On global existence and scattering for the wave maps equation

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Abstract

We prove global existence and scattering for the wave-maps equation in $n+1$ dimensions, $n = 2, 3$, for initial data which is small in the scale-invariant homogeneous Besov space $\dot{B}^{2,1}_{n/2} \times \dot{B}^{2,1}_{n/2-1}$. This result was proved in an earlier paper [15] for $n \geq 4$.

1 Introduction

Let (M, g) be a compact Riemmanian manifold without boundary. A wave map is a function from the Minkovski space $\mathbb{R}^n \times \mathbb{R}$ into M,

$$\phi : \mathbb{R}^n \times \mathbb{R} \to M,$$

which is locally a critical point for the functional

$$F(\phi) = \int_{\mathbb{R}^{n+1}} < \partial^i \phi, \partial_i \phi >_g dtdx \ .$$

Using local coordinates in M, the equations for ϕ can be written as

$$\Box \phi^\alpha + \Gamma^\alpha_{jk}(\phi) Q_0(\phi^j, \phi^k) = 0 \tag{1}$$

where the quadratic form Q_0 is given by

$$Q_0(u, v) = \partial^i u \partial_i v = -u_t v_t + u_x v_x.$$

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This quadratic form exhibits certain cancellation properties in estimates and is called a null-form. One simple way to see this cancellation is in the following decomposition of Q_0:

$$2Q_0(u,v) = \Box(uv) - u\Box v - v\Box u, \tag{2}$$

which will be used later in this article. As long as the solutions stay continuous it suffices to work in local coordinates; hence, in the sequel we shall work on the vector valued equation

$$\Box\phi + \Gamma(\phi)Q_0(\phi,\phi) = 0. \tag{3}$$

with Cauchy data at time $t = 0$,

$$\phi(0) = f_0, \quad \phi_t(0) = f_1 \tag{4}$$

Normally one chooses the initial data in a Sobolev space, $(f_0, f_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$. The solutions ϕ to (3) are dimensionless, i.e. the equation is invariant with respect to the transformation $\phi(t,x) \to \phi(\lambda t, \lambda x)$. Hence, within the above family of initial data spaces, the scale-invariant one is $\dot H^{\frac{n}{2}} \times \dot H^{\frac{n}{2}-1}$. For $s > \frac{n}{2}$ the equation (3) (4) was shown to be locally well-posed by Klainerman-Machedon [8] for $n \geq 3$ and later by Klainerman-Selberg [10] for $n = 2$. One should note that the difficulty decreases with the dimension, so the problem in 2 dimensions is the hardest. The special case $n = 1$ is also considered in [4].

On the global well-posedness side, global solutions were proved to exist for small smooth data decaying at infinity; see Klainerman [6] for $n \geq 3$, and Sideris [13] for related results, including the case $n = 2$. Recently counterexamples to global well-posedness for large data in dimension $n \geq 3$ were also found, see [12], [2]. In dimension $n = 2$ the conjecture is that there is no blow-up and the problem is globally well-posed even for large data (under reasonable assumptions on the target manifold). For spherically symmetric initial data this was proved in [3].

Consequently, the interesting remaining problem is to understand what happens in the scale invariant setting $s = \frac{n}{2}$. However, since $H^{\frac{n}{2}}$ does not embed into L^∞, the problem is not local with respect to the target space M; hence, any positive result should take into account the geometry of M. One way to avoid this difficulty is to substitute $\dot H^{\frac{n}{2}} \times \dot H^{\frac{n}{2}-1}$ by a Besov space $\dot B^{2,1}_{\frac{n}{2}} \times \dot B^{2,1}_{\frac{n}{2}-1}$ which is slightly smaller but has the same scaling. This space has the advantage that it embeds into L^∞, which makes the problem local and independent of the geometry.

In [15] we proved that in dimension $n \geq 4$ the wave maps equation is globally well-posed for initial data which is small in the above Besov space. Here we prove that the same result holds in dimensions $n = 2, 3$.

The main well-posedness result for the wave-maps equation is

Theorem 1. *a) There exist $C, D > 0$ so that for any initial data satisfying*

$$\|(f_0, f_1)\|_{\dot B^{2,1}_{\frac{n}{2}} \times \dot B^{2,1}_{\frac{n}{2}-1}} < C, \tag{5}$$

there exists a global solution ϕ to (3), (4) satisfying

$$\|(\phi(t), \phi_t(t))\|_{\dot{B}^{2,1}_{\frac{n}{2}} \times \dot{B}^{2,1}_{\frac{n}{2}-1}} \leq D,$$

which is the unique limit of smooth solutions. Furthermore, the solution depends smoothly on the initial data.

b) If we assume in addition that $(f_0, f_1) \in \dot{H}^s \times \dot{H}^{s-1}$, $s > \frac{n}{2}$, then

$$\|(\phi(t), \phi_t(t))\|_{\dot{H}^s \times \dot{H}^{s-1}} \leq c\|(f_0, f_1)\|_{\dot{H}^s \times \dot{H}^{s-1}}. \tag{6}$$

For large data we can take advantage of the finite speed of propagation and observe that the data is in effect small if restricted to a ball of sufficiently small radius. Consequently, we get local existence, but with a lifespan that fully depends on the initial data, and not only on its size.

Remark 1.1. *This result shows that in order for blowup to occur, a solution to the wave-maps equation needs to concentrate in a light cone with respect to the $\dot{B}^{2,1}_{\frac{n}{2}} \times \dot{B}^{2,1}_{\frac{n}{2}-1}$ norm.*

The scattering result is exactly what one would expect:

Theorem 2. *Given any sufficiently small initial data $(f_0, f_1) \in \dot{B}^{2,1}_{\frac{n}{2}} \times \dot{B}^{2,1}_{\frac{n}{2}-1}$ there exist unique functions $(f_0^+, f_1^+), (f_0^-, f_1^-) \in \dot{B}^{2,1}_{\frac{n}{2}} \times \dot{B}^{2,1}_{\frac{n}{2}-1}$ so that the solutions ϕ^+, ϕ^- to the homogeneous wave equation with initial data (f_0^+, f_1^+), respectively (f_0^-, f_1^-) satisfy*

$$\lim_{t\to\infty} \|\phi^+(t) - \phi(t)\|_{\dot{B}^{2,1}_{\frac{n}{2}}} + \|\phi_t^+(t) - \phi_t(t)\|_{\dot{B}^{2,1}_{\frac{n}{2}-1}} = 0$$

$$\lim_{t\to-\infty} \|\phi^-(t) - \phi(t)\|_{\dot{B}^{2,1}_{\frac{n}{2}}} + \|\phi_t^-(t) - \phi_t(t)\|_{\dot{B}^{2,1}_{\frac{n}{2}-1}} = 0$$

Furthermore, the map from (f_0, f_1) into (f_0^+, f_1^+), respectively (f_0^-, f_1^-) is a local diffeomorphism both in $\dot{B}^{2,1}_{\frac{n}{2}} \times \dot{B}^{2,1}_{\frac{n}{2}-1}$ and in $\dot{H}^s \times \dot{H}^{s-1}$, $s > \frac{n}{2}$.

To prove Theorem 1 we follow the same procedure as in [15]. We set up the problem as a fixed point problem and use a fixed point argument in appropriate function spaces. Below we sketch the fixed point argument and collect the properties which our function spaces are required to satisfy. The rest of the article is devoted to the construction of these spaces.

In [15] the function spaces are constructed by putting together the appropriate $X^{s,\theta}$ and "classical" solutions[1] to the inhomogeneous wave equation. This approach no longer works in dimensions 2 and 3. Instead, here we work with classical solutions to the inhomogeneous wave equation with respect to characteristic directions. Thus, in Section 3 we do some preparatory work on the wave equation in characteristic coordinates. In Section 4 we prove the multiplicative estimates for solutions to the homogeneous wave equation. In Section 5

[1] i.e. with $L^1(L^2)$ inhomogeneous term

we use a technique we call the "trace method" to extend this estimates to the $X^{s,\theta}$ spaces. In Section 6 we introduce the dyadic components of the function spaces F, F^s and we prove the appropriate multiplicative estimates. Finally, in Section 7 we put together the dyadic estimates and prove (iii),(iv),(iii)', (iv)' (stated below).

First set up the problem so that we can use a fixed point argument. Let V be the parametrix for the wave equation with zero Cauchy data at time 0, i.e. $u = Vf$ iff u solves

$$\Box u = f,\ u(0) = 0, u_t(0) = 0$$

Denote by ϕ_0 the solution to the wave equation with the same initial data as ϕ. Then the equation (3) can be rewritten as

$$\phi = \phi_0 + VN(\phi) \tag{7}$$

where N represents the nonlinear term in (3).

Suppose we want to use a fixed point argument in some translation invariant space F of distributions to solve (7). To start with, we need to know that $\phi_0 \in F$, i.e.

(i) F contains the solutions to the homogeneous wave equation with $\dot{B}^{2,1}_{n/2} \times \dot{B}^{2,1}_{n/2-1}$ initial data.

Then we want the solutions we get to have the right regularity, i.e.

(ii) $F \subset C(\dot{B}^{2,1}_{n/2}) \cap \dot{C}^1(\dot{B}^{2,1}_{n/2-1})$.

Finally, we want VN to map F into F. We will split this into two. Define the space $\Box F$ as $\Box$ applied to functions in F, with the induced norm. Then the properties (i), (ii) immediately imply that

$$V : \Box F \to F$$

Hence we still need to know that N maps F into $\Box F$. By (2) the nonlinearity N has the form

$$N(u) = f(u)\Box u + g(u)\Box u^2$$

This special nonlinearity has the correct mapping properties if

(iii) F is an algebra.

(iv) $F \cdot \Box F \subset \Box F$.

If the properties (i)-(iv) above hold, then the fixed point argument yields a unique solution for small initial data. To study solutions with more regular initial data one needs to find a similar space F^s with the following properties:

(i)' F^s contains the solutions to the homogeneous wave equation with $\dot{H}^s \times \dot{H}^{s-1}$ initial data.

(ii)' $F^s \subset C(\dot{H}^s) \cap \dot{C}^1(\dot{H}^{s-1})$.

(iii)' $F \cap F^s$ is an algebra.

Using a simple scaling argument, one can show that this is equivalent to the estimate

$$\|uv\|_{F^s} \leq c(\|u\|_F \|v\|_{F^s} + \|u\|_{F^s} \|v\|_F) \tag{8}$$

(iv)' $(F \cap F^s) \cdot \Box(F \cap F^s) \subset \Box(F \cap F^s)$.

Given (iv), this is equivalent to the estimate

$$\|uv\|_{\Box F^s} \leq c(\|u\|_F\|v\|_{\Box F^s} + \|u\|_{F^s}\|v\|_{\Box F}) \tag{9}$$

The proof of Theorem 1 is concluded if we find spaces F, F^s with the above properties. This is done in the following sections.

The scattering result, in this setup, is a straightforward consequence of our choice of spaces. The group $S(t)$ associated to the homogeneous wave equation

$$S(t)(u(0), u_t(0)) = (u(t), u_t(t)), \qquad \Box u = 0$$

is given by

$$S(t) = \frac{1}{2}Q^{-1}\begin{pmatrix} e^{it|D|} & 0 \\ 0 & e^{-it|D|} \end{pmatrix} Q, \qquad Q = \begin{pmatrix} i|D| & 1 \\ -i|D| & 1 \end{pmatrix}$$

Hence, for an arbitrary function u, the limit of $S(-t)(u(t), u_t(t))$ exists in $\dot{B}^{2,1}_{n/2} \times \dot{B}^{2,1}_{n/2-1}$, respectively $\dot{H}^s \times \dot{H}^{s-1}$ iff the following four limits exist in $\dot{B}^{2,1}_{n/2-1}$, respectively $\dot{H}^{s-1}$:

$$\lim_{t\to\pm\infty} e^{-it|D|}(i|D|u(t) - u_t(t)), \quad \lim_{t\to\pm\infty} e^{it|D|}(i|D|u(t) + u_t(t)) \tag{10}$$

Using Proposition 6.2 we shall prove that these limits exist for all functions in F, respectively F^s. Since the fixed point argument yields a solution which depends smoothly on the initial data, it follows that the maps from (f_0, f_1) into (f_0^+, f_1^+), respectively (f_0^-, f_1^-), are smooth. Since the nonlinearity is quadratic, the linearisation of both maps at 0 is the identity, therefore they are local diffeomorphisms in the appropriate spaces.

