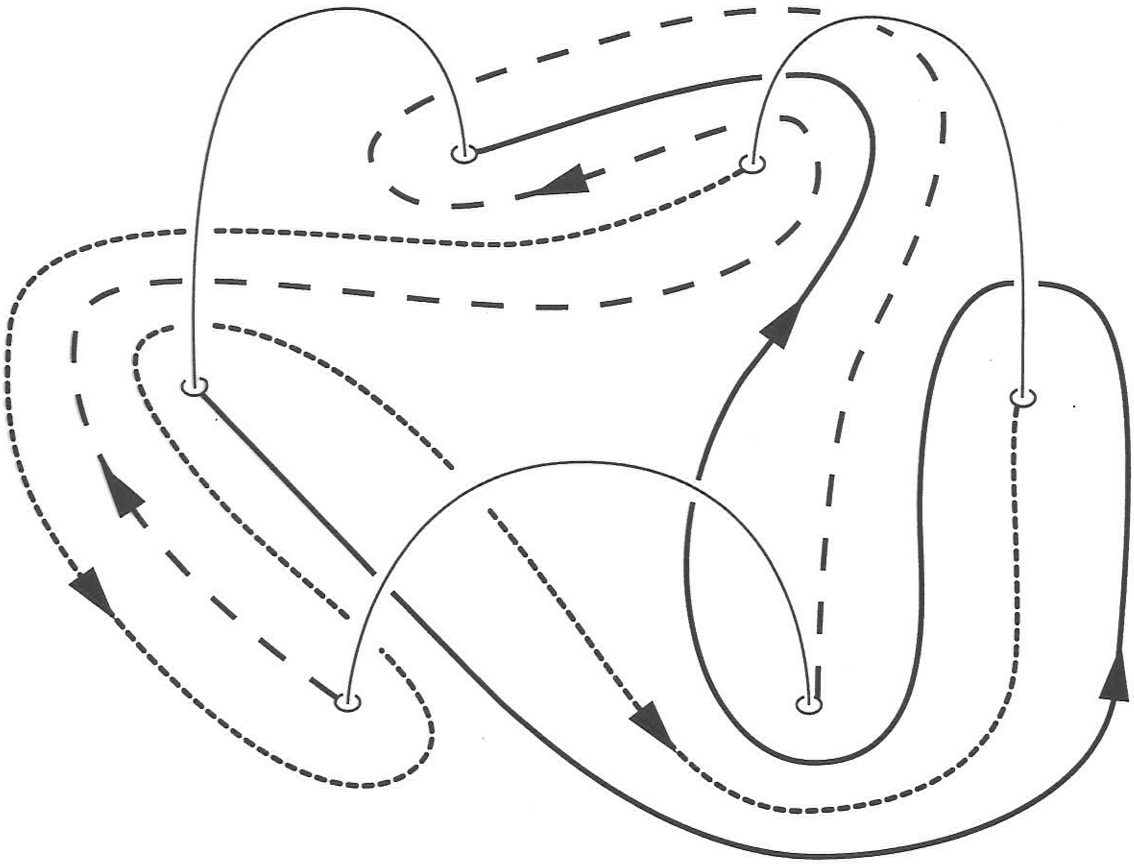
Intersection theory for Whitney Towers



21st annual workshop in Geometric Topology.

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Thanks

to organizers and participants

Credits

Rob Schneiderman, Jim Conant
Tim Cochran, Kent Orr
Kazuo Habiro, Slava Krushkal
Mike Freedman, Andrew Casson
Bob Edwards, Jim Cannon,
John Stallings, John Milnor,
R. H. Bing, H. Whitney

Lecture 1

- formulate problem and a theorem
- explain basic notions

  ⋮
- 4-dimensional Jacobi Identity

Lecture 2

- Proof of the theorem: Milnor's μ-invariants and Grope duality.
- formulate main conjectures in the theory.

Lecture 3

Survey on results about

- Grope cobordism (in 3-space)
- Grope concordance (in 4-space)

2-spheres in 4-manifolds

joint with Rob Schneiderman

Q: Given $[A_1], \dots, [A_m] \in \pi_d(M^{2d})$.

Can one represent these classes by disjoint embeddings $A_k : S^d \hookrightarrow M$?

A: $d > 2$ the answer is yes $\Leftrightarrow$

Wall's $\lambda(A_i, A_j) \in \mathbb{Z}\pi$

$\mu(A_i) \in \mathbb{Z}\pi / g \sim g^{-1}$

vanish $\forall i, j$. $\pi := \pi_1(M)$.

$d = 1$ is open!

$\underline{d \geq 2}$: λ & μ vanish $\Longleftrightarrow$

all (self-) intersections can be paired by (framed, embedded) Whitney-disks W^r_{ij}.

If $\mathring{W}^r_{ij} \cap A_k = \cancel{\emptyset}$ done by the <u>Whitney</u> <u>move</u>!

If $d = 2$, want to measure the failure of the W.-move by counting the intersections between W^r_{ij} and A_k ....

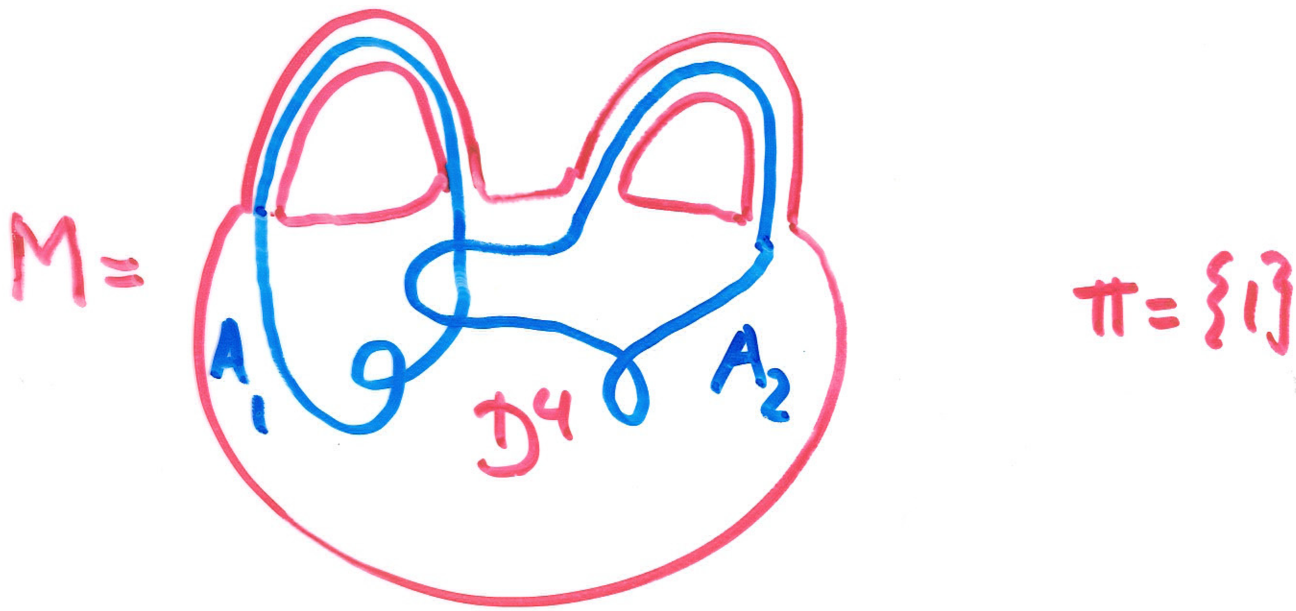
Need a new formalism, an intersection invariant for <u>Whitney</u> <u>towers</u>

