closed* 2*-manifold.*

dim 1: $\mathrm{SW}_E(Y \to X)$ *is canonically isomorphic to* $\mathbb{R}$ *for a closed spin* 1*-manifold and all elements* $\mathrm{SW}_E(\Sigma \to X)$ *with* $\partial\Sigma = Y$ *are mapped to* 1*.*

dim 0: *The set of objects* $\mathrm{ob}(\mathrm{SW}_E(E_x)) = SO(E_x, \mathbb{R}^n)$ *of the category for a point* $x \in X$ *comes with a nontrivial real line bundle* ξ *and isomorphisms* $\mathrm{Mor}(b_1, b_2) \cong \mathrm{Hom}(\xi_{b_1}, \xi_{b_2})$ *which are compatible with composition in the category.*

Moreover, the last item is equivalent to the usual definition of a spin structure, and so all the other items follow from it.

Proof. The 2-dimensional statement follows from the fact that $w_2(E) = 0$, and the isomorphism in dimension 1 is induced by the relative second Stiefel-Whitney class. To see why the last item is the usual definition of a spin structure on E_x, recall that the real line bundle is the same information as a double covering $\mathrm{Spin}(E_x, \mathbb{R}^n)$, and that the isomorphisms between the morphism spaces follow from the group structures on $SO(n)$ and $\mathrm{Spin}(n)$. □

Spin connections as trivializations of relative, real Dirac theory. For the next lemma, we recall from Remark 2.2.6 that for a inner product space E_x of dimension n, the isometries $O(E_x, \mathbb{R}^n)$ are homeomorphic to the space $\mathcal{L}(x)$ of Lagrangians of $E_x \perp -\mathbb{R}^n$.

Lemma 5.1.4. *A spin connection $\mathcal{S}$ on an oriented bundle $E \to X$ with metric and connection gives a trivialization of the relative, real Dirac theory on* $\mathrm{Maps}(\cdot\ , X)$ *in the following sense:*

dim 2: $\mathrm{index}(D_{f^*E}) = n \cdot \mathrm{index}(D_\Sigma) \in \mathbb{Z}/2$ *if Σ is a closed 2-manifold and $f : \Sigma \to X$ is used to twist the Dirac operator on Σ by E.*

dim 1: *For $f : Y \to X$, Y a closed spin 1-manifold, there is an isomorphism $\mathcal{S}(f)$: $\mathrm{Pf}(f^*E) \cong \mathrm{Pf}^n(Y)$, taking twisted to untwisted indices of Dirac operators of surfaces Σ with $\partial\Sigma = Y$.*

dim 0: *For each $x \in X$, there is a graded irreducible $C(E_x) - C_n$ bimodule $\mathcal{S}(x) = S(E_x)$ which gives a nontrivial line bundle over the connected component of $\mathcal{L}(x)$ (given by the orientation of E_x). Here the line over a Lagrangian $L \in \mathcal{L}(x)$ is $\mathrm{Hom}_{C(x)}(\mathcal{S}(x), F(L))$, where $C(x) = C(E_x) \otimes C_{-n}$ (and hence $\mathcal{S}(x)$ is a left $C(x)$-module). Moreover, for each path γ from x_1 to x_2, the spin structure on E induces an isomorphism between the following two left modules over $C(x_1)^{\mathrm{op}} \otimes C(x_2)$:*

$$\mathcal{S}(\gamma) : F(\gamma) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{R}}(\mathcal{S}(x_1), \mathcal{S}(x_2))$$

where $F(\gamma)$ is the relative Fock module from Definition 2.4.5. When two paths are composed along one point, then the gluing laws from Proposition 5.0.6 hold.

Note that the bimodules $\mathcal{S}(x) = S(E_x)$ fit together to give the C_n-linear spinor bundle $S(E)$, so we have finally motivated our Definition 2.3.1 of spin structures. The vacuum vectors in the Fock modules $F(\gamma^*E)$ define a parallel transport in $S(E)$. It is interesting to note that these vacuum vectors exist even for an oriented vector bundle E (with metric and connection) but it is the spin structure in the sense above which makes it possible to view them as a parallel transport in the spinor bundle.

Proof. The result follows directly from Lemmas 5.1.2 and 5.1.3. In dimension 0 one defines $S(E_x)$ in the following way: For a given Lagrangian L we have a Fock space $F(L)$ but also the line ξ_L from Lemma 5.1.3 (since L is the graph of a unique isometry). Moreover, given two Lagrangians L_i we have given isomorphisms

$$F(L_1) \otimes \xi_{L_1} \cong F(L_2) \otimes \xi_{L_2}$$

which are associative with respect to a third Lagrangian. Therefore, we may define $S(E_x)$ as the direct limit of this system of bimodules. Note that $S(E_x)$ is then canonically isomorphic to each bimodule of the form $F(L) \otimes \xi_L$ and so one can recover the line bundle ξ from $S(E_x)$. In fact, the bimodule and the line bundle ξ contain the exact same information. □

5.2 String connections and Chern-Simons theory

We want to explain the steps analogous to the ones in the previous section with $w_2 \in H^2(BSO(n);\mathbb{Z}/2)$ replaced by a "level" $\ell \in H^4(BG;\mathbb{Z})$. The most interesting case for us is the generator $p_1/2$ of $H^4(B\,\mathrm{Spin}(n);\mathbb{Z})$ which will lead to string structures. The analogue of Stiefel-Whitney theory is (classical) Chern-Simons theory which we briefly recall, following [Fr1]. We shall restrict to the case where the domain manifolds are spin as this is the only case we need for our applications.