2 Notations

Coordinates. We denote by $\mathcal{X} = (t, x)$ the usual coordinates in $\mathbb{R} \times \mathbb{R}^n$ and by $\Xi = (\tau, \xi)$ the corresponding Fourier variable. The symbol of $\Box$ is

$$\Box(\Xi) = \tau^2 - \xi^2.$$

Its characteristic set is the cone

$$K = \{\tau^2 = \xi^2\}$$

Alternatively we introduce characteristic coordinates with respect to arbitrary characteristic directions. Given a length 1 covector Θ on the cone K,

$$\Theta = \frac{1}{\sqrt{2}}(1, \zeta), \qquad |\zeta| = 1$$

we define the associated orthogonal coordinates $(t_\Theta, x_\Theta) = (t_\Theta, x_\Theta^1, x'_\Theta)$ by

$$t_\Theta = \mathcal{X} \cdot \Theta = \frac{1}{\sqrt{2}}(t + x \cdot \zeta), \qquad x_\Theta^1 = \frac{1}{\sqrt{2}}(t - x \cdot \zeta)$$

The corresponding Fourier variables are called $(\tau_\Theta, \xi_\Theta) = (\tau_\Theta, \xi_\Theta^1, \xi'_\Theta)$ and the symbol of $\Box$ has the form

$$\Box(\Xi) = 2\tau_\Theta\xi_\Theta^1 - {\xi'_\Theta}^2$$

Sets. Now we describe the way we partition the Fourier space. Start with an overlapping dyadic decomposition,

$$A_\lambda = \{\frac{\lambda}{4} \leq |\Xi| \leq 4\lambda\}$$

and the corresponding balls

$$\tilde{A}_\lambda = \{|\Xi| \leq 4\lambda\}$$

A second decomposition is based on the size of the symbol of $\Box$,

$$B_\nu = \{\frac{\nu}{4} \leq |\Box(\Xi)| \leq 4\nu\}$$

respectively

$$\tilde{B}_\nu = \{|\Box(\Xi)| \leq 4\nu\}$$

Combining these two partitions yields a decomposition of the sets A_λ with respect to the distance to the cone. More precisely, for $\mu \leq \lambda$ the sets

$$A_{\lambda,\mu} = A_\lambda \cap B_{\lambda\mu}, \qquad \tilde{A}_{\lambda,\mu} = A_\lambda \cap \tilde{B}_{\lambda\mu}$$

represent the dyadic region at frequency λ and distance μ from the cone, respectively the region at frequency λ and *within* distance μ from the cone.

Finally, the third partition we consider is a conical partition with respect to the angular variable. To decide what is the right scale for this partition, observe that a sector on the cone of angle α fits between two hyperplanes at angle $O(\alpha^2)$. Consequently, given α we consider an overlapping partition of an α^2 conical neighborhood of the cone into roughly α^{1-n} angle α sectors C_α^j, which have angular dimension $\alpha^{n-1} \times \alpha^2$:

$$\{\angle(\Xi, K) \leq \alpha^2\} = \bigcup_{j \in J_\alpha} C_\alpha^j$$

Here $|J_\alpha| \approx \alpha^{1-n}$ and for $i, j \in J_\alpha$ we define

$$|i - j| := [\alpha^{-1} \cdot \text{angular distance}(C_\alpha^i, C_\alpha^j)]$$

The intersection of such an α-sector with a dyadic region A_λ is denoted by

$$A_{\lambda,\alpha}^j = A_\lambda \cap C_\alpha^j$$

and has roughly the shape of a parallelepiped of size $\lambda \times (\alpha\lambda)^{n-1} \times \alpha^2\lambda$.

Multipliers. Here we define multipliers corresponding to the sets defined above. Given $s \in C_0^\infty((1/2, 2))$ so that

$$\sum_{j \in \mathbb{Z}} s\left(\frac{x}{2^j}\right) = 1$$

set

$$\tilde{s}(x) = \sum_{j \leq 0} s\left(\frac{x}{2^j}\right).$$

Then define the standard Paley-Littlewood operators S_λ, $\tilde{S}_\lambda$ with symbols

$$s_\lambda = s\left(\frac{|\Xi|}{\lambda}\right), \quad \tilde{s}_\lambda = \tilde{s}\left(\frac{|\Xi|}{\lambda}\right)$$

supported in A_λ, respectively $\tilde{A}_\lambda$. We also need truncation operators with respect to the distance to the characteristic cone K. Start with the operators $Q_\nu, \tilde{Q}_\nu$ with symbols

$$q_\nu(\Xi) = s(\frac{\Box(\Xi)}{\nu}), \qquad \tilde{q}_\nu(\Xi) = \tilde{s}(\frac{\Box(\Xi)}{\nu})$$

supported in B_ν, respectively $\tilde{B}_\nu$. Then the operators

$$S_{\lambda,\mu} = S_\lambda Q_{\lambda\mu}, \qquad \tilde{S}_{\lambda,\mu} = S_\lambda \tilde{Q}_{\lambda\mu}$$

have symbols supported in $A_{\lambda,\mu}$, respectively $\tilde{A}_{\lambda,\mu}$.

Finally, corresponding to the decomposition of an α^2 conical neighborhood of the cone K into α-sectors we consider a decomposition on the operator level,

$$\tilde{s}(4\alpha^{-2}\angle(\Xi, K)) = \sum_{j \in J_\alpha} R_\alpha^j$$

where for each j the symbol r_α^j of the multiplier R_α^j is supported in the α-sector C_α^j. Localizing these multipliers to frequency λ we define the operators

$$S_{\lambda,\alpha}^j = S_\lambda R_\alpha^j$$

whose symbol is supported in $A_{\lambda,\alpha}^j$.

Function spaces. Denote by $\overset{\circ}{X}{}^s$ the space of solutions to the homogeneous wave equation with $\dot{H}^s \times \dot{H}^{s-1}$ initial data. The Fourier transform of such functions are weighted L^2 distributions supported on the characteristic cone. The homogeneous $X^{s,\theta}$ spaces are multiplier weighted L^2 spaces, with norms

$$\|u\|_{s,\theta} = \|\widehat{u}(\tau,\xi)||\xi| + |\tau||^s||\tau| - |\xi||^\theta\|_{L^2}$$

Inhomogeneous versions of these spaces have been introduced earlier in the study of the KdV equation in [1, 5], nonlinear Schröedinger in [1] and the wave equation in [7].

Next we want to define dyadic counterparts of these spaces. Within a given dyadic region A_λ the index s is superfluous since it only adds a factor of λ^{-s} to the norm. Hence we dispense with it and set

$$\overset{\circ}{X}_\lambda = \{u \in \overset{\circ}{X}{}^0;\ \hat{u} \text{ is supported in } A_\lambda\}$$

$$X_\lambda^\theta = \{u \in X^{0,\theta};\ \hat{u} \text{ is supported in } A_\lambda\}$$

However, the spaces X_λ^θ are not good enough for our purposes. Consequently, we modify them by changing the way we do the summation with respect to dyadic pieces relative to the distance to the cone. Thus define the spaces $X_\lambda^{\theta,p}$ of functions with Fourier transform supported in A_λ and norm

$$\|u\|^p_{X_\lambda^{\theta,p}} = \sum_{\mu \ dyadic} \mu^{\theta p} \|Q_{\lambda\mu} u\|^p_{L^2}$$

These spaces are well-defined as spaces of distributions only if $\theta < \frac{1}{2}$ or $(\theta, p) = (\frac{1}{2}, 1)$. It is the latter case we are interested in. Observe that the functions in the space $X_\lambda^{\frac{1}{2},1}$ are uniquely defined modulo $\overset{\circ}{X}_\lambda$, i.e. modulo L^2 solutions to the homogeneous wave equation. This ambiguity is fixed if we think of $X_\lambda^{\frac{1}{2},1}$ functions as the sum of their dyadic parts and include $\overset{\circ}{X}_\lambda$ as one such (limiting) dyadic part.

3 Characteristic coordinates and energy estimates

Given a characteristic covector Θ we use the characteristic coordinates $(t_\Theta, x^1_\Theta, x'_\Theta)$ introduced in the previous section. The Fourier variable is denoted by $(\tau_\Theta, \xi^1_\Theta, \xi'_\Theta)$ and the corresponding differentiation operators are $D_{t_\Theta}, D_{x^1_\Theta}, D_{x'_\Theta}$.

Our first result shows how to measure the L^2 solutions for the wave equation with respect to characteristic coordinates.

Proposition 3.1. *Let $u \in \overset{\circ}{X}{}^0$ be an L^2 solution to the homogeneous wave equation. Then $\frac{D_{x^1_\Theta}}{|D_{x_\Theta}|} u \in L^\infty(L^2)$ and*

$$\|u\|_{\overset{\circ}{X}{}^0} \approx \|\frac{D_{x^1_\Theta}}{|D_{x_\Theta}|} u\|_{L^\infty_{t_\Theta}(L^2_{\xi_\Theta})}$$

Proof. If $u \in \overset{\circ}{X}{}^0$ then we can represent its Fourier transform as

$$\hat{u} = f d\sigma, \qquad f \in L^2(d\sigma)$$

where $d\sigma$ is the surface measure on the characteristic cone K. We need to write $d\sigma$ in terms of $d\xi_\Theta$. The cone has the equation

$$\tau_\Theta = \frac{|\xi'_\Theta|^2}{2\xi^1_\Theta}$$

therefore

$$d\sigma = (1 + \frac{|\xi'_\Theta|^2}{{\xi^1_\Theta}^2} + \frac{|\xi'_\Theta|^4}{4{\xi^1_\Theta}^4})^{\frac{1}{2}} d\xi_\Theta = \frac{2|\xi^1_\Theta|^2 + |\xi'_\Theta|^2}{2|\xi^1_\Theta|^2} d\xi_\Theta \tag{11}$$

Hence the Fourier transform of the trace of u at time t_Θ satisfies

$$|\hat{u}(t_\Theta, \xi_\Theta)| = f(\xi_\Theta)\frac{2|\xi^1_\Theta|^2 + |\xi'_\Theta|^2}{2|\xi^1_\Theta|^2}$$

which, by (11), further gives

$$\|(\frac{2|\xi^1_\Theta|^2 + |\xi'_\Theta|^2}{2|\xi^1_\Theta|^2})^{-\frac{1}{2}}\hat{u}(t_\Theta, \xi_\Theta)\|_{L^2_{\xi_\Theta}} = \|f\|_{L^2(d\sigma)}$$

$\triangle$

To look at solutions to the inhomogeneous wave equation we define the forward parametrix $\Box_f^{-1}$ for the wave equation, which is a multiplier with symbol

$$\Box_f^{-1}(\tau, \xi) = \frac{1}{\Box(\tau + i0, \xi)}$$

Due to the choice of the characteristic coordinates, the symbol of $\Box_f^{-1}$ in these coordinates is simply

$$\Box_f^{-1}(\tau_\Theta, \xi_\Theta) = \frac{1}{\Box(\tau_\Theta + i0, \xi_\Theta)} = \frac{1}{2\tau_\Theta\xi^1_\Theta - (\xi'_\Theta)^2}$$

Then its kernel is defined by

$$K(t_\Theta, \xi_\Theta) = \frac{i}{2\xi^1_\Theta}e^{it_\Theta\frac{(\xi'_\Theta)^2}{2\xi^1_\Theta}}\chi_{\{t_\Theta \geq 0\}}$$

Consequently, we obtain the representation formula

$$\widehat{(\Box_f^{-1}f)}(t_\Theta, \xi_\Theta) = \int_{-\infty}^{t_\Theta}\frac{i}{2\xi^1_\Theta}e^{i(t_\Theta - \bar{t}_\Theta)\frac{(\xi'_\Theta)^2}{2\xi^1_\Theta}}\hat{f}(\bar{t}_\Theta, \xi_\Theta)d\bar{t}_\Theta \tag{12}$$

whenever the right hand side integral is well defined. To avoid distracting technicalities, here and in the sequel we assume that f is supported away from $\{\xi^1_\Theta = 0\}$. In particular this yields

Proposition 3.2. *The following estimate holds for* $\Box_f^{-1}$*:*

$$\|\frac{D_{x^1_\Theta}}{|D_{x'_\Theta}|}\Box_f^{-1}f\|_{L^\infty_{t_\Theta}(L^2_{x_\Theta})} \lesssim \||D_{x'_\Theta}|^{-1}f\|_{L^1_{t_\Theta}(L^2_{x_\Theta})}$$

Later we will not need the global form of these energy estimates. Instead we need frequency localized versions of these estimates. More precisely, we consider functions which are frequency localized in an α-sector $A^j_{\lambda,\alpha}$ which is at angle α from Θ. Then for $(\tau_\Theta, \xi_\Theta) \in \tilde{A}^j_{\lambda,\alpha}$ we estimate the weights in the estimates. For small α we clearly get $|\tau_\Theta| \approx \lambda$. Then in a section $\tau_\Theta = const$ the geometry looks like in the following picture:

Hence

$$|\xi^1_\Theta| \approx \alpha^2\lambda, \qquad |\xi'_\Theta| \approx \alpha\lambda$$

Thus, we obtain

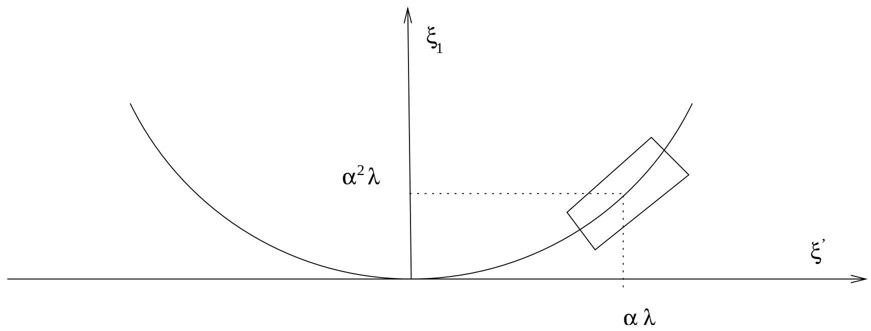

Figure 1: Section of an α sector at angle α from a characteristic direction.