Easiest case :

$L = (\ell_1, \dots, \ell_m)$ link in S^3

$M^4 := D^4 \cup$ 2-handles on ℓ_k

$A_k :=$ core$(h_k) \underset{\ell_k}{\cup}$ nullhomotopy(ℓ_k)



joint with

Theorem 1 : Rob Schneiderman

Assume a link $L = (\ell_1, \dots, \ell_m)$ bounds, in D^4, a Whitney tower $\mathcal{W}$ of class $n-1$, $n \geq 2$. Then

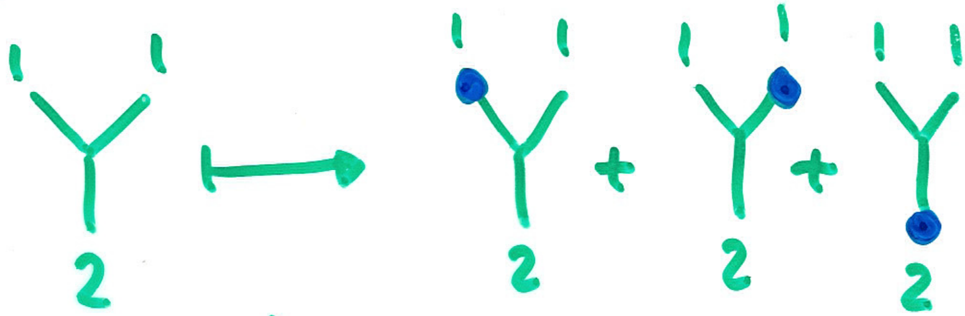
(a) All Milnor invariants of length up to $n-1$ vanish:

$$\mu_{<n}(L) = 0$$

(b) $$\mu_n(L) = \eta(\tau_n(\mathcal{W}))$$

$$\eta : \mathcal{T}_n(m) \longrightarrow \mathcal{D}_n(m)$$

$m = 2$
$n = 3$



(Sato-Levine inv...) $= 0$

Example: $n=2$ then $W =$ union of null homotopies for L always exists ($\mu_{<2}(L) \equiv 0$)

$\mu_2(L) =$ linking numbers

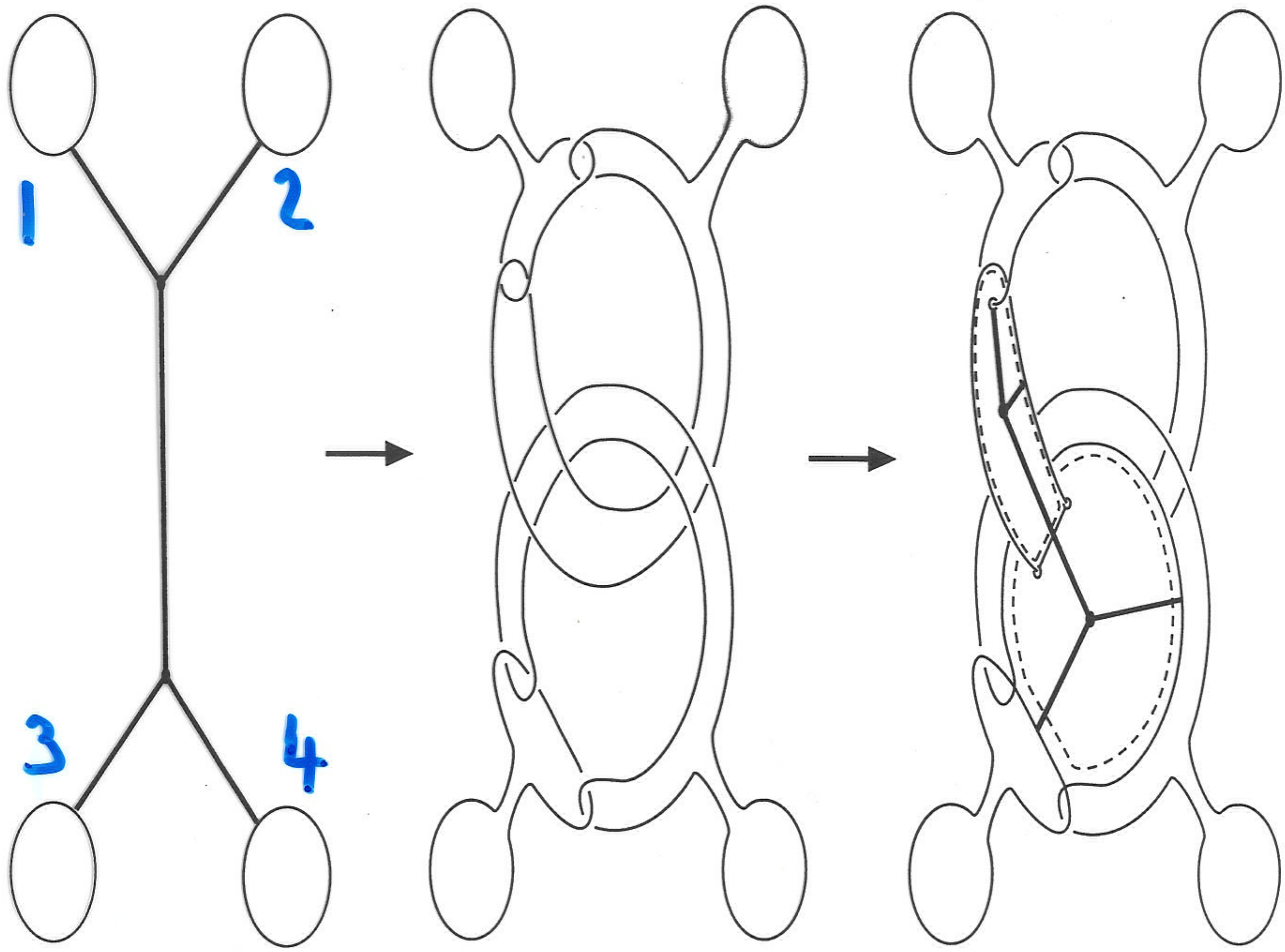
$$\tau_2(W) = \sum_{\substack{p \text{ intersection} \\ \text{point}}}$$