Let G be a compact Lie group and fix a level $\ell \in H^4(BG;\mathbb{Z})$. For $d = 0, \dots, 4$ we consider compact d-dimensional spin manifolds M^d together with connections $\mathfrak{a}$ on a G-principalbundle $E \to M$. The easiest invariant is defined for a closed 4-manifold M^4, and is given by the characteristic number $\langle c_E^*(\ell), [M]\rangle \in \mathbb{Z}$, where $c_E : M \to BG$ is a classifying map for E. It is independent of the connection and one might be tempted to view it as the analog of $\mathrm{SW}(E \to \Sigma) \in \mathbb{Z}/2$ of a closed surface Σ. However, this is not quite the right point of view. In fact, Chern-Simons theory is a 3-dimensional field theory

$$\mathrm{CS} = \mathrm{CS}_\ell : \mathcal{B}_3^G \longrightarrow \mathrm{Hilb}_{\mathbb{C}}$$

in the sense of Section 2.1, with geometric structure being given by G-bundles with connection. The value $\mathrm{CS}(M^3, \mathfrak{a}) \in S^1$ for a closed 3-manifold is obtained by extending the bundle and connection over a 4-manifold W with boundary M, and then integrating the Chern-Weil representative of ℓ over W. By the integrality of ℓ on closed 4-manifolds, it follows that one gets a well defined invariant in $S^1 = \mathbb{R}/\mathbb{Z}$, viewed as the unit circle in $\mathbb{C}$ (just like $\mathbb{Z}/2 = \{\pm 1\}$ was the unit circle in $\mathbb{R}$). Thus we think of this *Chern-Simons invariant* as the analogue of $\mathrm{SW}(E \to \Sigma)$. One can then use the tautological definitions explained in the previous sections to get the following values for the invariant $\mathrm{CS}(M^d, \mathfrak{a})$, leading to an *extended* 3-dimensional field theory.

d	M^d closed	$\partial M^d \neq \emptyset$
4	element in $\mathbb{Z}$	element in $\mathbb{R}$ reducing to invariant of ∂M
3	element in S^1	point in the hermitian line for ∂M
2	hermitian line	object in the $\mathbb{C}$-category for ∂M
1	$\mathbb{C}$-category	

By a $\mathbb{C}$-category we mean a category where all morphism spaces are hermitian lines. In Stiefel-Whitney theory we associated an $\mathbb{R}$-category to 0-manifolds. So $\mathbb{R}$ has been replaced by $\mathbb{C}$ and all dimensions have moved up by one. It will be crucial to understand the 0-dimensional case in Chern-Simons theory, where von Neumann algebras enter the picture.

Relative, complex Dirac theory. First we stick to dimensions 1 to 4 as above and explain the relation to Dirac operators.

Theorem 5.2.1. *For $G = \mathrm{Spin}(n)$ at level $\ell = p_1/2$, the above extended Chern-Simons theory is canonically isomorphic to relative, complex Dirac theory.*

In Dirac theory one has conformal structures on the spin manifolds M which enables one to define the Dirac operator D_M, as well as the twisted Dirac operator $D_{\mathfrak{a}}$. Here we use the fundamental representation of $\mathrm{Spin}(n)$ to translate a principal $\mathrm{Spin}(n)$bundle into a spin vector bundle, including the connections $\mathfrak{a}$. In the various dimensions $d = 1, \dots, 4$, relative, complex Dirac theory is given by the following table of classical actions. It is a (well known) consequence of our theorem that the relative theory is metric independent. Let M be a closed d-manifold and E an n-dimensional vector bundle E over M with connection $\mathfrak{a}$.

d	$\mathfrak{D}(M^d, \mathfrak{a}) \overset{\text{def}}{=}$
4	$\mathrm{index}^{rel}(M, \mathfrak{a}) \overset{\text{def}}{=} \frac{1}{2}\,\mathrm{index}(D_{\mathfrak{a}}) - \frac{n}{2}\,\mathrm{index}(D_M) \in \mathbb{Z}$
3	$\eta^{rel}(M^3, \mathfrak{a}) \in S^1$
2	$\mathrm{Pf}^{rel}(M^2, \mathfrak{a})$, a hermitian line
1	$[F^{rel}(M^1, \mathfrak{a})]$, a $\mathbb{C}$-category of representations

Proof of Theorem 5.2.1. The statement in dimension 4 follows from the index theorem (see below) which implies that the relative index in the above table equals the characteristic class $\langle p_1(E)/2, [M^4]\rangle$ on closed 4-manifolds. In dimension 3, we first need to explain the invariant η^{rel}. It is one half of the reduced η-invariant which shows up in the Atiyah-Patodi-Singer index theorem for 4-manifolds with boundary (where we are using the Dirac operator twisted by the virtual bundle $E \oplus -\underline{\mathbb{R}}^n$):

$$\mathrm{index}(D_{M^4,\mathfrak{a}}) - n \cdot \mathrm{index}(D_M) = \int_M \hat{A}(M)\widetilde{ch}(E \otimes \mathbb{C}, \mathfrak{a}) - \tilde{\eta}(\partial M, \mathfrak{a}) \ \in \mathbb{R}$$