Corollary 3.3. *Let u, f be functions whose Fourier transform is supported in an α sector $A^j_{\lambda,\alpha}$ at angle α with respect to Θ. Then*

a) If u is an L^2 solution to the homogeneous wave equation then $u \in L^\infty_{t_\Theta}(L^2_{x_\Theta})$ and

$$\|u\|_{\overset{\circ}{X}{}^0} \approx \alpha \|u\|_{L^\infty_{t_\Theta}(L^2_{x_\Theta})}$$

b) If u is the forward solution for $\Box u = f$, i.e. $u = \Box_f^{-1} f$, then

$$\alpha \|u\|_{L^\infty_{t_\Theta}(L^2_{x_\Theta})} \lesssim (\alpha\lambda)^{-1} \|f\|_{L^1_{t_\Theta}(L^2_{x_\Theta})}$$

Another consequence of the same arguments is

Proposition 3.4. *Let u be a solution to $\Box u = f$ whose Fourier transform is supported in an α sector $A^j_{\lambda,\alpha}$ at angle α with respect to Θ. Then u can be represented as*

$$u(t_\Theta, x_\Theta) = v + \int_{-\infty}^{\infty} u_{\bar t_\Theta}(t_\Theta, x_\Theta) \chi_{\{t_\Theta \geq \bar t_\Theta\}} d\bar t_\Theta$$

where v, $u_{\bar t_\Theta}$ are L^2 solutions to the homogeneous wave equation, frequency localized in an enlargement of $A^j_{\lambda,\alpha}$, and satisfying

$$\|v\|_{\overset{\circ}{X}_\lambda} + \int_{-\infty}^{\infty} \|u_{\bar t_\Theta}\|_{L^1_{\bar t_\Theta}(\overset{\circ}{X}_\lambda)} d\bar t_\Theta \lesssim (\alpha\lambda)^{-1} \|f\|_{L^1_{t_\Theta}(L^2_{x_\Theta})} + \alpha \|u\|_{L^\infty_{t_\Theta}(L^2_{x_\Theta})} \tag{13}$$

Here χ stands for the characteristic function of the indicated set.

In other words, this says that the solutions u to $\Box u = f$ which are frequency localized in an α-sector A^j_α can be represented as a superposition of L^2 solutions for the homogeneous wave equation, truncated across the hyperplanes $t_\Theta = \mathit{const}$. This simple observation will be very useful later on, in order to reduce estimates for solutions to the inhomogeneous wave equation to the corresponding estimates for solutions to the homogeneous wave equation.

Proof. Set

$$v = u - \Box_f^{-1} f$$

Then v solves the homogeneous wave equation and, by Corollary 3.3,

$$\begin{aligned}\|v\|_{\overset{\circ}{X}_\lambda} &\approx \alpha\|v\|_{L^\infty_{t_\Theta}(L^2_{x_\Theta})} \\ &\lesssim \alpha\|u\|_{L^\infty_{t_\Theta}(L^2_{x_\Theta})} + (\alpha\lambda)^{-1}\|f\|_{L^1_{t_\Theta}(L^2_{x_\Theta})}\end{aligned}$$

By (12), on the other hand, the function $\Box_f^{-1} f$ can be represented as

$$\Box_f^{-1} f = \int_{-\infty}^{\infty} u_{\bar{t}_\Theta}(t_\Theta, x_\Theta)\chi_{\{t_\Theta \geq \bar{t}_\Theta\}} d\bar{t}_\Theta$$

where

$$\widehat{u_{\bar{t}_\Theta}}(t_\Theta, \xi_\Theta) = \frac{i}{2\xi^1_\Theta} e^{i(t_\Theta - \bar{t}_\Theta)\frac{(\xi'_\Theta)^2}{2\xi^1_\Theta}} \hat{f}(\bar{t}_\Theta, \xi_\Theta)$$

Then $u_{\hat{t}_\Theta}$ is supported in the τ_Θ projection of A^j_α on the cone and satisfies

$$\begin{aligned}\|u_{\bar{t}_\Theta}\|_{\overset{\circ}{X}_\lambda} &\approx \alpha\|u_{\bar{t}_\Theta}\|_{L^\infty_{t_\Theta}(L^2_{x_\Theta})} \\ &\lesssim (\alpha\lambda)^{-1}\|f(\bar{t}_\Theta)\|_{L^2_{x_\Theta}}.\end{aligned}$$

Thus (13) follows. △

Next we relate these frequency localized solutions to the inhomogeneous wave equation to the $X_\lambda^{\frac{1}{2},\infty}$ spaces.

Proposition 3.5. *Let u be a function solving $\Box u = f$ whose Fourier transform is supported in an α-sector $A^j_{\lambda,\alpha}$ at angle α with respect to Θ. Then*

$$\|u\|_{X_\lambda^{\frac{1}{2},\infty}} \lesssim (\alpha\lambda)^{-1}\|f\|_{L^1_{t_\Theta}(L^2_{x_\Theta})} \tag{14}$$

Proof. Compute

$$|\hat{u}(\tau_\Theta, \xi_\Theta)| = \frac{|\hat{f}(\tau_\Theta, \xi_\Theta)|}{|\Box(\tau_\Theta, \xi_\Theta)|} \leq \frac{|g(\xi_\Theta)|}{|\Box(\tau_\Theta, \xi_\Theta)|}$$

where

$$g(\xi_\Theta) = \int_{-\infty}^{\infty} |\hat{f}(t_\Theta, \xi_\Theta)| dt$$

satisfies

$$\|g\|_{L^2_{\xi_\Theta}} \lesssim \|f\|_{L^1_{t_\Theta}(L^2_{x_\Theta})}$$

Then it remains to show that

$$d^{\frac{1}{2}}\|q_{\lambda d}(\tau_\Theta, \xi_\Theta)\frac{|g(\xi_\Theta)|}{|\Box(\tau_\Theta, \xi_\Theta)|}\|_{L^2} \lesssim (\alpha\lambda)^{-1}\|g\|_{L^2_{\xi_\Theta}}$$

uniformly in $d > 0$. (Here d stands for the distance to the cone.) This reduces to the following estimate for fixed ξ_Θ:

$$\int_{|\Box(\tau_\Theta,\xi_\Theta)|\approx d\lambda} \frac{d}{\Box^2(\tau_\Theta,\xi_\Theta)} d\tau_\Theta \lesssim (\alpha\lambda)^{-2}$$

The dyadic interval $\{\Box(\tau_\Theta,\xi_\Theta) \approx d\lambda\}$ has size $d\lambda {\xi_\Theta^1}^{-1}$ (in τ_Θ) therefore the above estimate is equivalent to

$$d\lambda {\xi_\Theta^1}^{-1} \frac{d}{\lambda^2 d^2} \lesssim (\alpha\lambda)^{-2}$$

and further to

$$\alpha^2\lambda \lesssim \xi_\Theta^1$$

which is true within the α sector $A_{\lambda,\alpha}^j$.

We conclude this section with a result which shows that the $X^{s,\theta}$ spaces behave well with respect to truncation across characteristic hyperplanes.

Proposition 3.6. *Let $\Theta \in char\ \Box$. Then*

$$\chi_{\{\mathcal{X}\cdot\Theta>\gamma\}} \cdot X^{s,\theta} \subset X^{s,\theta}$$

for $|s|, |\theta|, |s-\theta| < \frac{1}{2}$.

Proof. Without any restriction in generality we can take $\gamma = 0$. The weight corresponding to $X^{s,\theta}$ in our coordinates is

$$(|\tau_\Theta| + |\xi_\Theta|)^{s-\theta} |2\tau_\Theta\xi_\Theta^1 - {\xi_\Theta'}^2|^\theta$$

and the truncation function is $\chi_{\{t_\Theta \geq 0\}}$. We can take the Fourier transform in ξ_Θ and reduce the problem to the one-dimensional problem for fixed ξ_Θ. After redenoting the constants and switching the roles of the physical and Fourier variable, we need to prove that

$$H : L^2_\phi \to L^2_\phi$$

where H is the Hilbert transform and

$$\phi^{\frac{1}{2}}(x) = (|x| + a)^{s-\theta}|x-b|^\theta, \quad a \geq 0, \quad b \in \mathbb{R}$$

By duality we can take $s - \theta \geq 0$. Rescaling we can set $b = 4$. Then

$$(|x| + a)^{s-\theta}|x-4|^\theta \approx |x|^{s-\theta}|x-4|^\theta + a^{s-\theta}|x-4|^\theta$$

therefore the problem reduces to the case $a = 0$. Now (see Stein [14], chapter 5) it suffices to verify that $\phi^2 = |x|^{s-\theta}|x-4|^\theta$ is an A_2 weight. But this is a simple exercise which is left for the reader.

4 Product estimates for solutions to the homogeneous wave equation

In this section we prove the dyadic multiplicative estimates for solutions to the homogeneous wave equation. The estimates below are the sharp dyadic counterparts of estimates proved in [9] and in earlier papers of Klainerman-Machedon. However, the proof here is different. Our main estimate is

Theorem 3. *The following multiplicative estimates hold for solutions to the homogeneous wave equation:*

a) Let $\mu \ll \lambda$. Then

$$\overset{\circ}{X}_\lambda \cdot \overset{\circ}{X}_\mu \subset \mu^{\frac{n+1}{4}} X_\lambda^{\frac{3-n}{4}} \tag{15}$$

b) Let $\epsilon > 0$. Then

$$\overset{\circ}{X}_\lambda \cdot \overset{\circ}{X}_\lambda \subset \lambda^{\frac{1}{2}+\epsilon} X^{-\frac{n-1}{4}-\epsilon, \frac{3-n}{4}+2\epsilon} \tag{16}$$

Remark 4.1. *Observe that the product with a frequency μ function increases the support of the Fourier transform roughly by μ. Hence, the first estimate above is strictly speaking not true unless we enlarge the allowed support of the Fourier transform of the product. This can be achieved by adding to $X_\lambda^{\frac{3-n}{4}}$ the similar spaces corresponding to frequencies 2λ and $\lambda/2$. We shall neglect such harmless imprecisions here and in the sequel.*

Remark 4.2. *The estimate (16) follows easily by summation from the sharp dyadic estimate*

$$S_{\mu,d}(\overset{\circ}{X}_\lambda \cdot \overset{\circ}{X}_\lambda) \subset \lambda^{\frac{1}{2}} \mu^{\frac{n-1}{4}} X_\mu^{\frac{3-n}{4}} \tag{17}$$

which is proved below.

Proof. a) Let $u \in \overset{\circ}{X}_\lambda$, $v \in \overset{\circ}{X}_\mu$. They are L^2 solutions to the homogeneous wave equation therefore their Fourier transforms are L^2 distributions on the characteristic cone K at frequency λ, respectively μ. Then the Fourier transform of uv is supported at frequency λ, within distance μ of the cone.