$$= \sum_{\substack{\text{components} \\ i,j}} \mu_2(i,j) \cdot$$

Remark: For general M^4, these intersection numbers can be equipped with group elements

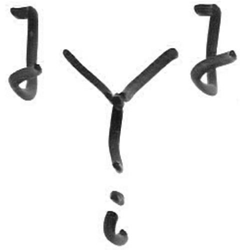
$$\rightsquigarrow \lambda(A_i, A_j) \in \mathbb{Z}\pi_1 M .$$

Corollary: Links with prescribed Milnor invariants are easy to see:



Bing - Cochran - Habiro links

$T_4(L_\sigma) = \sigma =$  $\in \mathcal{T}_4^{(4)}$

Levine's Conjecture: The map $\eta : T_n \longrightarrow \mathcal{D}_n$ has kernel generated by  , i.e. η is injective for even n.

"Corollary:" (under construction ....)

(a) $\mu_{\le 2k}(L) = 0 \iff$ in D^4, L bounds W. tower of class $2k$.

$\iff$ −〃− disjoint gropes −〃− .

(b) −〃− $2k+1$

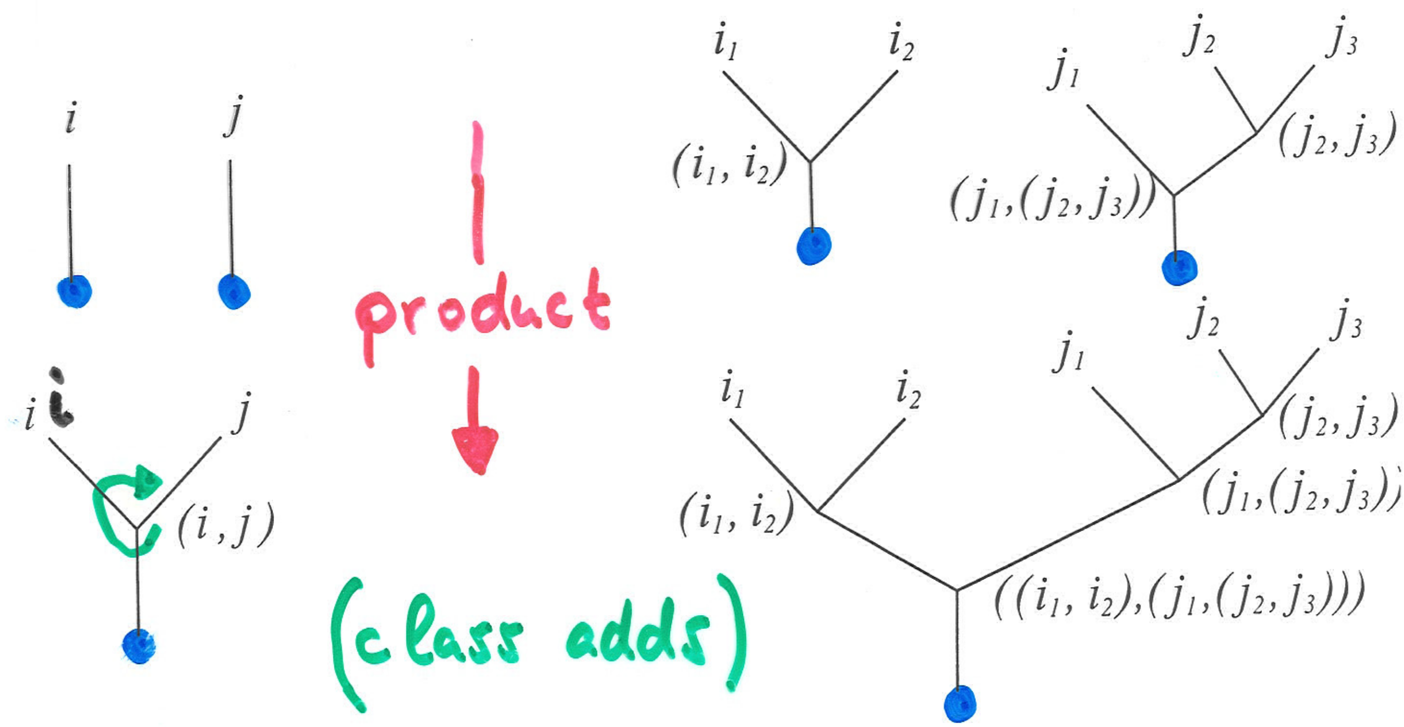
$\iff$ $\mu_{\le 2k+1}(L) = 0$ and an explicit list of $\mathbb{Z}/2$ - invariants vanish.

 $\hat{=}$ Arf(k_i)

Basics, Dim. 1

Def.: A **tree** is, for now, a uni-trivalent, contractible graph with vertex **orientations**, and **labels** from $\{1,2,\dots,m\}$ on the univalent vertices.