Both indices above are even dimensional because of a quaternion structure on the bundles (coming from the fact that the Clifford algebra C_4 is of quaternion type). Applying this observation together with the fact that the Chern character in degree 4 is given by $p_1(E \otimes \mathbb{C})/2 = p_1(E)$ one gets

$$\int_M p_1(E, \mathfrak{a}) \equiv \tilde{\eta}(\partial M, \mathfrak{a}) \qquad \mathrm{mod}\ 2\mathbb{Z}$$

Since we are assuming that E is a spin bundle, the left hand side is an even integer for closed M. Therefore, we may divide both sides by 2 to obtain a well defined invariant $\eta^{rel}(\partial M, \mathfrak{a})$ in $\mathbb{R}/\mathbb{Z}$ which equals $\mathrm{CS}(\partial M, \mathfrak{a})$.

In dimension 2, one needs to understand the Pfaffian line of the skew-adjoint operator $D^+_{\mathfrak{a}}$, as well as the corresponding relative Pfaffian line

$$\mathrm{Pf}^{rel}(M, \mathfrak{a}) = \overline{\mathrm{Pf}(\mathfrak{a})} \otimes \mathrm{Pf}(D_M)^{\otimes n}$$

in the above table. The main point is that the relative η-invariant above can be extended to 3-manifolds with boundary so that it takes values in this relative Pfaffian line. Therefore, one can define an isomorphism of hermitian lines $\mathrm{CS}(M^2, \mathfrak{a}) \to \mathrm{Pf}^{rel}(M^2, \mathfrak{a})$ by associating this relative η-invariant to a connection on $M^2 \times I$ (extending $\mathfrak{a}$ respectively the trivial connection).

Finally, for a closed 1-manifold, $[F^{rel}(M^1, \mathfrak{a})]$ is the isomorphism class of twisted Fock spaces explained in Definition 2.3.12. They can be defined from harmonic boundary values of twisted Dirac operators on $M \times I$. The isomorphism class of the bimodule does not depend on the extension of bundle and connection to $M \times I$. Each of these Fock spaces is a complex graded irreducible representation of the Clifford algebra

$$C^{rel}(M, \mathfrak{a}) = C(E, \mathfrak{a})^{\mathrm{op}} \otimes C(M)^{\otimes \dim(E)},$$

the latter replacing $C_n = C(\mathrm{pt})^{\otimes n}$ from Stiefel-Whitney theory. Given the isomorphism class of such a bimodule, there is an associated $\mathbb{C}$-category whose objects are actual representations in this isomorphism type, and whose morphisms are intertwiners. The equivalence of categories from $\mathrm{CS}(M, \mathfrak{a})$ to the $\mathbb{C}$-category defined by $[F^{rel}(M, \mathfrak{a})]$ is on objects given by sending a connection on $M^1 \times I$ (extending $\mathfrak{a}$ respectively the trivial connection) to the Fock space defined from harmonic boundary values of twisted Dirac operators on $M \times I$. By definition, this is an object in thecorrect category. To define the functor on morphisms, one uses the canonical isomorphism

$$\mathrm{Pf}^{rel}(M^1 \times S^1, \mathfrak{a}) \cong \mathrm{Hom}_{C^{rel}(M \times S^1, \mathfrak{a})}(F^{rel}(M \times I, \mathfrak{a}_0), F^{rel}(M \times I, \mathfrak{a}_1))$$

where $\mathfrak{a}$ is a connection on a bundle over $M \times S^1$ which is obtained by gluing together two connections $\mathfrak{a}_0, \mathfrak{a}_1$ on $M \times I$. □

5.3 Extending Chern-Simons theory to points

Fix a compact, simply connected, Lie group G and a level $\ell \in H^4(BG)$. Recall from Theorem 5.0.2 that there is a von Neumann algebras $A = A_{G,\ell}$ and a G-kernel (see Remark 5.4.3)

$$\widetilde{\Phi} : G \longrightarrow \mathrm{Out}(A) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(A)/\,\mathrm{Inn}(A)$$

canonically associated to (G, ℓ). Moreover, $\widetilde{\Phi}$ defines the extension G_ℓ of G by $PU(A) = \mathrm{Inn}(A)$ and lifts to a monomorphism $\Phi : G_\ell \to \mathrm{Aut}(A)$. We want to use these data to define the Chern-Simons invariant of a point.

On a G-bundle V over a point, we first pick a G_ℓ-structure. Recall from Definition 5.0.5 that this is an algebra A_V together with a G-equivariant map

$$\alpha_V : \mathrm{Iso}_G(G, V) \longrightarrow \mathrm{Out}(A, A_V) \stackrel{\mathrm{def}}{=} \mathrm{Iso}(A, A_V)/\,\mathrm{Inn}(A)$$

It turns out that the $\mathrm{Out}(A)$-torsor $\mathrm{Out}(A, A_V)$ is actually defined independently of the choice of such a G_ℓ-structure.