Given two vectors Ξ_1, Ξ_2 on the cone K so that

$$|\Xi_1| \approx \lambda, \quad |\Xi_2| \approx \mu, \quad \angle(\Xi_1, \Xi_2) \approx \alpha$$

their sum $\Xi_1 + \Xi_2$ will be roughly at distance

$$d = d(\Xi_1 + \Xi_2, K) \approx \alpha^2 \mu \tag{18}$$

from the cone. This shows that the output at distance d from the cone comes from α-sectors on the cone, at angle α. Furthermore, if we look at all such possible sums,

$$A^i_{\lambda,\alpha} + A^j_{\mu,\alpha}, \qquad |i-j| \approx 2$$

then they have the finite intersection property, i.e. there exists N which depends only on the dimension so that each such sum intersects at most N others. This property is referred to below as "angular orthogonality".

The first step in the proof is to use the orthogonality with respect to d and the angular orthogonality to reduce the problem to the case when u, v have Fourier transform supported in α sectors. More precisely, we claim that (15) follows from the following dyadic estimate

$$\|u \cdot v\|_{L^2} \lesssim \mu^{\frac{n-1}{2}} \alpha^{\frac{n-3}{2}} \|u\|_{\overset{\circ}{X}_\lambda} \|v\|_{\overset{\circ}{X}_\mu}, \tag{19}$$

for $\hat{u}$, $\hat{v}$ supported in α-sectors $A^i_{\lambda,\alpha}$, $A^j_{\mu,\alpha}$ at angle α (i.e. with $|i-j| \approx 2$).

Indeed, if (19) holds then

$$\begin{aligned}
\|S_\lambda(u \cdot v)\|^2_{X_\lambda^{\frac{3-n}{4}}} &\approx \sum_{d \ dyadic} d^{\frac{3-n}{2}} \|S_{\lambda,d}(u \cdot v)\|^2_{L^2} && \text{(orthogonality)} \\
&= \sum_{d \ dyadic} d^{\frac{3-n}{2}} \|S_{\lambda,d}(\sum_{|i-j|=2}^{\alpha^2\mu = d} S^i_{\lambda,\alpha} u \cdot S^j_{\mu,\alpha} v)\|^2_{L^2} && \text{(distance estimate in (18))} \\
&\lesssim \sum_{d \ dyadic} d^{\frac{3-n}{2}} \sum_{|i-j|=2}^{\alpha^2\mu = d} \|S^i_{\lambda,\alpha} u \cdot S^j_{\mu,\alpha} v\|^2_{L^2} && \text{(angular orthogonality)} \\
&\lesssim \sum_{d \ dyadic} d^{\frac{3-n}{2}} \mu^{n-1} \alpha^{n-3} \sum_{|i-j|=2}^{\alpha^2\mu = d} \|S^i_{\lambda,\alpha} u\|^2_{\overset{\circ}{X}_\lambda} \|S^j_{\mu,\alpha} v\|^2_{\overset{\circ}{X}_\mu} && \text{(use (19))} \\
&\approx \mu^{\frac{n+1}{2}} \sum_{\alpha \ dyadic} \sum_{|i-j|=2} \|S^i_{\lambda,\alpha} u\|^2_{\overset{\circ}{X}_\lambda} \|S^j_{\mu,\alpha} v\|^2_{\overset{\circ}{X}_\mu} && \text{(use(18))} \\
&\approx \mu^{\frac{n+1}{2}} \|u\|^2_{\overset{\circ}{X}_\lambda} \|v\|^2_{\overset{\circ}{X}_\mu}
\end{aligned}$$

It remains to prove (19). Now the supports of $\hat{u}, \hat{v}$ are contained in α-sectors on the cone of size $\lambda \times (\alpha\lambda)^{n-1}$, respectively $\mu \times (\alpha\mu)^{n-1}$. One could easily prove this directly by interpreting it as a convolution estimate in the Fourier space. However, anticipating the arguments used later in the paper, we choose to do it differently. The idea is to represent v as a superposition of traveling waves and then to combine this with the characteristic energy estimates for u proved in Corollary 3.3(a). The first step is carried out in the following lemma:

Lemma 4.3. *Let $v \in \overset{\circ}{X}_\mu$, with Fourier transform supported in $A^j_{\mu,\alpha}$. Then v can be represented as a superposition of traveling waves,*

$$v(\mathcal{X}) = \int_{C^j_\alpha \cap K \cap S(0,1)} v_\Theta(\mathcal{X} \cdot \Theta) d\Theta$$

where

$$\|v_\Theta\|_{L^2_\Theta(L^2)} \lesssim \mu^{\frac{n-1}{2}} \|v\|_{\overset{\circ}{X}_\mu}, \qquad \|v_\Theta\|_{L^1_\Theta(L^2)} \lesssim (\alpha\mu)^{\frac{n-1}{2}} \|v\|_{\overset{\circ}{X}_\mu} \tag{20}$$

To prove this, represent v as

$$v(\mathcal{X}) = \int e^{i\mathcal{X}\Xi} f(\Xi)\delta_K, \qquad f \in L^2(\delta_K)$$

where δ_K is the surface measure on the cone. In polar coordinates this becomes

$$v(\mathcal{X}) = \int_{C_\alpha^j \cap K \cap S(0,1)} \int_0^\infty e^{ir\mathcal{X}\cdot\Theta} f(r\Theta) r^{n-1} dr d\Theta$$

Then it suffices to set

$$v_\Theta(s) = \int_0^\infty e^{irs} f(r\Theta) r^{n-1} dr$$

and estimate its L^2 norm using Plancherel's theorem,

$$\int |v_\Theta(s)|^2 ds d\Theta \approx \int |f|^2(r\Theta) r^{2(n-1)} dr d\Theta \approx \int |f|^2(\Xi) r^{n-1} \delta_K$$

Since $r \approx \mu$ in the support of f, this implies that

$$\|v_\Theta\|_{L^2_\Theta(L^2)} \approx \mu^{\frac{n-1}{2}} \|f\|_{L^2(K)} \approx \mu^{\frac{n-1}{2}} \|v\|_{\overset{\circ}{X}_\mu}$$

i.e. the first part of (20). For the second part of (20) it suffices to observe that the support $C_\alpha^j \cap K \cap S(0,1)$ of v_Θ with respect to Θ has size α^{n-1}. $\triangle$

Now we prove (19). We have

$$\begin{aligned}
\|uv\|_{L^2} &\leq \int_{C_\alpha^j \cap K \cap S(0,1)} \|u(\mathcal{X}) v_\Theta(\mathcal{X}\cdot\Theta)\|_{L^2} d\Theta \\
&\leq \int_{C_\alpha^j \cap K \cap S(0,1)} \|u\|_{L^\infty_{t_\Theta}(L^2_{x_\Theta})} \|v_\Theta\|_{L^2} d\Theta \\
&\lesssim \alpha^{-1} \|u\|_{\overset{\circ}{X}_\lambda} \|v_\Theta\|_{L^1_\Theta(L^2)} && \text{(use Corollary 3.3(a) for } u) \\
&\lesssim \mu^{\frac{n-1}{2}} \alpha^{\frac{n-3}{2}} \|u\|_{\overset{\circ}{X}_\lambda} \|v\|_{\overset{\circ}{X}_\mu} && \text{(use (20) for } v)
\end{aligned}$$

b) Let $u, v \in \overset{\circ}{X}_\lambda$. Given two vectors Ξ_1, Ξ_2 on the cone K so that

$$|\Xi_1| \approx \lambda, \quad |\Xi_2| \approx \lambda, \quad |\Xi_1 + \Xi_2| \approx \mu, \quad \angle(\Xi_1, \Xi_2) \approx \alpha$$

their sum $\Xi_1 + \Xi_2$ will satisfy

$$\Box(\Xi_1 + \Xi_2) \approx \alpha^2 \lambda^2$$

It can be in a dyadic annulus A_μ only if $\mu \gtrsim \alpha\lambda$, in which case it is at distance

$$d \approx \frac{\alpha^2 \lambda^2}{\mu} \tag{21}$$

from the cone.

As in part (a), we claim that it suffices to prove the estimate

$$\|S_\mu(u\cdot v)\| \lesssim \alpha^{-\frac{1}{2}}\mu^{\frac{1}{2}}(\alpha\lambda)^{\frac{n-2}{2}}\|u\|_{\overset{\circ}{X}_\lambda}\|v\|_{\overset{\circ}{X}_\lambda} \tag{22}$$

in the special case when $\hat{u}$, $\hat{v}$ are supported in α-sectors of the cone $A^i_{\lambda,\alpha}$, respectively $A^j_{\lambda,\alpha}$ which are separated by an angle α, i.e. $|i-j|\approx 2$.

Indeed, if (22) holds then for α, d as above we have

$$\begin{aligned}
\|S_{\mu,d}(u\cdot v)\|^2_{0,\frac{3-n}{4}} &\approx d^{\frac{3-n}{2}}\|S_{\mu,d}(u\cdot v)\|^2 \\
&= d^{\frac{3-n}{2}}\|S_{\mu,d}(\sum_{|i-j|=2} S^i_{\lambda,\alpha}u\cdot S^j_{\lambda,\alpha}v)\|^2 \\
&\lesssim d^{\frac{3-n}{2}}(\sum_{|i-j|=2}\|S_\mu(S^i_{\lambda,\alpha}u\cdot S^j_{\lambda,\alpha}v)\|)^2 \\
&\lesssim d^{\frac{3-n}{2}}\alpha^{3-n}\lambda^{n-2}\mu(\sum_{|i-j|=2}\|S^i_{\lambda,\alpha}u\|_{\overset{\circ}{X}_\lambda}\|S^j_{\lambda,\alpha}v\|_{\overset{\circ}{X}_\lambda})^2 \qquad \text{(use (22))} \\
&\lesssim \lambda\mu^{\frac{n-1}{2}}(\sum_i\|S^i_{\lambda,\alpha}u\|^2_{\overset{\circ}{X}_\lambda})(\sum_j\|S^j_{\lambda,\alpha}v\|^2_{\overset{\circ}{X}_\lambda}) \qquad \text{(use (21))} \\
&\approx \lambda\mu^{\frac{n-1}{2}}\|u\|^2_{\overset{\circ}{X}_\lambda}\|v\|^2_{\overset{\circ}{X}_\lambda}
\end{aligned}$$

which gives (17).

The notable difference compared to part (a) is that here we loose the angular orthogonality; to compensate for this we relax the norm in the target space by introducing the parameter ϵ, thus eliminating the need for summation with respect to μ, d.

It remains to prove (22). Unfortunately a direct application of the method in part (a) yields a weaker bound, with an additional $\alpha^{-\frac{1}{2}}$ factor. This indicates that there is some additional orthogonality which we have not yet used. However, the method in part (a) does give the sharp bound

$$\|\hat{u}\hat{v}\|_{L^2} \lesssim \alpha^{-1}(\alpha^2\lambda)^{\frac{n-1}{2}}\|u\|_{\overset{\circ}{X}_\lambda}\|v\|_{\overset{\circ}{X}_\lambda} \tag{23}$$

for $\hat{u}$, $\hat{v}$ supported in α^2-sectors at angle α. It remains to show that (23) implies (22). To achieve this we first partition the α-sectors on the cone with respect to the radial direction into $\alpha\lambda$-cubes on the cone; then we cut each such cube into α^2-sectors. The complete argument follows.

The supports $A^i_{\lambda,\alpha}$ and $A^j_{\lambda,\alpha}$ of $\hat{u}$ and $\hat{v}$ have size $\lambda\times(\alpha\lambda)^{n-1}$. Since $A^i_{\lambda,\alpha}$ and $A^j_{\lambda,\alpha}$ are at angle α, it follows that they are contained in parallel parallelepipeds of comparable size $\lambda\times(\alpha\lambda)^{n-1}\times\alpha^2\lambda$. Then their convolution has size $\lambda\times(\alpha\lambda)^{n-1}\times\alpha^2\lambda$.

The support of the convolution intersects the A_μ annulus only if $\mu\geq\alpha\lambda$. The intersection is then contained in a $\mu\times(\alpha\lambda)^{n-1}\times\alpha^2\lambda$ cube. Consequently (22) follows from the estimate

$$\|\hat{u}\hat{v}\|_{L^2(C)} \lesssim \alpha^{-\frac{1}{2}}(\alpha\lambda)^{\frac{n-1}{2}}\|u\|_{\overset{\circ}{X}_\lambda}\|v\|_{\overset{\circ}{X}_\lambda} \tag{24}$$

for all cubes C of size $(\alpha\lambda)^n \times \alpha^2\lambda$.