A rooted tree comes with a preferred univalent vertex •



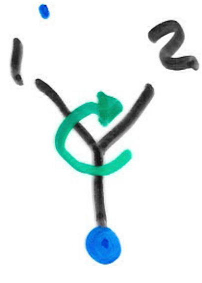
$\mathcal{M}_n$:= free abelian group on such trees (rooted)

n := number of tips (excluding root) =: class

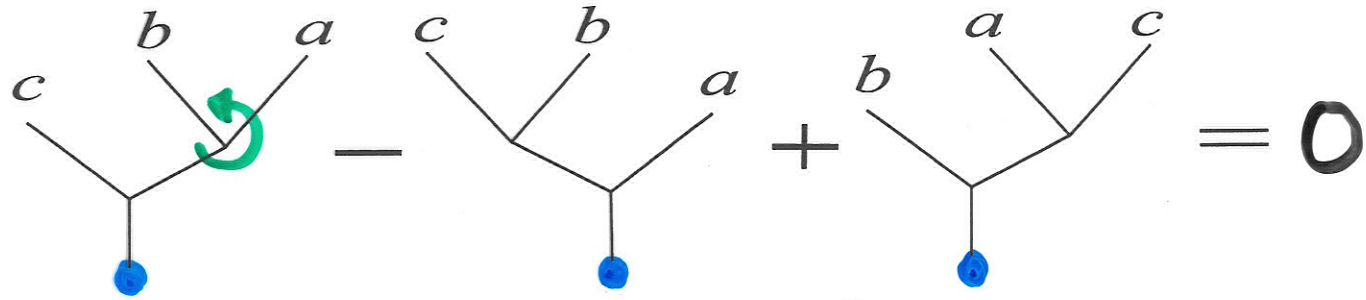
$\mathcal{M} = \bigoplus_{n \geq 0} \mathcal{M}_n$ is the <u>free algebra</u> (non-associative) on m generators.

$$\mathcal{L} := \mathcal{M} \Big/ \begin{array}{l} \text{antisymmetry AS} \\ \text{Jacobi identity IHX} \end{array}$$

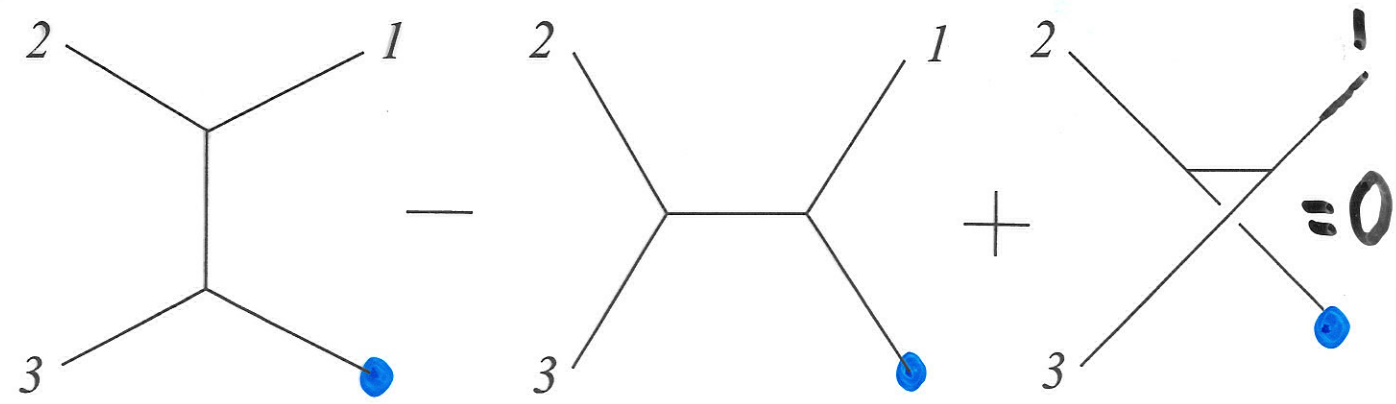
is the <u>free Lie algebra</u> on m generators.

$[x_1, x_2] =$  $\underset{AS}{=} -$ 

Jacobi Identity



$$[[a,b],c] - \underbrace{[a,[b,c]]}_{= -[[b,c],a]} + [[c,a],b] = 0$$



IHX- relation

Basics, Dim. 2

Algebraic Fact: Let F be the free group on m generators. Then

$$F_n / F_{n+1} \cong \mathcal{L}_n \quad \text{where}$$

$F_2 := [F,F]$, $F_{n+1} := [F, F_n]$

is the lower central series.

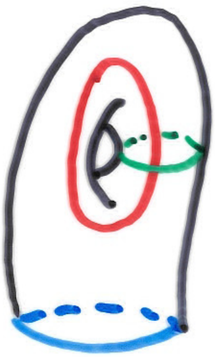
multiplication in F_n $\hat{=}$ addition in $\mathcal{L}_n$

group commutator in F_n $\hat{=}$ bracket in $\mathcal{L}_n$

$[a,b] := a \cdot b \cdot a^{-1} \cdot b^{-1}$ for

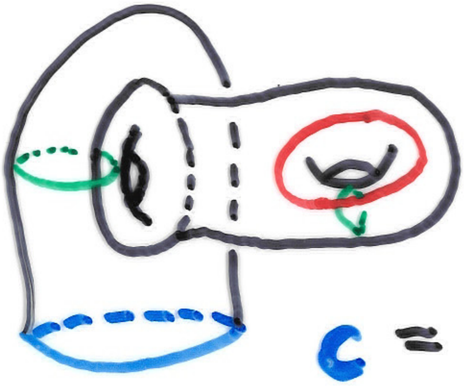
group elements a, b: group $\rightsquigarrow$ graded Lie alg.

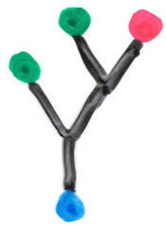
Abstract Gropes



height 1 & class 2

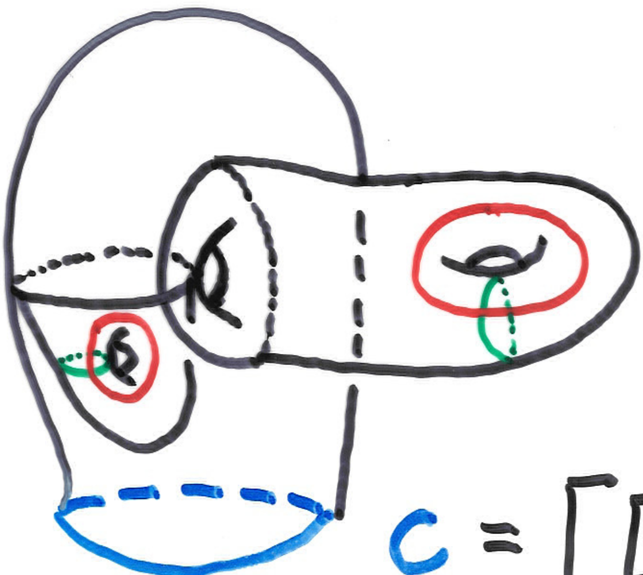
$c = [{\color{green}\bullet}, {\color{red}\bullet}] \in \pi^{(1)} = \pi_2 \trianglelefteq \pi$