Definition 5.3.1. We define $CS(V)$ to be the $\mathrm{Out}(A)$-torsor $\mathrm{Out}(A, A_V)$.

The above independence argument really shows that there is a map

$$\mathrm{Iso}_G(V_1, V_2) \longrightarrow \mathrm{Out}(A_{V_1}, A_{V_2})$$

which is well defined without choosing G_ℓ-structures. In particular, without knowing what the algebras A_{V_i} really are. When applying this map to the parallel transport of a G-connection $\mathfrak{a}$ an interval I, we get the value $CS(I, \mathfrak{a})$.

To motivate why this definition really extends Chern-Simons theory to points, we propose a whole new picture of the theory.

Chern-Simons theory revisited. We propose a rigidified picture of the Chern-Simons actions motivated by our definition in dimension 0. For a given (G, ℓ), the Chern-Simons invariant $CS(M^d, \mathfrak{a})$ for connected d-manifolds would then take values in mathematical objects listed in the table below. Note that the values for closed spin manifolds are special cases of manifolds with boundary, i.e. the entries in the middle column are subsets of the entries in the right hand column.

d	M^d closed	$\partial M^d \neq \emptyset$
4	$\mathbb{Z}$	$\mathbb{R}$
3	S^1	$U(A)$
2	$PU(A)$	$\mathrm{Aut}(A)$
1	$\mathrm{Out}(A)$	$\mathrm{Out}(A)$-equivariant maps
0	space of $\mathrm{Out}(A)$-torsors	

Here $A = A_{G,\ell}$ is the von Neumann algebra discussed in the previous section. The guiding principle in the table above is that for closed connected d-manifolds M, the Chern-Simons invariant $CS(M^d, \mathfrak{a})$ should be a point in a particular version of an Eilenberg-MacLane space $K(\mathbb{Z}, 4-d)$ (whereas for manifolds M with boundary one gets a point in the corresponding contractible space). More precisely, for $K(\mathbb{Z}, 4-d)$ we used the models

$$\mathbb{Z}, \quad S^1 = \mathbb{R}/\mathbb{Z}, \quad \mathrm{Inn}(A) \cong PU(A) = U(A)/S^1, \quad \mathrm{Out}(A) = \mathrm{Aut}(A)/\,\mathrm{Inn}(A),$$

for $d = 4, 3, 2, 1$. Our model of a $K(\mathbb{Z}, 4)$ is the space of $\mathrm{Out}(A)$-torsors. This is only a conjectural picture of classical Chern-Simons theory but it should be clear why it rigidifies the definitions in Section 5.2: Every point in $PU(A)$ defines a hermitian line via the S^1-torsor of inverse images in $U(A)$. Moreover, every point in $g \in \mathrm{Out}(A)$ defines an isomorphism class of $[F_g]$ of A–A-bimodules by twisting the standard bimodule $L^2(A)$ by an automorphism in $\mathrm{Aut}(A)$ lying above g. This defines the $\mathbb{C}$-category of $A - A$-bimodules isomorphic to $[F_g]$.

Remark 5.3.2. The homomorphism $\widetilde{\Phi} : G \longrightarrow \mathrm{Out}(A)$ should be viewed as follows: An element in G gives a G-bundle with connection $\mathfrak{a}_g$ on S^1 via the clutching construction. Then $\widetilde{\Phi}(g) = \mathrm{CS}(S^1, \mathfrak{a}_g)$. In the previous section we explained a similar construction which gives $\mathrm{CS}(I, \mathfrak{a})$, and also the value of CS on points. Thus we haven't explained the definition of the rigidified Chern-Simons invariant only for surfaces.

We will not seriously need the new picture of Chern-Simons theory in the following because we really only want to explain what a 'trivialization' of it is. But we do spell out the basic results which are necessary to make this picture precise. Since we couldn't find these statements in the literature (only the analogous results for II_1-factors were known before), were originally formulated them as conjectures. We recently learned from Antony Wassermann that they are also true in the III_1-context. His argument for the contractibility of $U(A)$ is a variation of the argument for type I given in [DD], for $\mathrm{Aut}(A)$ he reduces the problem to the type II case.

Theorem 5.3.3. (Wassermann). *If A is a hyperfinite type III_1-factor then the unitary group $U(A)$ is contractible in the weak (or equivalently strong) operator topology. Moreover, the automorphism group* $\mathrm{Aut}(A)$ *is also contractible in the topology of pointwise norm convergence in the predual of A.*

It follows that $PU(A) = U(A)/\mathbb{T}$ is a $K(\mathbb{Z}, 2)$. One has to be more careful with the topology on $\mathrm{Out}(A) = \mathrm{Aut}(A)/PU(A)$ because with the quotient topology this is not a Hausdorff space (using the above topologies, $PU(A)$ is *not* closed in $\mathrm{Aut}(A)$). A possible strategy could be to *define* a continuous map $X \to \mathrm{Out}(A)$ to be any old map but together with local continuous sections to $\mathrm{Aut}(A)$.