By orthogonality it suffices to do this when u, v have Fourier transform supported in $(\alpha\lambda)^n$ cubes on the cone, at angle α. Since they are at angle α, both can be embedded into parallel parallelepipeds Q^i, Q^j of size $(\alpha\lambda)^n \times \alpha^2\lambda$, whose "radial" direction is given by some characteristic direction Θ at angle α with respect to both $A^i_{\lambda,\alpha}$ and $A^j_{\lambda,\alpha}$.

For the last reduction, we decompose each such parallelepiped into α^{-n} parallel parallelepipeds of size $\alpha\lambda \times (\alpha^2\lambda)^{n-1} \times \alpha^3\lambda$ oriented in the radial direction,

$$Q^i = \bigcup_{k\in J} Q^i_k \qquad Q^j = \bigcup_{k\in J} Q^j_k$$

where J stands for the lattice points in an n-dimensional cube with sides α^{-1}, using "parallel" labeling for the two partitions. Denote by J^i, respectively J^j the subsets of J which correspond to parallelipipeds which intersect the cone (or more general, which intersect an $\alpha^3\lambda$ neighbourhood of the cone).

The projection of this decomposition on the plane $\tau = 0$ is shown in Figure 2. The solid lines represent the two α-sectors and their radial and angular decompositions, while the dotted lines represent Q^i, Q^j and their decomposition. The projections on a plane transversal to the radial direction Θ are shown in Figure 3.

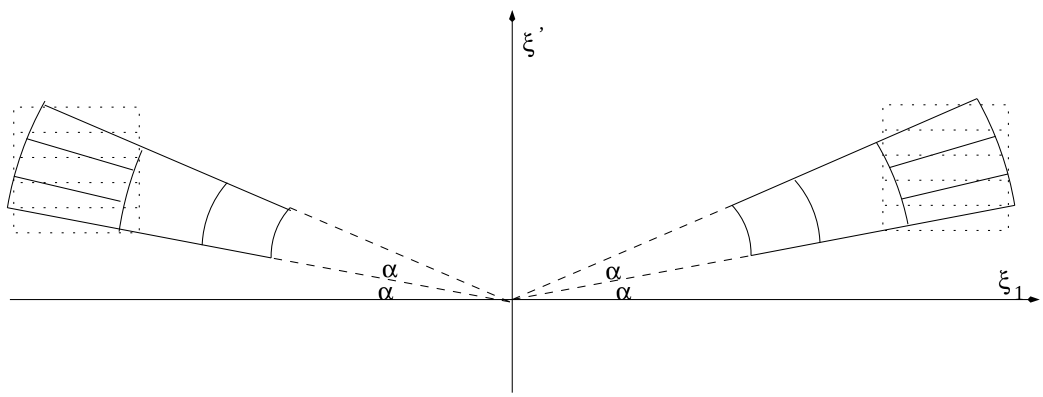

Figure 2: The decomposition for a pair of α-sectors at angle α for $n = 2$

The essential features of this decomposition are described in the following Lemma:

Lemma 4.4. *i) The intersection of each parallelepiped Q^i_k, respectively Q^j_k with the cone is contained in (an enlargement of) an α^2-sector.*

ii) For each $k \in J^i$, $l \in J^j$ the sum $Q^i_k + Q^j_l$ is contained in (an enlargement of) a same size parallelepiped situated at the position $m = k + l$ in a similar lattice of parallelepipeds. Furthermore, each such sum m occurs at most $O(\alpha^{2-n})$ times as we vary $k \in J^i$, $l \in J^j$.

Part (i) of the Lemma shows that this decomposition is equivalent to a decomposition of the part of the cone within Q^i, Q^j into α^2 sectors, and will allow us to use (23). Part (ii), on the other hand, provides the necessary orthogonality in our argument.

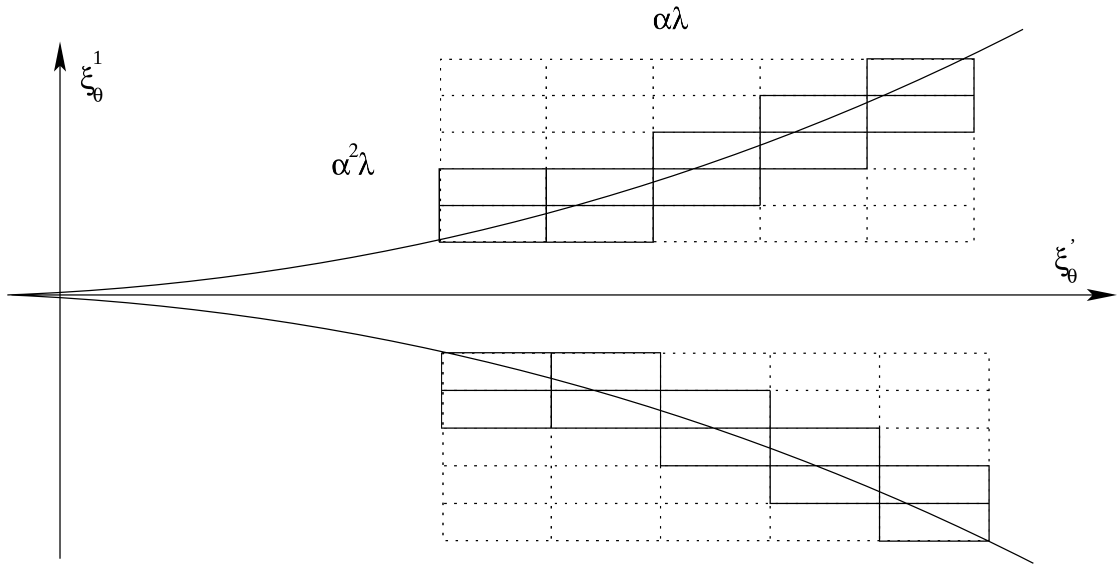

Figure 3: Transversal projection of the decomposition for a pair of $\alpha\lambda$ cubes on the cone at angle α for $n = 2$

Proof. For part (i) it suffices to observe that within the regions Q_i, Q_j the radial direction is at angle α with respect to Θ and at angle α^2 with respect to the plane $\xi_\Theta^1 = 0$. Hence the difference between a radial displacement of $\alpha\lambda$ (the lenght of our small parallelipipeds Q_k^i, Q_k^j) and a similar displacement in the direction Θ is of the order of $\alpha^2\lambda$ in the ξ'_Θ direction and of $\alpha^3\lambda$ in the ξ_Θ^1 direction. But this is comparable with the dimensions of the parallelipipeds Q_k^i, Q_k^j.

Part (ii) comes from a simple transversality argument. It suffices to consider the transversal sections of Q_k^i, Q_k^j, as shown in Figure 3. Then we need to look at the $\alpha^3\lambda \times (\alpha^2\lambda)^{n-1}$ parralelipipeds within an $\alpha^3\lambda$ neighbourhood of sections of the parabolas

$$\xi_\Theta^1 {t_\Theta}^i = (\xi'_\Theta)^2, \qquad \xi_\Theta^1 {t_\Theta}^j = (\xi'_\Theta)^2,$$

where ${t_\Theta}^i$, ${t_\Theta}^j$ have size of the order of λ and opposite signs. If we denote by U^i, respectively U^j the $\alpha^3\lambda$ neighbourhood of the sections of the parabolas within Q^i, Q^j, then we have to determine when their sum belongs to a fixed $\alpha^3\lambda \times (\alpha^2\lambda)^{n-1}$ parrallelipiped. This is equivalent to determining the intersection of U^i with a translate of $-U^j$. Since U^i and U^j are at angle α, this intersection is transversal. In dimension $n = 2$ it is contained in (an enlargement of) a $\alpha^3\lambda \times \alpha^2\lambda$ parallelipiped, which intersects finitely many $\alpha^3\lambda \times \alpha^2\lambda$ parallelipipeds (independent of α). In dimension $n = 3$ the intersection is contained inside a curved parallelipiped of size $\alpha^3\lambda \times \alpha^2\lambda \times \alpha\lambda$, which can be covered with roughly α^{-1} parallelipipipeds of size $\alpha^3\lambda \times (\alpha^2\lambda)^2$. $\triangle$

Corresponding to the decomposition of Q^i and Q^j we split $\hat{u}$ and $\hat{v}$ into

$$\hat{u} = \sum_{k \in J^i} \hat{u}_k, \qquad \hat{v} = \sum_{k \in J^j} \hat{v}_k$$

Now we can prove (24):

$$\begin{aligned}
\|uv\|_{L^2}^2 &= \| \sum_{k\in J^i,\ l\in J^j} u_k v_l \|_{L^2}^2 \\
&\approx \sum_{m\in 2J} \| \sum_{k\in J^i,\ l\in J^j}^{k+l=m} u_k v_l \|_{L^2}^2 && \text{(use (ii) above)} \\
&\leq \sum_{m\in 2J} (\sum_{k\in J^i,\ l\in J^j}^{k+l=m} \|u_k v_l\|_{L^2})^2 \\
&\lesssim \alpha^{2n-4}\lambda^{n-1} \sum_{m\in 2J} (\sum_{k\in J^i,\ l\in J^j}^{k+l=m} \|u_k\|_{\overset{\circ}{X}_\lambda} \|v_l\|_{\overset{\circ}{X}_\lambda})^2 && \text{(use (i) and (23))} \\
&\lesssim \alpha^{2n-4}\lambda^{n-1} \sum_{m\in 2J} |\alpha|^{-n+2} \sum_{k+l=m} \|u_k\|_{\overset{\circ}{X}_\lambda}^2 \|v_l\|_{\overset{\circ}{X}_\lambda}^2 && \text{(use (ii))} \\
&= \alpha^{n-2}\lambda^{n-1}\|u\|_{\overset{\circ}{X}_\lambda}^2 \|v\|_{\overset{\circ}{X}_\lambda}^2
\end{aligned}$$

△

5 The trace method

The aim of this section is to introduce the so-called "trace method" which allows us to transfer estimates from solutions to the homogeneous wave equation to the $X_\lambda^{\frac{1}{2},1}$ spaces. As a corollary we obtain the extension of the estimates in Theorem 3 to the $X_\lambda^{\frac{1}{2},1}$ spaces.

Given an arbitrary Lipschitz function

$$\phi : D(\phi) \subset \mathbb{R}^n \to \mathbb{R}$$

we denote by X_ϕ^θ, respectively $X_\phi^{\frac{1}{2},1}$ the spaces of functions with Fourier transform supported in $\mathbb{R} \times D(\phi)$ and norms

$$\|u\|_{X_\phi^\theta} = \| |\tau - \phi(\xi)|^\theta \hat{u}(\tau,\xi)\|_{L^2}$$

respectively

$$\|u\|_{X_\phi^{\frac{1}{2},1}} = \sum_{\lambda\ dyadic} \lambda^{\frac{1}{2}} \|s(\lambda^{-1}(\tau - \phi(\xi)))\hat{u}(\tau,\xi)\|_{L^2}$$

Also denote by $\overset{\circ}{X}_\phi$ the space of functions with Fourier transform in $L^2(\delta_{\{\tau-\phi(\xi)\}})$. Then our main result is

Proposition 5.1. *Let ϕ, ψ be two Lipschitz functions defined on subsets of $\mathbb{R}^n$ with values in $\mathbb{R}$. Given a continuous function χ, denote by T_χ the multiplication operator by χ. Assume that $T_\chi : \overset{\circ}{X}_\phi \to X^\theta_\psi$ is bounded operator for some $0 \le \theta \le \frac{1}{2}$. Then T_χ is also bounded from $X^{\frac{1}{2},1}_\phi$ into X^θ_ψ and*

$$\|T\|_{L(X^{\frac{1}{2},1}_\phi, X^\theta_\psi)} \lesssim \|T\|_{L(\overset{\circ}{X}_\phi, X^\theta_\psi)}$$

Observe that the result remains true for any operator T commuting with time Fourier translations, $Te^{it\tau} = e^{it\tau}T$. Here we consider a more restricting context only in order to avoid distracting technicalities.

Proof. Replacing T_χ by

$$S(t) = e^{it\psi(D_x)}\chi(t,x)e^{-it\phi(D_x)}$$

the problem reduces to the similar one for S in the case $\phi = \psi = 0$.