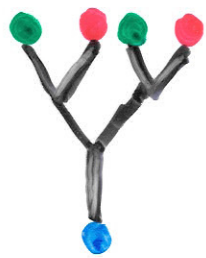




class 3

$c = [{\color{green}\bullet}, [{\color{green}\bullet}, {\color{red}\bullet}]] \in \pi_3$





height 2

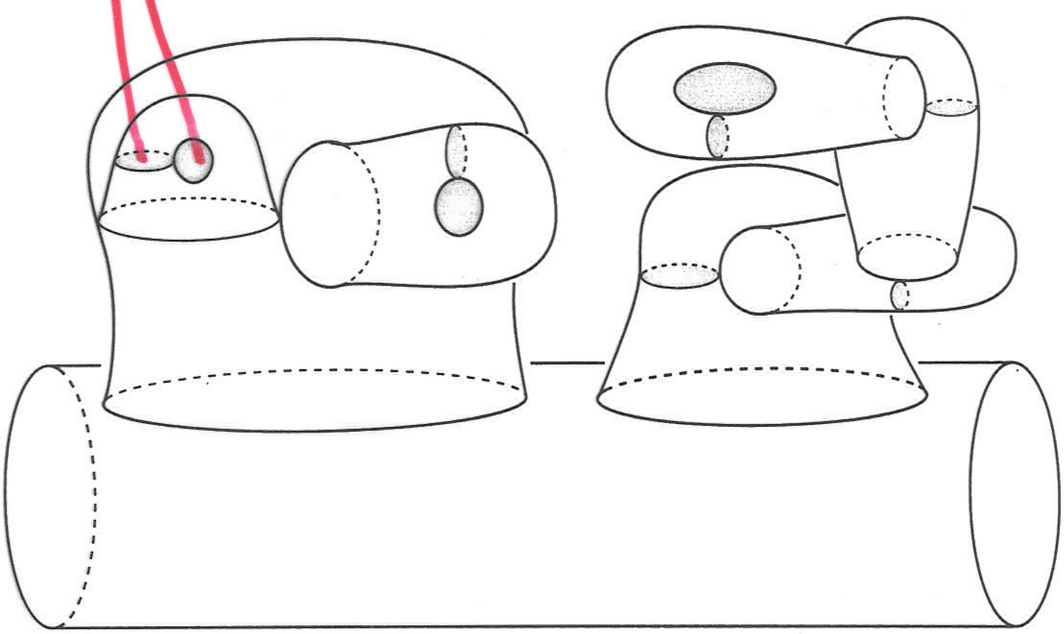
$c = [[{\color{green}\bullet}, {\color{red}\bullet}], [{\color{green}\bullet}, {\color{red}\bullet}]] \in \pi^{(2)}$

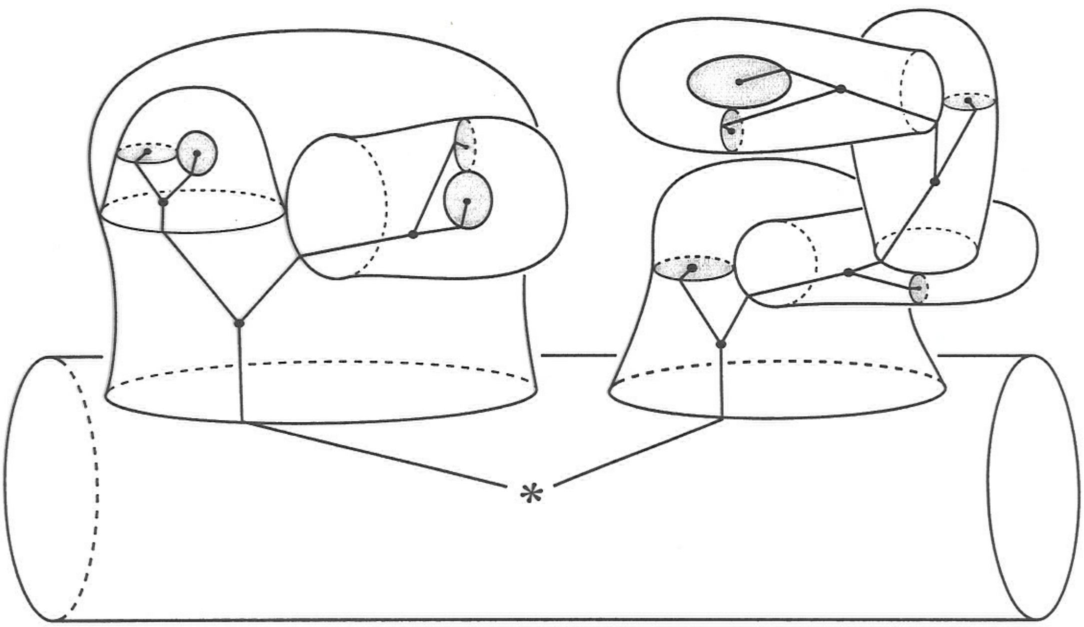
$\wedge\!\!\mid$

etc. class 4 π_4

Variations : • genus > 1,
• more ∂- components
(annulus-like grope)