String connections as trivializations of Chern-Simons theory. Let $E \to X$ be a principal G-bundle with connection. We get a Chern-Simons theory for E by restricting to those bundles with connection on spin manifolds M^d which come from piecewise smooth maps $M \to X$ via pullback. Thus the new geometric structures on M are $\mathrm{Maps}(M, X)$ and we get the Chern-Simons theory CS_E.

Definition 5.3.4. Let G_ℓ be the group extension of G at level ℓ constructed in Section 5.4. Then a *geometric* G_ℓ-structure $\mathcal{S}$ on E is a trivialization of the extended Chern-Simons theory CS_E. For a closed spin manifold M in dimension d this amounts to the following 'lifts' of the Chern-Simons action on piecewise smooth maps $f : M^d \to X$:

d	values of $\mathcal{S}(f) = \mathcal{S}(f : M^d \to X)$
4	the equation $\mathrm{CS}(f) = 0$, no extra structure!
3	$\mathcal{S}(f) \in \mathbb{R}$ reduces to $\mathrm{CS}(f) \in \mathbb{R}/\mathbb{Z}$
2	$\mathcal{S}(f)$ is a point in the line $\mathrm{CS}(f)$
1	$\mathcal{S}(f)$ is an object in the $\mathbb{C}$-category $\mathrm{CS}(f)$
0	for $x \in X$, $\mathcal{S}(x)$ is a $G_\ell - E_x$-pointed factor

The last line uses Definition 5.0.5. There are also data associated for manifolds M with boundary, and these data must fit together when gluing manifolds and connections. Note that for $d \leq 3$, $\mathcal{S}(f : M^d \to X)$ takes values in the same objects as $\mathrm{CS}(F)$ if $F : W^{d+1} \to X$ extends f, i.e. $\partial W = M$. By construction, they both project to the same point in the corresponding quotient given by $\mathrm{CS}(f)$. The geometric G_ℓ-structure on $F : W^{d+1} \to X$ gives by definition a point in this latter group. For example, if $d = 2$ then $\mathcal{S}(F)$ is the element of S^1 such that

$$\mathcal{S}(F) \cdot \mathcal{S}(f) = \mathrm{CS}(F) \in \mathrm{CS}(f)$$

Finally, we assume that these data fit together to give bundles (respectively sections in these bundles) over the relevant mapping spaces.

Note that the data associated to points combine exactly to a G_ℓ-structure on E as explained in Definition 5.0.3. Thus a geometric G_ℓ-structure has an underlying (topological) G_ℓ-structure.

Theorem 5.3.5. *Every principal G-bundle with G_ℓ-structure admits a geometric G_ℓ-structure, unique up to isomorphism.*

In fact, the 'space' of geometric G_ℓ-structures is probably contractible. The proof of this theorem will appear elsewhere but it is important to note that the construction uses a 'thickening' procedure at every level, i.e. one crosses all manifolds M^d with I and extends the bundle f^*E with connection over $M \times I$ in a way that it restricts to the trivial bundle on the other end. So one seriously has to use the fact that all the structures explained above are really 'relative', i.e. twisted tensor untwisted, structures.

In the case $G = \mathrm{Spin}(n)$ and $\ell = p_1/2$ we need a more geometric interpretation. This is given by the following result which incorporates Definition 5.0.9. There, a geometric $\mathrm{String}(n)_{p_1/2}$ structure was called a *string connection* and we stick to this name.

Corollary 5.3.6. *Given an n-dimensional vector bundle $E \to X$ with spin connection. Then a string connection $\mathcal{S}$ on E induces the following data for closed conformal spin manifolds M^d. In the table below, $D_{M,f}$ is the conformal Dirac operator twisted by $f^*(E)$ for a piecewise smooth map $f : M \to X$ and the data fit together to give bundles (respectively sections in these bundles) over the relevant mapping spaces.*

d	values of $\mathcal{S}(f) = \mathcal{S}(f : M^d \to X)$
4	the equation $\mathrm{index}(D_{f^*E}) = n\,\mathrm{index}(D_M)$, no extra structure!
3	$\mathcal{S}(f) \in \mathbb{R}$ reduces to $\eta^{rel}(M, f^*E) \in \mathbb{R}/\mathbb{Z}$
2	an isomorphism $\mathcal{S}(f) : \mathrm{Pf}(f^*E) \cong \mathrm{Pf}(M)^{\otimes n}$
1	a representation $\mathcal{S}(f)$ isomorphic to $[F(f)]$
0	for $x \in X$, $\mathcal{S}(x)$ is a $\mathrm{String}(n) - E_x$-pointed factor

Again, the last line uses Definition 5.0.5 and the data in dimension 0 give exactly a string structure on E.

The precise gluing conditions in dimensions $0, 1, 2$ were explained in Definition 5.0.9 and that's all we shall need. The main point is that the von Neumann algebras for 0-manifolds can be used to decompose the representations of closed 1-manifolds into the Connes fusion of bimodules. That's the locality condition we need for our purposes of constructing a cohomology theory in the end. Note that by Theorem 5.3.5 such string connections exist and are up to isomorphism determined by the topological datum of a string structure.

Remark 5.3.7. In this Section 5.2 we have not taken care of the actions of diffeomorphisms of d-manifolds, $d = 0, 1, 2, 3, 4$. This is certainly necessary if one wants the correct theory and we have formulated the precise conditions only in Definition 5.0.9 (which is important for elliptic objects). However, we felt that the theory just presented is complicated enough as it stands and that the interested reader will be able to fill the gaps if necessary.