Observe that in this special case $\overset{\circ}{X}_0$ contains the functions in L^2_x which are independent of t, while

$$X^{\frac{1}{2},1}_0 = \dot{B}^{2,1}_{\frac{1}{2}}(L^2_x), \qquad X^\theta_0 = \dot{H}^\theta(L^2_x)$$

Given a function $u \in X^{\frac{1}{2},1}_0$, we interpret its image through S as the trace

$$S(t)u(t) = S(t)u(s)_{|s=t}$$

Since the diagonal trace of $\dot{B}^{2,1}_{\frac{1}{2}}(\dot{H}^\theta)$ is in $\dot{H}^\theta$ for $0 \le \theta < \frac{1}{2}$, we obtain

$$\begin{aligned} \|S(t)u(t)\|_{\dot{H}^\theta_t(L^2_x)} &\lesssim \|S(t)u(s)\|_{\dot{B}^{2,1}_{\frac{1}{2}}(\dot{H}^\theta_t(L^2_x))} \\ &\lesssim \|u(s)\|_{\dot{B}^{2,1}_{\frac{1}{2}}(L^2_x)}\|S\|_{L(\overset{\circ}{X}_0, X^\theta_0)} \end{aligned}$$

△

The special case when ϕ and ψ are $\pm|\xi|$ corresponds to product estimates for the $X^{2,1}_\lambda$ spaces. The analogue of Theorem 3 is

Theorem 4. *Let $n = 2, 3$. Then the following multiplicative estimates hold:*

a) Let $\mu \ll \lambda$. Then

$$X^{\frac{1}{2},1}_\lambda \cdot X^{\frac{1}{2},1}_\mu \subset \mu^{\frac{n+1}{4}} X^{\frac{3-n}{4}}_\lambda \cap \mu^{\frac{n}{2}} X^{\frac{1}{2},1}_\lambda \tag{25}$$

b) Let $\mu \le 4\lambda$. Then

$$X^{\frac{1}{2},1}_\lambda \cdot X^{\frac{1}{2},1}_\mu \subset \lambda^{\frac{1}{2}+\epsilon} X^{-\frac{n-1}{4}-\epsilon, \frac{3-n}{4}+2\epsilon}, \qquad \epsilon > 0 \tag{26}$$

Proof. a) It suffices to prove the estimate when the frequency λ factor is supported in a fixed dyadic region $A_{\lambda,d}$ (in frequency).

a1) If $d > 10\mu$ then the product is also supported at distance $O(d)$ from the cone. But for functions with Fourier transform supported at distance $O(d)$ from the cone the $X_\lambda^{\frac{1}{2},1}$ norm is equivalent to the $d^{-\frac{1}{2}}L^2$ norm. Hence it suffices to show that

$$L^2 \cdot X_\mu^{\frac{1}{2},1} \subset \mu^{\frac{n}{2}} L^2$$

which follows from the second part of the next Lemma.

Lemma 5.2. *For all $\mu > 0$ the following two embeddings hold:*

$$X_\mu^{\frac{1}{2},1} \subset C_t^0(L_x^2) \tag{27}$$

(this is the analogue of the energy estimate for solutions to the homogeneous wave equation), and

$$X_\mu^{\frac{1}{2},1} \subset \mu^{\frac{n}{2}} L^\infty \tag{28}$$

Proof. The first part follows from the straightforward inequality

$$\|\hat{u}\|_{L^1_\tau(L^2_\xi)} \lesssim \|u\|_{X_\mu^{\frac{1}{2},1}}$$

The second embedding is a consequence of the first one due to the Sobolev embeddings. $\triangle$

a2) If $d < 10\mu$ then the Fourier transform of the product is also supported within distance $O(\mu)$ from the cone. For such functions the $\mu^{\frac{n+1}{4}} X_\lambda^{0,\frac{3-n}{4}}$ norm is stronger than the $\mu^{-\frac{n}{2}} X_\lambda^{\frac{1}{2},1}$ norm, therefore it suffices to prove that

$$X_\lambda^{\frac{1}{2},1} \cdot X_\mu^{\frac{1}{2},1} \subset \mu^{\frac{n+1}{4}} X_\lambda^{0,\frac{3-n}{4}}$$

If both factors have Fourier transform supported outside a conic neighborhood of $\{|\xi| = 0\}$ then the result follows from Theorem 3 by applying Proposition 5.1 once for each factor if we successively take

$$\phi(\xi) = \pm|\xi|, \quad D(\phi) = \{\lambda/4 \leq |\xi| \leq 4\lambda\}, \quad \psi(\xi) = \pm|\xi|, \quad D(\psi) = \{\lambda/8 \leq |\xi| \leq 8\lambda\}$$

respectively

$$\phi(\xi) = \pm|\xi|, \quad D(\phi) = \{\mu/4 \leq |\xi| \leq 4\mu\}, \quad \psi(\xi) = \pm|\xi|, \quad D(\psi) = \{\lambda/8 \leq |\xi| \leq 8\lambda\}$$

The factors with Fourier transform supported within a conic neighborhood of $\{|\xi| = 0\}$ can be easily dealt with via a Lorentz transform which maps the conic neighborhood of $\{|\xi| = 0\}$ away from the line $\{|\xi| = 0\}$.

b) This follows directly from Theorem 3 and Proposition 5.1 if we use a Lorentz transform to deal with factors which are frequency localized in a small conical neighborhood of the line $\{|\xi| = 0\}$.

6 Dyadic function spaces

At fixed frequency λ we build our spaces F_λ of functions with Fourier transform supported in $B(0,4\lambda)\setminus B(0,\lambda/4)$ as follows:

$$F_\lambda = X_\lambda^{\frac{1}{2},1} + \sum_{\alpha\ dyadic}^{\alpha\le 1} Y_{\lambda,\alpha}$$

with norm

$$\|u\|_{F_\lambda} = \inf\{\|u_0\|_{X_\lambda^{\frac{1}{2},1}} + \sum_{\alpha=2^{-l}}^{l\ge 1} \|u_\alpha\|_{Y_{\lambda,\alpha}};\ u = u_0 + \sum_{\alpha=2^{-l}}^{l\ge 1} u_\alpha\}$$

The $Y_{\lambda,\alpha}$ spaces are atomic spaces. A function u is an $Y_{\lambda,\alpha}$ atom iff

(a) $\hat{u}$ is supported in $\tilde{A}_{\lambda,\frac{1}{20}\alpha^2\lambda}$.

(b) For each j there exists a characteristic direction Θ_j, at angle $\alpha_j \ge \alpha$ from the support of $\tilde{S}^j_{\lambda,\alpha}$, so that

$$\sum_j (\lambda\alpha_j)^{-2}\|\Box R^j_\alpha u\|^2_{L^1_{t_{\Theta_j}}(L^2_{x_{\Theta_j}})} \le 1$$

$$\sum_j \alpha_j^2 \|R^j_\alpha u\|^2_{L^\infty_{t_{\Theta_j}}(L^2_{x_{\Theta_j}})} \le 1$$

Then the $Y_{\lambda,\alpha}$ norm is defined by

$$\|u\|_{Y_{\lambda,\alpha}} = \inf\{\sum_{k=1}^\infty |a_k|;\ u = \sum_{k=1}^\infty a_k u_k,\quad u_k \text{ are } Y_{\lambda,\alpha} \text{ atoms }\}$$

In other words, the $Y_{\lambda,\alpha}$ spaces consist of frequency localized classical solutions to the inhomogeneous wave equation with respect to characteristic directions, square summed in the frequency space. These are exactly the bounds which are satisfied by L^2 solutions to the homogeneous wave equation. To see that one only needs to orthogonally decompose its Fourier transform with respect to the α sectors and then write the characteristic energy estimate in the sector R^j_α with respect to the direction Θ_j as in Section 3. Thus we have the embedding

$$\overset{\circ}{X}_\lambda \subset Y_{\lambda,\alpha}$$

On the other hand, one can naively see that these spaces are nested modulo $X_\lambda^{\frac{1}{2},1}$, in the sense that

$$Y_{\lambda,2\alpha} \subset Y_{\lambda,\alpha} + X_\lambda^{\frac{1}{2},1}$$

The $Y_{\lambda,\alpha}$ space allows for a wider range of directions Θ for each α-sector, therefore in this regard $Y_{\lambda,\alpha}$ is larger than $Y_{\lambda,2\alpha}$. However, the functions in $Y_{\lambda,\alpha}$ have Fourier transform supported closer to the cone, therefore we need the $X_\lambda^{\frac{1}{2},1}$ space to account for the part of $Y_{\lambda,2\alpha}$ which has Fourier transform at distance roughly $\alpha^2\lambda$ from the cone.

We start with a few simple properties of the F_λ spaces. First we relate them to the $X_\lambda^{\frac{1}{2},\infty}$ space.

Proposition 6.1. *The following embedding holds:*

$$F_\lambda \subset X_\lambda^{\frac{1}{2},\infty} \tag{29}$$

Proof. This property is straightforward for $X_\lambda^{\frac{1}{2},1}$. It remains to prove it for the $Y_{\lambda,\alpha}$ spaces. Let u be a $Y_{\lambda,\alpha}$ atom as above. Decompose it as

$$u = \sum_j R_\alpha^j u$$

By orthogonality, it suffice to prove the result for one of the pieces. But $R_\alpha^j u$ is frequency localized in the α-sector C_α^j at angle $\alpha_j > \alpha$ from Θ, therefore the conclusion follows from Theorem 3.5. $\triangle$

Our second result shows that the functions in F_λ satisfy the "energy estimate" with respect to time-like directions. It also provides the main ingredient in the proof of the scattering result.

Proposition 6.2. *The following embedding holds:*

$$F_\lambda \subset L^\infty(L^2) \tag{30}$$

Furthermore, the limits

$$\lim_{t\to\infty} e^{\pm it|D|}(u_t \pm i|D|u)$$

exist in λL^2 for all $u \in F_\lambda$.

Using the Sobolev embeddings we also obtain

Corollary 6.3. *The following embedding holds:*

$$F_\lambda \subset \lambda^{\frac{n}{2}} L^\infty \tag{31}$$

Proof. For the the $X_\lambda^{\frac{1}{2},1}$ space the result follows from Proposition 5.2. Then it remains to prove it for $Y_{\lambda,\alpha}$. Hence start with a $Y_{\lambda,\alpha}$ atom

$$u = \sum_j R_\alpha^j u$$

The orthogonal projections of the α-sectors $A_{\lambda,\alpha}^j$ on the plane $\tau = 0$ are almost disjoint. Then by orthogonality it suffices to prove the result for a single piece $R_\alpha^j u$. But, by Proposition 3.4, $R_\alpha^j u$ is a superposition of truncated L^2 solutions to the wave equation therefore (30) follows.

The second part of the result requires a more detailed analysis. To fix the signs, suppose we look at the limit

$$\lim_{t\to\infty} e^{it|D|}(u_t + i|D|u)$$

We need to consider two cases:

a) If $\hat{u}$ is supported away from $\tau - |\xi| = 0$ then we use the embedding (29) to obtain $(u_t + i|D|u) \in \lambda^{\frac{1}{2}} L^2$. Hence its limit at infinity is 0.

b) If $\hat{u}$ is supported near $\tau - |\xi| = 0$ then

$$u_t \pm i|D|u \in \lambda Y_{\lambda,\alpha}$$

therefore it suffices to show that the limit

$$\lim_{t\to\infty} e^{it|D|} u$$

exists in L^2 when u is in $Y_{\lambda,\alpha}$, supported in a α sector near $\tau = |\xi|$. But then u is a superposition of L^2 solutions to the wave equation truncated across characteristic hyperplanes. Then the conclusion follows if we prove that, as $t \to \pm\infty$, such a truncated solution approaches either 0 or the non-truncated solution for any L^2 initial data. By density it suffices to prove this for smooth compactly supported initial data. Indeed, suppose the initial data is supported in the unit ball and we truncate it using the characteristic function $\chi_{\{t<x_1\}}$. Then the corresponding solution decays like $t^{-\frac{n-1}{2}}$. Hence

$$|u(t)|^2_{L^2(t<x_1)} \lesssim \int_{t<x_1,|x|<t+1} t^{1-n} dx \lesssim t^{\frac{1-n}{2}}$$

which implies that

$$\lim_{t\to\infty} |u(t)|_{L^2(t<x_1)} = 0$$

If instead we truncate by $\chi_{\{t>x_1\}}$ then the truncated solution will approach the untruncated one as t approaches ∞. △

The next result is an extension of Lemma 4.3; it shows that $Y_{\lambda,\theta}$ functions which are frequency localized in an α-sector, $\alpha > \theta$, can be represented as a superposition of truncated traveling waves.