caps



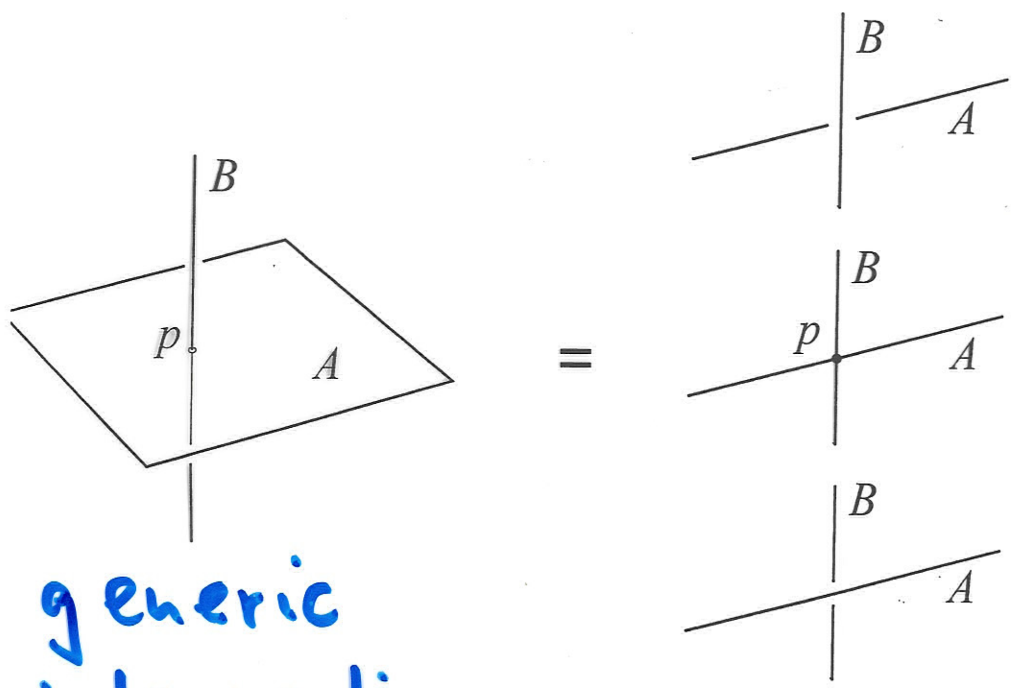


$$\mathrm{tree}(G) \subseteq \overset{\shortparallel}{G}$$

base point

surface of genus 1 $\rightsquigarrow$ trivalent $\overset{\shortparallel}{\text{vertex}}$

Basics, Dim. 4



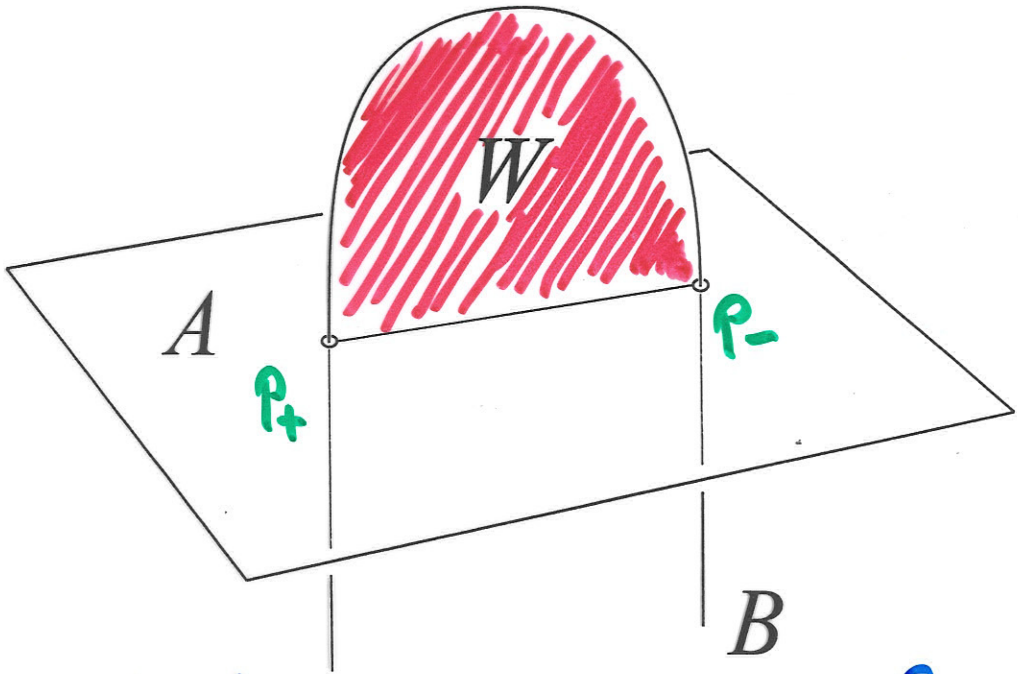
generic intersection

more symmetric movie.

$A \subseteq$ present

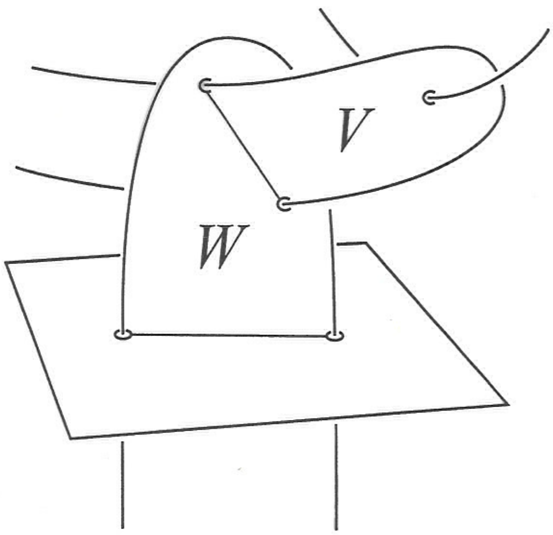
$B \subseteq \mathbb{R}^4$ continues into past & future

Whitney disks

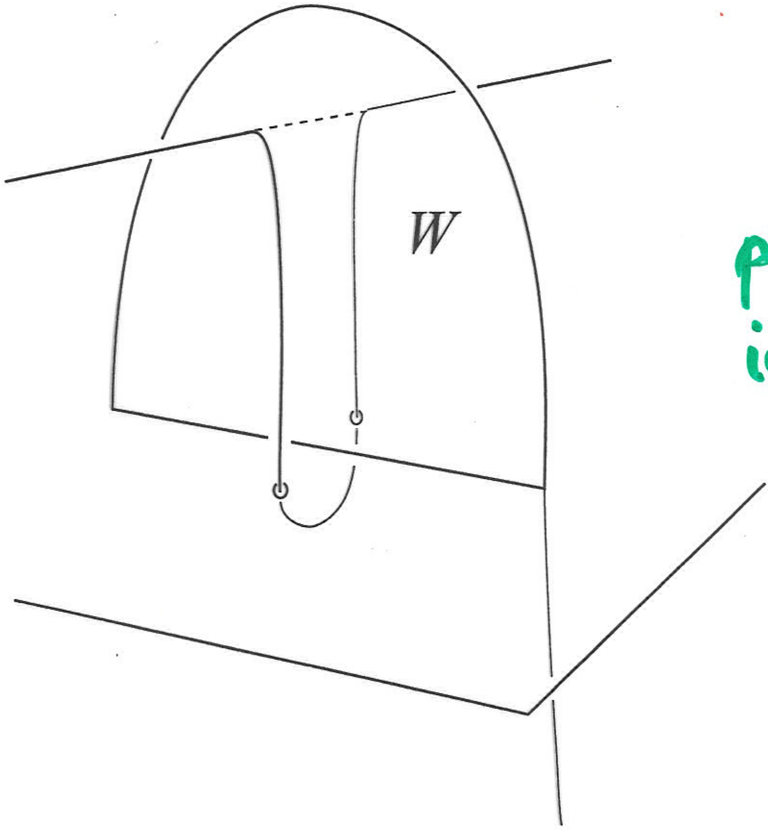


used to remove p_+ & p_-.