5.4 Type III_1-factors and compact Lie groups

In this section we discuss canonical extensions of topological groups

$$1 \longrightarrow PU(A_\rho) \longrightarrow G_\rho \longrightarrow G \longrightarrow 1, \tag{5.4.1}$$

one for each projective unitary representation ρ of the loop group LG of a Lie group G. The above extensions were first found for $G = \mathrm{Spin}(n)$ and ρ the positive energy vacuum representation at level $\ell = p_1/2$. We used 'local fermions' in the construction, and arrived at the groups $\mathrm{String}(n) = G_\rho$. Antony Wassermann explained to us the more general construction (in terms of 'local loops') which we shall discuss below.

In the extension above, A_ρ is a certain von Neumann algebra, the 'local loop algebra', and one can form the projective unitary group $PU(A_\rho) = U(A_\rho)/\mathbb{T}$. If $U(A_\rho)$ is contractible, the projective group has the homotopy type of a $K(\mathbb{Z}, 2)$. In that case one gets a boundary map

$$\pi_3 G \longrightarrow \pi_2 PU(A_\rho) \cong \mathbb{Z}$$

which we call the *level* of ρ. In the special case where G is compact and ρ is the vacuum representation of LG at level $\ell \in H^4(BG)$, this leads to an extension $G_\ell \to G$ which was used in Theorem 5.0.2. By Wassermann's Theorem 5.3.3, the unitary group is contractible in this case.

Lemma 5.4.2. *If G is simply connected and compact, then the two notions of level above agree in the sense that*

$$\ell \in H^4(BG) \cong \mathrm{Hom}(\pi_3 G, \mathbb{Z})$$

gives the boundary map $\pi_3 G \to \pi_2 PU(A_\rho) \cong \mathbb{Z}$ in extension 5.4.1 if ρ is the positive energy vacuum representation of LG at level ℓ.

The proof is given at the end of this section. It is interesting to remark that the 'local equivalence' result in [Wa, p.502] implies that the construction leads to canonically isomorphic algebras A_ρ and groups G_ρ if one uses any other positive energy representation of LG at the same level ℓ.

Remark 5.4.3. The extension 5.4.1 is constructed as a pullback from a homomorphism $G \to \mathrm{Out}(A_\rho)$. Such homomorphisms are also called *G-kernels* and they were first studied by Connes in [Co2]. He showed that for G a finite cyclic group, G-kernels into the hyperfinite II_1 factor are classified (up to conjugation) by an obstruction in $H^3(G;\mathbb{T}) \cong H^4(BG)$. This result was extended in Jones' thesis to arbitrary finite groups [J1]. In a sense, the above construction is anextension of this theory to compact groups (and hyperfinite III_1 factors). More precisely, Wassermann pointed out that the extensions 5.4.1 are extensions of Polish groups and by a general theorem have therefore Borel sections. There is then an obstruction cocycle in C. Moore's [Mo] third *Borel* cohomology of G which measures the nontriviality of the extension. By a a result of D. Wigner [Wig], one in fact has $H^4(BG) \cong H^3_{Borel}(G;\mathbb{T})$. In the simply connected case (and for tori), Wassermann has checked that the obstruction cocycle in Borel cohomology actually agrees with the level $\ell \in H^4(BG)$. This lead Wassermann to a similar classification as for finite groups, using the unique minimal action (cf. [PW]) of the constant loops on A_ρ.

For our applications to homotopy theory, this Borel cocycle is not as important as the boundary map on homotopy groups in Lemma 5.4.2. However, it might be an important tool in the understanding of non-simply connected groups because the isomorphism $H^4(BG) \cong H^3_{Borel}(G;\mathbb{T})$ continues to hold for all compact Lie groups (even non-connected).

Remark 5.4.4. One drawback with this more general construction is that the von Neumann algebras A_ρ are not graded, whereas our original construction in terms of local fermions gives graded algebras via the usual grading of Clifford algebras. Whenever such a grading is needed, we shall revert freely to this other construction.

Remark 5.4.5. The 'free loop group' LG is the group consisting of all piecewise smooth (and continuous) loops. The important fact is that the theory of positive energy representations of smooth loop groups extends to these larger groups (cf. [PS] and [J4]).

Let ρ be a projective unitary representation of LG, i.e., a continuous homomorphism $\rho\colon LG \to PU(H)$ from LG to the projective unitary group $PU(H) \stackrel{\mathrm{def}}{=} U(H)/\mathbb{T}$ of some complex Hilbert space H. This group carries the quotient topology of the weak (or equivalently strong) operator topology on $U(H)$. Note that by definition, we are assuming that ρ is defined for all piecewise smooth loops in G. Pulling back the canonical circle group extension

$$1 \longrightarrow \mathbb{T} \longrightarrow U(H) \longrightarrow PU(H) \longrightarrow 1$$

via ρ, we obtain an extension $\mathbb{T} \longrightarrow \tilde{L}G \longrightarrow LG$, and a unitary representation $\tilde{\rho}\colon \tilde{L}G \to U(H)$.