Proposition 6.4. *Let $\alpha > \theta$. Let $u \in Y_{\lambda,\theta}$ so that $\hat{u}$ is supported in an α-sector $A^i_{\lambda,\alpha}$. Then u can be represented as*

$$u = \int_D u_\Theta d\Theta \tag{32}$$

where $D \subset K \cup S(0,1)$ is a $\frac{\theta}{10}$ enlargement of $C^i_\alpha \cup K \cup S(0,1)$ and

$$\int_D \|u_\Theta\|_{L^2_{t_\Theta}(L^\infty_{x_\Theta})} d\Theta \lesssim (\alpha\lambda)^{\frac{n-1}{2}} \|u\|_{Y_{\lambda,\theta}} \tag{33}$$

Proof. It suffices to prove the result for an $Y_{\lambda,\theta}$ atom. Decompose the α sector into roughly $(\alpha/\theta)^{n-1}$ θ-sectors. Because $u \in Y_{\lambda,\theta}$, we can represent it as the square summable superposition of $(\alpha/\theta)^{n-1}$ components supported in θ-sectors. Then we can reduce the problem to the case of a single θ-sector, i.e. to the case when $\alpha = \theta$.

But according to Proposition 3.4, an $Y_{\lambda,\theta}$ atom u which is frequency localized in an θ-sector $A^i_{\lambda,\theta}$ can be expressed as superpositions of solutions for the homogeneous wave equation, truncated on characteristic hyperplanes. These solutions are frequency localized in the Θ projection on the cone of the support of $\hat{u}$, with respect to a characteristic direction Θ at angle at least θ from $A^i_{\lambda,\theta}$. But by the definition of $Y_{\lambda,\theta}$, $\hat{u}$ is supported within $\frac{1}{20}\theta^2\lambda$ from the cone. Then its Θ projection on the cone is contained within a $\frac{\theta}{10}\lambda$ neighborhood of $A^i_{\lambda,\theta} \cap K$.

The solutions for the homogeneous wave equation can, in turn, be represented by Lemma 4.3 as a superposition of traveling waves. This yields the desired conclusion. $\triangle$

Now we can define the spaces $\Box Y_{\lambda,\alpha}, \Box F_\lambda$ with the induced norm. Their characterization is the expected one:

Proposition 6.5. *i) The space $\Box Y_{\lambda,\alpha}$ is an atomic space, where a function f is an $\Box Y_{\lambda,\alpha}$ atom if*

(a) $\hat{f}$ is supported in $\tilde{A}_{\lambda,\frac{1}{20}\alpha^2\lambda}$.

(b) For each j there exists a characteristic direction Θ_j, at angle $\alpha_j \geq \alpha$ from the support of $\tilde{S}^j_{\lambda,\alpha}$, so that

$$\sum_j (\lambda\alpha_j)^{-2} \|R^j_\alpha f\|^2_{L^1_{t_{\Theta_j}}(L^2_{x_{\Theta_j}})} \leq 1$$

ii) The operator $\Box : Y_{\lambda,\alpha} \to \Box Y_{\lambda,\alpha}$ is right invertible and $Ker \Box = \overset{\circ}{X}_\lambda$.

iii) $\Box F_\lambda = \lambda X^{-\frac{1}{2},1} + \sum \Box Y_{\lambda,\alpha}$. In addition, the operator $\Box : F_\lambda \to \Box F_\lambda$ is right invertible and $Ker \Box = \overset{\circ}{X}_\lambda$.

Proof. Clearly the functions in $\Box Y_{\lambda,\theta}$ are contained in the atomic space determined by $\Box Y_{\lambda,\alpha}$ atoms defined as in (a),(b). To prove the converse, and also half of (ii), it suffices to construct a right inverse to $\Box$ which maps $\Box Y_{\lambda,\alpha}$ atoms into uniformly bounded multiples of $Y_{\lambda,\alpha}$ atoms. But due to Corollary 3.3(b) such an inverse is the forward parametrix $\Box^{-1}_f$ for the wave operator . The proof of part (iii) is then straightforward. By (30), the elements in the kernel are $L^\infty_t(L^2_x)$ solutions to the homogeneous wave equation, i.e. exactly the functions in $\overset{\circ}{X}_\lambda$. $\triangle$

Now introduce the spaces G_λ of functions with Fourier transform supported in $\frac{\lambda}{4} \leq |\xi| \leq 4\lambda$,

$$G_\lambda = \bigcap_{\theta \ dyadic} Z_{\lambda,\theta} \cap X^{\frac{1}{2},\infty}_\lambda$$

with norm

$$\|u\|_{G_\lambda} = \max\{\|u\|_{X^{0,\frac{1}{2},\infty}}, \sup_{\theta \ dyadic} \|u\|_{Z_{\lambda,\theta}}\}$$

where the $Z_{\lambda,\theta}$ norms are defined for $\theta \lesssim 1$ by

$$\|u\|_{Z_{\lambda,\theta}}^2 = \sum_j \sup_{\alpha \geq \theta} \alpha^2 \|R_\theta^j u\|_{L_\Theta^\infty(L^2)}^2$$

where α is the angle between the characteristic direction Θ and the support of $Q_{\lambda,\theta}^j$.

Formally we have

$$G_\lambda = \lambda(\Box F_\lambda)'$$

However, we have to be a bit cautious here due to the restriction on the support of the Fourier transform. We can stay out of trouble if we modify a bit the support of the Fourier transform:

Proposition 6.6. *The following embeddings hold:*

a) $G_\lambda \subset \lambda(\Box F_\lambda)'$.

b) $\lambda S_\lambda(\Box F_\lambda)' \subset G_\lambda$.

Remark 6.7. *A stronger form of the above Proposition is*

a) The space G_λ *is a closed subspace of* $\lambda(\Box F_{2\lambda} + \Box F_{\frac{\lambda}{2}})'$.

b) The space $\lambda(\Box F_\lambda)'$ *is a quotient space of* $G_{2\lambda} + G_{\frac{\lambda}{2}}$.

However, the result in the Proposition is easier to prove and is well adapted for the duality arguments we use later on.

Proof. The proof of part (a) is straightforward as one only needs to test G_λ functions against the building blocks of $(\Box F_\lambda)$, i.e. $X_\lambda^{-\frac{1}{2},1}$ and $\Box Y_{\lambda,\alpha}$.

For part (b), let $f \in \lambda(\Box F_\lambda)'$, with norm 1. We can interpret $S_\lambda f$ as a distribution if for any test function u we set

$$S_\lambda f(u) = f(S_\lambda u)$$

First we test it against $X_\lambda^{-\frac{1}{2},1}$:

$$|(S_\lambda f)(u)| \leq \|S_\lambda u\|_{X_\lambda^{-\frac{1}{2},1}}$$

This implies that

$$\|S_\lambda f\|_{X_\lambda^{\frac{1}{2},\infty}} \lesssim 1$$

Next we test it against $Y_{\lambda,\alpha}$ type functions. For each j we choose a direction Θ_j at angle $\alpha_j > \alpha$ from the α-sector C_α^j and a function $u_j \in L_{t_{\Theta_j}}^1(L_{x_{\Theta_j}}^2)$. Then set

$$u = \sum_j S_{\lambda,\alpha}^j u_j$$

The operators $S_{\lambda,\alpha}^j$ are uniformly bounded in $L_{t_{\Theta_j}}^1(L_{x_{\Theta_j}}^2)$. Then it is not difficult to show that $u \in \Box F_\lambda$, and

$$\|u\|_{\Box F_\lambda}^2 \lesssim \sum_j (\lambda\alpha_j)^{-2} \|u_j\|_{L_{t_{\Theta_j}}^1(L_{x_{\Theta_j}}^2)}^2$$

Hence we must also have

$$|f(u)|^2 \lesssim \sum_j \alpha_j^{-2} \|u_j\|^2_{L^1_{t_{\Theta_j}}(L^2_{x_{\Theta_j}})}$$

and further

$$|\sum_j (S^j_{\lambda,\alpha} f)(u_j)|^2 \lesssim \sum_j \alpha_j^{-2} \|u_j\|^2_{L^1_{t_{\Theta_j}}(L^2_{x_{\Theta_j}})}$$

This holds for all choices of u_j in the above spaces, therefore we obtain

$$\sum_j (\alpha_j)^2 \|S^j_{\lambda,\alpha} f\|^2_{L^\infty_{t_{\Theta_j}}(L^2_{x_{\Theta_j}})} \lesssim 1$$

for all possible choices of characteristic directions Θ_j at angle $\alpha_j > \alpha$ from C^j_α. Since $S^j_{\lambda,\alpha} = R^j_\alpha S_\lambda$, this says exactly that

$$\|S_\lambda f\|_{Z_{\lambda,\alpha}} \lesssim 1$$

and further

$$\|S_\lambda f\|_{G_\lambda} \lesssim 1$$

△

We start the study of these spaces with the embedding into L^∞:

Proposition 6.8. *The following embeddings hold:*

$$G_\lambda \subset \lambda^{\frac{n}{2}} L^\infty$$

$$F_\lambda \subset G_\lambda$$

The first embedding follows easily if we use the $Z_{\lambda,\theta}$ norm for $\theta \approx 1$. The proof of the second embedding is more involved. Start with

$$X_\lambda^{\frac{1}{2},1} \subset Z_{\lambda,\theta}$$

By orthogonality this reduces to a θ-sector. Then this is the dual to (14).

It remains to prove the embedding

$$Y_{\lambda,\theta} \subset Z_{\lambda,\alpha}$$

a) Suppose that $\theta < \alpha$. By orthogonality it suffices to prove the embedding for $Y_{\lambda,\theta}$ atoms which are frequency localized within an α-sector $A^j_{\lambda,\alpha}$. This is equivalent to the uniform estimate

$$\beta \|u\|_{L^\infty_{t_\Theta}(L^2_{x_\Theta})} \lesssim \|u\|_{Y_{\lambda,\theta}}$$

when u is frequency localized in $A^j_{\lambda,\alpha}$ and the characteristic direction Θ is at angle $\beta \geq \alpha$ from $A^j_{\lambda,\alpha}$.

We can decompose u into pieces which are frequency localized in θ-sectors. The θ sectors's projections in the direction Θ are almost disjoint, therefore by orthogonality it suffices to prove the above estimate for functions with Fourier transform supported in a θ sector. But the part of $Y_{\lambda,\theta}$ within a θ sector is a superposition of truncated L^2 solutions to the wave equation, which by Corollary 3.3 satisfy the appropriate $L^\infty_\Theta(L^2)$ bound.

b) If $\theta > \alpha$ then we reason in a dual way. In this case the problem reduces to a θ sector. We can truncate it at distance $\alpha^2\lambda$ from the cone, and then split it in α sectors. By orthogonality the problem reduces to α sectors and we conclude as before.

Next we prove that our function spaces behave nicely with respect to truncation away from the cone.

Proposition 6.9. *The following estimates hold uniformly in $\lambda, \nu > 0$:*

$$\|\tilde{Q}_\nu f\|_{F_\lambda} \lesssim \|f\|_{F_\lambda} \tag{34}$$

$$\|(1-\tilde{Q}_\nu) f\|_{\Box F_\lambda} \lesssim \nu^{-1}\|f\|_{F_\lambda} \tag{35}$$

$$\|\tilde{Q}_\nu f\|_{G_\lambda} \lesssim \|f\|_{G_\lambda} \tag{36}$$

Proof. Observe that (36) follows from (34) by duality. Then it suffices to prove (34) and (35). Both properties are trivial for the $X_\lambda^{\frac{1}{2},1}$ space; also for the $Y_{\lambda,\alpha}$ space if the thickness of the support of the Fourier transform of its elements is less than $\lambda^{-1}\nu$, i.e. if $\alpha^2\lambda^2 < \nu$. It remains to prove these properties for $Y_{\lambda,\alpha}$ atoms with $\alpha^2\lambda^2 \geq \nu$.