Problem: Other stuff can intersect W.

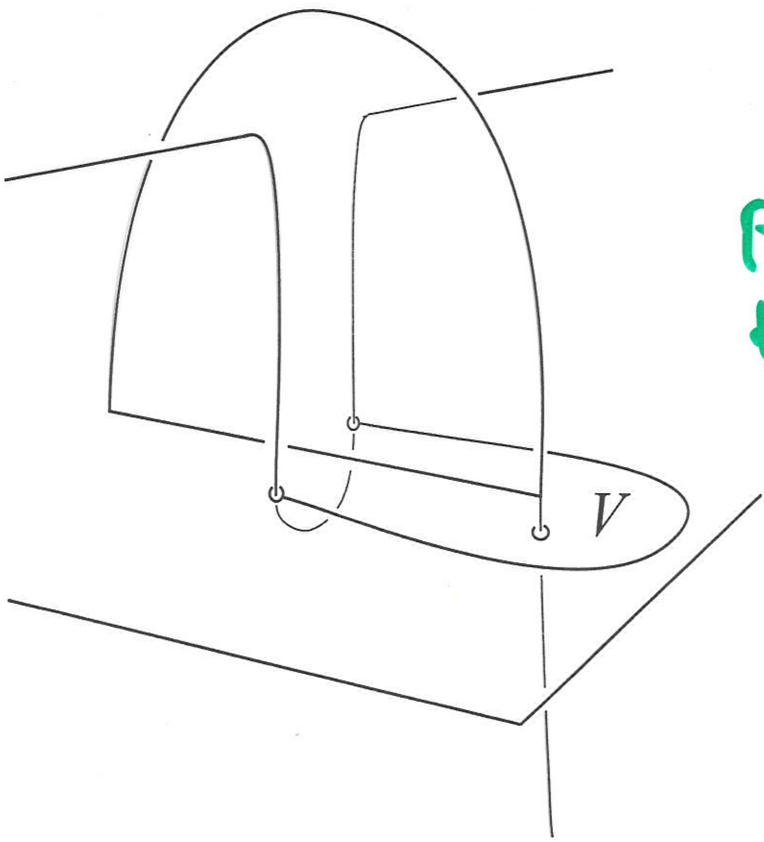


V = W. disk of order 2.

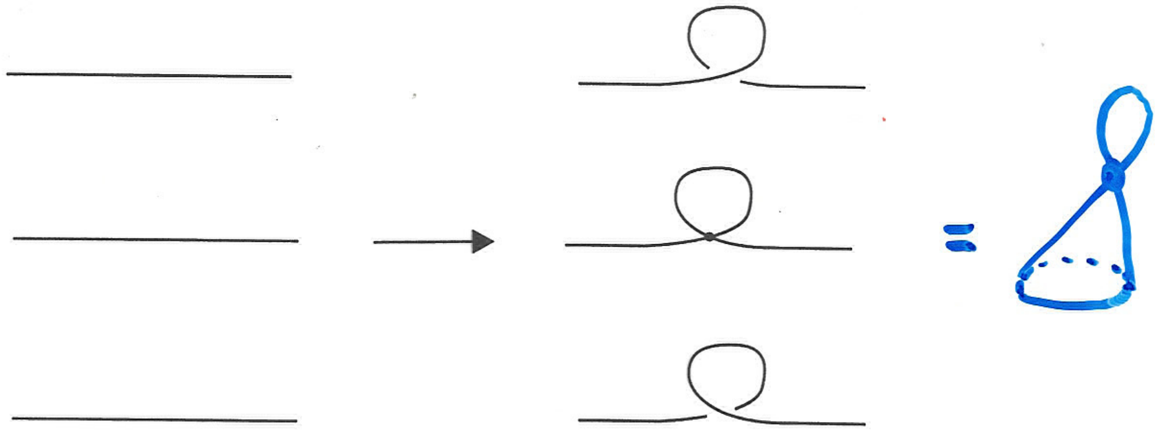


pushing intersections down

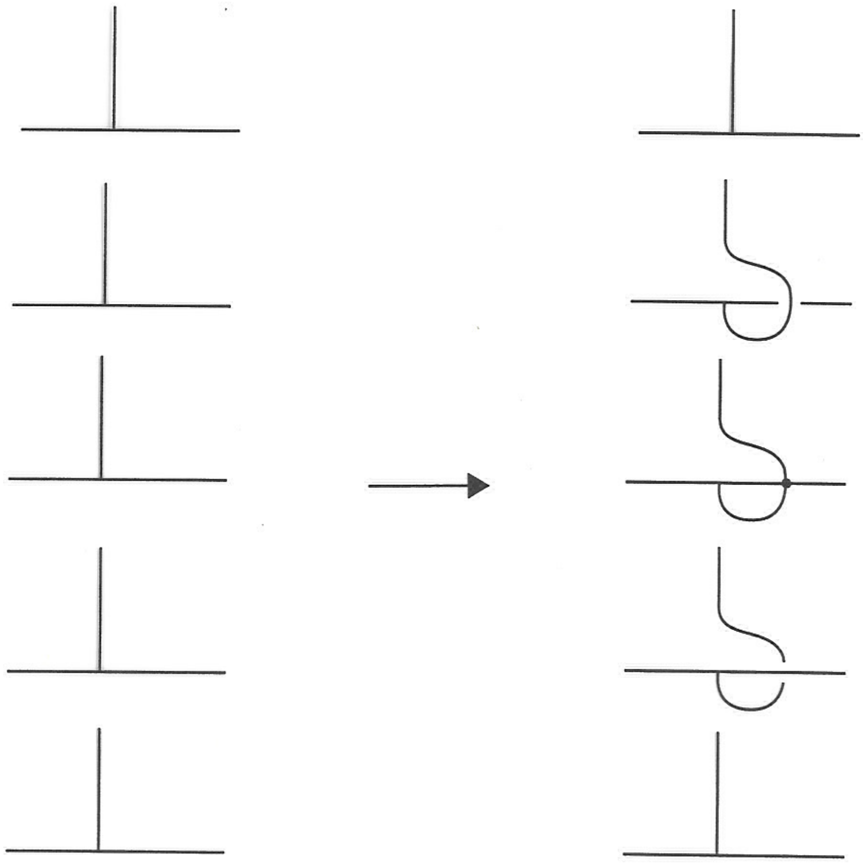
some simple manouvers....