Let $I \subset S^1$ be the upper semi-circle consisting of all $z \in S^1$ with non-negative imaginary part. Let $L_I G \subset LG$ be the subgroup consisting of those loops $\gamma\colon S^1 \to G$ with support in I (i.e., $\gamma(z)$ is the identity element of G for $z \notin I$). Let $\tilde{L}_I G < \tilde{L}G$ be the preimage of $L_I G$. Define

$$A_\rho \stackrel{\text{def}}{=} \tilde{\rho}(\tilde{L}_I G)'' \subset B(H).$$

to be the von Neumann algebra generated by the operators $\tilde{\rho}(\gamma)$ with $\gamma \in \tilde{L}_I G$. Recall that von Neumann's double commutant theorem implies that this is precisely the weak operator closure (in the algebra $B(H)$ of all bounded operators on H) of linear combinations of group elements $\tilde{L}_I G$.

To construct the group extension (5.4.1) we start with the group extension

$$1 \longrightarrow L_I G \longrightarrow P^I_{\mathbb{1}} G \longrightarrow G \longrightarrow 1, \tag{5.4.6}$$

where $P^I_{\mathbb{1}} G = \{\gamma\colon I \to G \mid \gamma(1) = \mathbb{1}\}$, the left map is given by restriction to $I \subset S^1$ (alternatively we can think of $L_I G$ as maps $\gamma\colon I \to G$ with $\gamma(1) = \gamma(-1) = \mathbb{1}$), and the right map is given by evaluation at $z = -1$. The idea is to modify this extension by replacing the normal subgroup $L_I G$ by the projective unitary group $PU(A_\rho)$ of the von Neumann algebra A_ρ (the unitary group $U(A_\rho) \subset A_\rho$ consists of all $a \in A_\rho$ with $aa^* = a^*a = 1$), using the homomorphism

$$\rho\colon L_I G \longrightarrow PU(A_\rho),$$

given by restricting the representation ρ to $L_I G \subset LG$. We note that by definition of $A_\rho \subset B(H)$, we have $\rho(L_I G) \subset PU(A_\rho) \subset PU(H)$.

We next observe that $P^I_{\mathbb{1}} G$ acts on $L_I G$ by conjugation and that this action extends to a left action on $PU(A_\rho)$. In fact, this action exists for the group $P^I G$ of all piecewise smooth path $I \to G$ (of which $P^I_{\mathbb{1}} G$ is a subgroup): To describe how $\delta \in P^I G$ acts on $PU(A_\rho)$, extend $\delta\colon I \to G$ to a piecewise smooth loop $\gamma\colon S^1 \to G$ and pick a lift $\tilde{\gamma} \in \tilde{L}G$ of $\gamma \in LG$. We decree that $\delta \in P^I G$ acts on $PU(A_\rho)$ via

$$[a] \mapsto [\tilde{\rho}(\tilde{\gamma}) a \tilde{\rho}(\tilde{\gamma}^{-1})].$$

Here $a \in U(A_\rho) \subset B(H)$ is a representative for $[a] \in PU(A_\rho)$. It is clear that $\tilde{\rho}(\tilde{\gamma}) a \tilde{\rho}(\tilde{\gamma}^{-1})$ is a unitary element in $B(H)$; to see that it is in fact in A_ρ, we may assume that a is of the form $a = \tilde{\rho}(\tilde{\gamma}_0)$ for some $\tilde{\gamma}_0 \in \tilde{L}_I G$ (these elements generate A_ρ as von Neumann algebra). Then $\tilde{\rho}(\tilde{\gamma}) a \tilde{\rho}(\tilde{\gamma}^{-1}) = \tilde{\rho}(\tilde{\gamma}\tilde{\gamma}_0\tilde{\gamma}^{-1})$, which shows that this element is in fact in A_ρ and that it is independent of how we extend the path $\delta\colon I \to G$ to a loop $\gamma\colon S^1 \to G$, since $\gamma_0(z) = 1$ for $z \notin I$.

Lemma 5.4.7. *With the above left action of $P^I G$ on $PU(A_\rho)$, the representation $\rho : L_I G \to PU(A_\rho)$ is $P^I G$-equivariant. Therefore, there is a well defined monomorphism*

$$r : L_I G \longrightarrow PU(A_\rho) \rtimes P^I G, \quad r(\gamma) \stackrel{def}{=} (\rho(\gamma^{-1}), \gamma)$$

into the semidirect product, whose image is a normal subgroup.

Before giving the proof of this Lemma, we note that writing the semidirect product in the order given, one indeed needs a *left* action of the right hand group on the left hand group. This follows from the equality

$$(u_1 g_1)(u_2 g_2) = u_1 (g_1 u_2 g_1^{-1}) g_1 g_2$$

because $u \mapsto g u g^{-1}$ is a left action on $u \in U$.

Proof. The first statement is obvious from our definition of the action on $PU(A_\rho)$. To check that r is a homomorphism, we compute

$$\begin{aligned} r(\gamma_1) r(\gamma_2) &= (\rho(\gamma_1^{-1}), \gamma_1)(\rho(\gamma_2^{-1}), \gamma_2) \\ &= (\rho(\gamma_1^{-1})[\rho(\gamma_1)\rho(\gamma_2^{-1})\rho(\gamma_1^{-1})], \gamma_1\gamma_2) \\ &= (\rho(\gamma_2^{-1})\rho(\gamma_1^{-1}), \gamma_1\gamma_2) = (\rho(\gamma_1\gamma_2)^{-1}, \gamma_1\gamma_2) \\ &= r(\gamma_1\gamma_2) \end{aligned}$$

To check that the image of r is normal, it suffices to check invariance under the two subgroups $PU(A_\rho)$ and $P^I G$. For the latter, invariance follows directly from the $P^I G$-equivariance of ρ. For the former, we check

$$\begin{aligned} (u^{-1}, 1)(\rho(\gamma^{-1}), \gamma)(u, 1) &= (u^{-1}\rho(\gamma^{-1}), \gamma)(u, 1) \\ &= (u^{-1}\rho(\gamma^{-1})\rho(\gamma) u \rho(\gamma)^{-1}, \gamma) \\ &= (r(\gamma^{-1}), \gamma) \end{aligned}$$

This actually shows that the two subgroups $r(L_I G)$ and $PU(A_\rho)$ commute in the semidirect product group. Finally, projecting to the second factor $P^I G$ one sees that r is injective. □

Definition 5.4.8. We define the group G_ρ to be the quotient of $PU(A_\rho) \rtimes P^I_{\mathbb{1}} G$ by the normal subgroup $r(L_I G)$, in short

$$G_\rho \stackrel{\text{def}}{=} PU(A_\rho) \rtimes_{L_I G} P^I_{\mathbb{1}} G$$

Then there is a projection onto G by sending $[u, \gamma]$ to $\gamma(-1)$ which has kernel $PU(A_\rho)$. This gives the extension in 5.4.1.

The representation of G_ρ into $\mathrm{Aut}(A_\rho)$. We observe that there is a group extension

$$G_\rho \longrightarrow PU(A_\rho) \rtimes_{L_I G} P^I G \longrightarrow G$$

where the right hand map sends $[u, \gamma]$ to $\gamma(1)$. This extension splits because we can map g to $[\mathbb{1}, \gamma(g)]$, where $\gamma(g)$ is the constant path with value g. This implies the isomorphism

$$G_\rho \rtimes G \cong PU(A_\rho) \rtimes_{L_I G} P^I G$$

with the action of G on G_ρ defined by the previous split extension. Note that after projecting G_ρ to G this action becomes the conjugation action of G on G because the splitting used constant paths.

Lemma 5.4.9. *There is a homomorphism*

$$\Phi : PU(A_\rho) \rtimes_{L_I G} P^I G \longrightarrow \mathrm{Aut}(A_\rho) \quad \Phi([u], \gamma) \stackrel{def}{=} c_u \circ \phi(\gamma)$$

where c_u is conjugation by $u \in U(A_\rho)$ and $\phi(\gamma)$ is the previously defined action of $P^I G$ on A_ρ (which was so far only used for its induced action on $PU(A_\rho)$).

Proof. The statement follows (by calculations very similar to the ones given above) from the fact that

$$\phi(\gamma) \circ c_u = c_{\rho(\gamma) u \rho(\gamma)^{-1}} \circ \phi(\gamma)$$

□

We summarize the above results as follows.

Proposition 5.4.10. *There is a homomorphism $\Phi : G_\rho \rtimes G \longrightarrow \mathrm{Aut}(A_\rho)$ which reduces to the conjugation action $PU(A_\rho) \twoheadrightarrow \mathrm{Inn}(A_\rho) \subset \mathrm{Aut}(A_\rho)$ on*

$$PU(A_\rho) = \ker(G_\rho \longrightarrow G) = \ker(G_\rho \rtimes G \longrightarrow G \rtimes G)$$

The action of G on G in the right hand semidirect product is given by conjugation. This implies that the correct way to think about the homomorphism Φ is as follows: It is a homomorphism $\Phi_0 : G_\rho \to \mathrm{Aut}(A_\rho)$, together with a lift to $\mathrm{Aut}(A_\rho)$ of the conjugation action of G on $\mathrm{Out}(A_\rho)$ (which is given via $\widetilde{\Phi}_0 : G \to \mathrm{Out}(A_\rho)$).

Proof of Lemma 5.4.2. Since $P^I_{\mathbb{1}} G$ is contractible, the boundary maps in extension 5.4.6 are isomorphisms. Therefore, we need to show that $\rho_* : \pi_2 L_I G \to \pi_2 PU(A_\rho)$ is the same map as the level $\ell \in H^4(BG)$. If G is simply connected the latter can be expressed as the induced map $\rho_* : \pi_2 LG \to \pi_2 PU(H)$. Note that we use the same letter ρ for the original representation $\rho : LG \to PU(H)$ as well as for its restriction to $L_I G$. Now the inclusion $L_I G \hookrightarrow LG$ induces an isomorphism on π_2 and so does the inclusion $PU(A_\rho) \hookrightarrow PU(H)$. For the latter one has to know that $U(A_\rho)$ is contractible by Theorem 5.3.3 (which is well known for $U(H)$). Putting this information together, one gets the claim of our lemma. □

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