pushing to the side



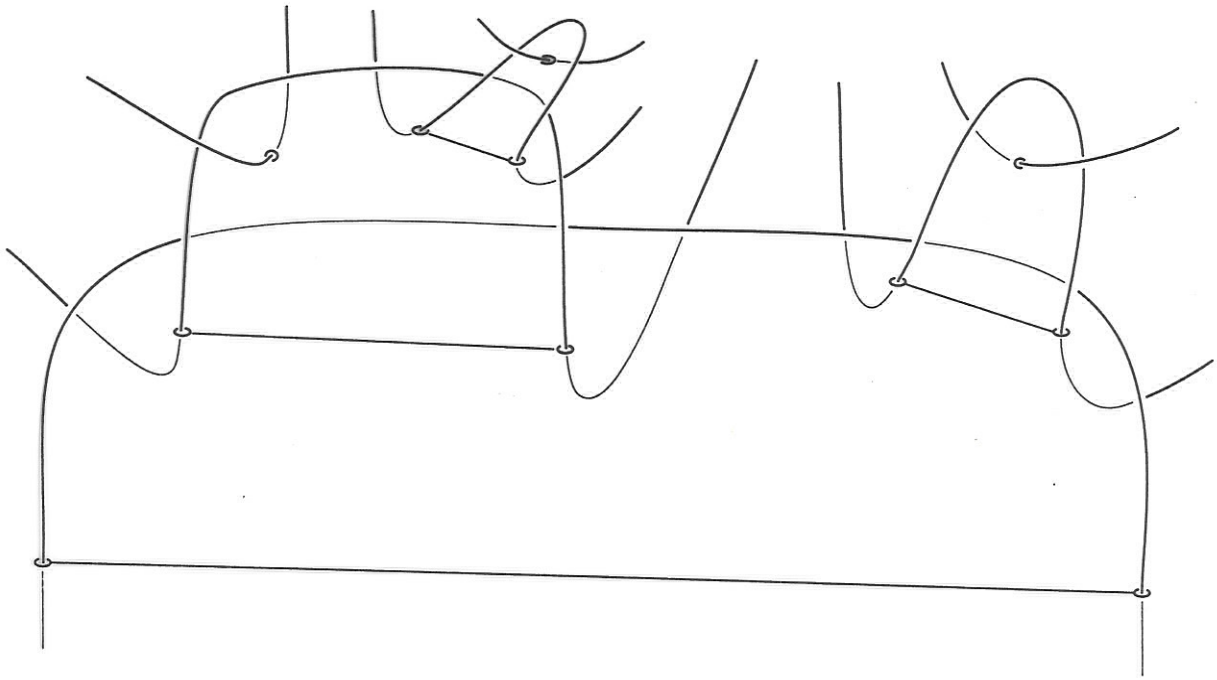
interior twist

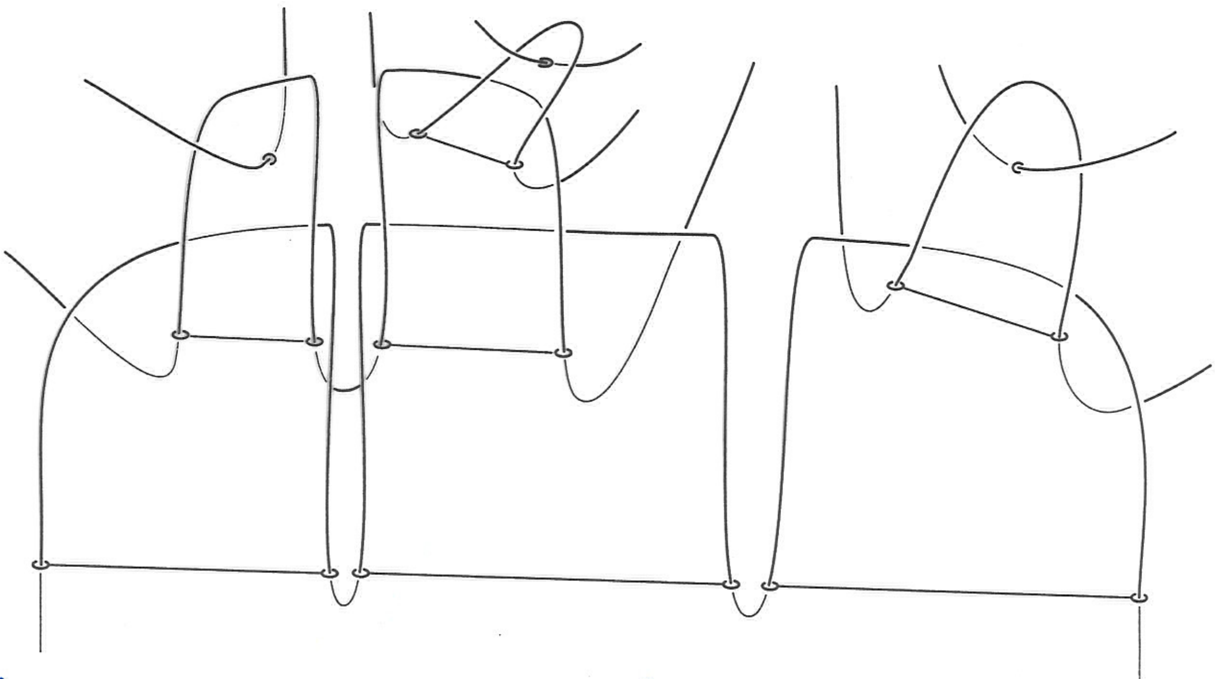


boundary twist

May assume that W.-disks are <u>framed</u>, <u>embedded</u>, <u>disjoint</u> <u>on ∂</u>.

Splitting a Whitney tower





May assume each W.-disk contains only one intersection point or arc!