These would in turn follow from the following $L^1(L^2)$ bounds and the dual $L^\infty(L^2)$ bounds:

$$\|\tilde{Q}_\nu u\|_{L^1_{t_\Theta}(L^2_{x_\Theta})} \lesssim \|u\|_{L^1_{t_\Theta}(L^2_{x_\Theta})}$$

$$\|(1-\tilde{Q}_\nu)\Box^{-1} u\|_{L^1_{t_\Theta}(L^2_{x_\Theta})} \lesssim \nu^{-1}\|u\|_{L^1_{t_\Theta}(L^2_{x_\Theta})}$$

for $\hat{u}$ supported in an α-sector $A^j_{\lambda,\alpha}$ at angle $\beta \geq \alpha$ from the characteristic direction Θ.

In characteristic coordinates, $\tilde{Q}_\nu$ has symbol

$$\tilde{q}_\nu = \tilde{s}\left(\frac{2\tau_\Theta \xi^1_\Theta - (\xi'_\Theta)^2}{\nu}\right)$$

and kernel

$$K(t_\Theta, \xi_\Theta) = \frac{\nu}{2\xi^1_\Theta}\hat{\tilde{s}}\left(\frac{\nu t_\Theta}{2\xi^1_\Theta}\right)e^{it_\Theta\frac{|\xi'_\Theta|^2}{2\xi^1_\Theta}}$$

According to the analysis in Section 3, within the α-sector $A^j_{\lambda,\alpha}$ we have

$$|\xi^1_\Theta| \approx \beta^2\lambda, \qquad |\xi'_\Theta| \approx \beta\lambda$$

For ξ_Θ restricted to this range we have $K \in L^1_{t_\Theta}(L^\infty_{\xi_\Theta})$ which implies the first bound. A similar analysis for the operator $(1-\tilde{Q}_\nu)\Box^{-1}$ yields the kernel

$$H(t_\Theta,\xi_\Theta) = \nu^{-1}\frac{1}{2\xi^1_\Theta}\hat{w}(\frac{t}{2\xi^1_\Theta})e^{it\frac{|\xi'_\Theta|^2}{2\xi^1_\Theta}} \qquad w(t) = \frac{1-\tilde{s}(t)}{t}$$

Then for ξ_Θ in the same range as above we get

$$\|H(t_\Theta,\xi_\Theta)\|_{L^1_{t_\Theta}(L^\infty_{\xi_\Theta})} \leq \nu^{-1}$$

which implies the second bound. $\triangle$

We continue with a simple multiplicative estimate which shows that multiplication by L^∞ functions at frequency μ leave our function spaces unchanged away from a μ neighborhood of the cone.

Proposition 6.10. *Let $\mu \ll \lambda$. Then*

$$(1-\tilde{Q}_{10\lambda\mu})F_\lambda \cdot L^\infty_\mu \subset F_\lambda \tag{37}$$

$$(1-\tilde{Q}_{10\lambda\mu})G_\lambda \cdot L^\infty_\mu \subset G_\lambda \tag{38}$$

Proof. Observe first that in both cases the product is frequency localized outside a 5μ neighborhood of the cone. By Proposition 6.5, (37) is equivalent to

$$\Box((1-\tilde{Q}_{10\lambda\mu})F_\lambda \cdot L^\infty_\mu) \subset \Box F_\lambda \tag{39}$$

If we use now the Leibnitz rule and observe that a derivative at frequency λ is roughly equivalent to multiplication by λ, then we get

$$\Box((1-\tilde{Q}_{10\lambda\mu})F_\lambda \cdot L^\infty_\mu) \subset \Box(1-\tilde{Q}_{10\lambda\mu})F_\lambda \cdot L^\infty_\mu + \lambda\mu(1-\tilde{Q}_{10\lambda\mu})F_\lambda \cdot L^\infty_\mu$$

By (35) the first factor in the second right hand side term can be replaced by $\Box(1-\tilde{Q}_{10\lambda\mu})F_\lambda$. Then (39) reduces to

$$(1-\tilde{Q}_{10\lambda\mu})\Box F_\lambda \cdot L^\infty_\mu \subset \Box F_\lambda \tag{40}$$

Furthermore, if we prove this then (38) also follows by duality.

The proof of (40) for the $X_\lambda^{-\frac{1}{2},1}$ space is straightforward; it essentially repeats the orthogonality argument in part (a1) of Theorem 4. It remains to look at $\Box Y_{\lambda,\alpha}$. Observe first that $(1-\tilde{Q}_{10\lambda\mu})Y_{\lambda,\alpha} = 0$ unless $\lambda\alpha^2 > 10\mu$.

The dyadic pieces of $\Box Y_{\lambda,\alpha}$ atoms are $L^1(L^2)$ functions supported in $\lambda \times (\lambda\alpha)^{n-1} \times \frac{1}{20}\lambda\alpha^2$ cubes. Multiplication by L^∞_μ clearly preserves these spaces; the only problem is that the support of the Fourier transform increases by μ.

However, we know that $\lambda\alpha^2 > 10\mu$. Then the increase in the size of the support within distance $\frac{1}{40}\alpha^2\lambda$ from the cone can be taken care of by $Y_{\lambda,\frac{1}{2}\alpha}$; the contribution outside this region goes into the $X_\lambda^{\frac{1}{2},1}$ space.

Now we prove the main result of this section, which deals with the multiplicative properties of the spaces F_λ, G_λ. The estimates in the next theorem mirror the earlier ones in Theorems 3, 4 for the $\overset{\circ}{X}_\lambda$ spaces, respectively the $X_\lambda^{\frac{1}{2},1}$ spaces.

Theorem 5. *a) Let $\mu \ll \lambda$ and $\epsilon > 0$, small. Then*

$$F_\lambda \cdot F_\mu \subset \mu^{\frac{n}{2}} F_\lambda \cap \mu^{\frac{n+1}{4}+\epsilon} X_\lambda^{\frac{3-n}{4}+\epsilon}, \tag{41}$$

$$G_\lambda F_\mu \subset \mu^{\frac{n}{2}} G_\lambda \cap \mu^{\frac{n+1}{4}+\epsilon} X_\lambda^{\frac{3-n}{4}+\epsilon} \tag{42}$$

and

$$F_\lambda G_\mu \subset \lambda^{\frac{n-1}{2}} \left(\mu^{\frac{1}{2}} G_\lambda \cap \mu^{\frac{n-3}{4}+\epsilon} X_\lambda^{\frac{3-n}{4}+\epsilon} \right) \tag{43}$$

b) Let $\mu \leq \lambda$ and $\epsilon > 0$, small. Then

$$F_\lambda \cdot F_\lambda \subset \lambda^{\frac{1}{2}+2\epsilon} X^{\frac{1-n}{4}+\epsilon, \frac{3-n}{4}+\epsilon} \tag{44}$$

$$S_\mu(F_\lambda \cdot G_\lambda) \subset \lambda \mu^{\frac{n-3}{4}+\epsilon} X_\mu^{\frac{3-n}{4}+\epsilon} \tag{45}$$

Note that this theorem is a bit weaker than Theorems 3,5. The reason for that is that we cannot cut orthogonally the sectors in G_λ into shorter pieces. In principle this can be fixed by taking shorter dyadic pieces in the definition of F_λ, G_λ. However, this would make the proofs considerably more complicated.

Proof of (41),(42). Due to the embeddings

$$X_\lambda^{\frac{1}{2},1} \subset F_\lambda \subset X_\lambda^{\frac{1}{2},\infty}$$

it follows that the $\mu^{\frac{n}{2}} F_\lambda$ norm is stronger than the $\mu^{\frac{n+1}{4}+\epsilon} X_\lambda^{0,\frac{3-n}{4}+\epsilon}$ norm on functions which are frequency localized at distance at least $O(\mu)$ from the cone but is weaker on functions which are frequency localized within distance $O(\mu)$ from the cone. Then it is natural to cut the first factor in (41) in frequency in a piece at distance 5μ from the cone and a piece at distance at most 20μ from the cone,

$$F_\lambda = \tilde{Q}_{10\lambda\mu} F_\lambda + (1 - \tilde{Q}_{10\lambda\mu}) F_\lambda$$

The Fourier transform of $(1 - \tilde{Q}_{10\lambda\mu}) F_\lambda \cdot F_\mu$ is still at distance at least $O(\mu)$ from the cone. Then it suffices to show that

$$(1 - \tilde{Q}_{10\lambda\mu}) F_\lambda \cdot \mu^{-\frac{n}{2}} F_\mu \subset F_\lambda$$

This follows from the embedding (31) and Proposition 6.10.

It remains to look at the piece at distance $O(\mu)$ from the cone and show that

$$\tilde{Q}_{10\lambda\mu} F_\lambda \cdot F_\mu \subset \mu^{\frac{n+1}{4}+\epsilon} X_\lambda^{\frac{3-n}{4}+\epsilon} \tag{46}$$

Using a similar argument, (42) reduces to the stronger estimate

$$\tilde{Q}_{10\lambda\mu} G_\lambda \cdot F_\mu \subset \mu^{\frac{n+1}{4}+\epsilon} X_\lambda^{\frac{3-n}{4}+\epsilon} \tag{47}$$

This is the counterpart to the estimate (15) for solutions to the homogeneous wave equation. In this case the output is within distance μ from the cone.

Now we prove (47). If the second factor is in $X_\mu^{\frac{1}{2},1}$ then, using the trace method, the problem reduces to the case when that component is in effect a solution to the homogeneous wave equation. Hence, it remains to show that

$$\tilde{Q}_{10\lambda\mu} G_\lambda \cdot Y_{\mu,\theta} \subset \mu^{\frac{n+1}{4}+\epsilon} X_\lambda^{\frac{3-n}{4}+\epsilon}$$

By modifying ϵ, this reduces to the similar estimate within a dyadic region with respect to the distance to the cone. More precisely, we let $\alpha > 0$ and estimate the Fourier transform of the product in a dyadic region at distance $d \approx \alpha^2 \mu$ from the cone. It suffices to show that

$$Q_{\alpha^2\lambda\mu}(\tilde{Q}_{\lambda\mu} G_\lambda \cdot Y_{\mu,\theta}) \subset \mu^{\frac{n+1}{4}+\epsilon} X_\lambda^{\frac{3-n}{4}+\epsilon}$$

Truncate the frequency λ factor at distance $\frac{1}{10}\alpha^2\mu$ from the cone,

$$\tilde{Q}_{10\lambda\mu} G_\lambda = \tilde{Q}_{\frac{1}{10}\alpha^2\lambda\mu} G_\lambda + (1 - \tilde{Q}_{\frac{1}{10}\alpha^2\lambda\mu})\tilde{Q}_{10\lambda\mu} G_\lambda$$

For the second component we use the embedding $G_\lambda \subset X_\lambda^{\frac{1}{2},\infty}$. We would like to get a bound for it in the smaller space $X_\lambda^{\frac{1}{2},1}$. To achieve that we have to count the dyadic pieces in the support of $(1 - \tilde{Q}_{\frac{1}{10}\alpha^2\lambda\mu})\tilde{Q}_{10\lambda\mu} Y_{\lambda,\theta_1}$; we have roughly

$$\ln \frac{1}{\alpha^2} \lesssim \ln \frac{\mu}{d}$$

such pieces, therefore

$$(1 - \tilde{Q}_{\frac{1}{10}\alpha^2\lambda\mu})\tilde{Q}_{10\lambda\mu} G_\lambda \subset \ln \frac{\mu}{d}\ X_\lambda^{\frac{1}{2},1}$$

Hence, using the trace method, the estimate corresponding to the second component of G_λ reduces to the case when the G_λ factor is a solution to the homogeneous wave equation[2].

If we also substitute G_λ by $Z_{\lambda,\alpha}$ then it suffices to prove that

$$Q_{\alpha^2\lambda\mu}(\tilde{Q}_{\frac{1}{10}\alpha^2\lambda\mu} Z_{\lambda,\alpha} \cdot Y_{\mu,\theta}) \subset \mu^{\frac{n+1}{4}} X_\lambda^{\frac{3-n}{4}} \tag{48}$$

[2] Of course here we loose a factor of $\ln \frac{\mu}{d}$, but we can account for that by modifying the ϵ in the Theorem

It is enough to consider the case when the second factor is an $Y_{\mu,\theta}$ atom. If $\theta > \alpha$ then the output within distance $\alpha^2\mu$ from the cone is generated by pairs of sectors at angle no more than θ. Thus we reduce the problem by orthogonality to θ-sectors. But by Proposition 3.4 the dyadic pieces of $Y_{\mu,\theta}$ within θ sectors are superpositions of truncated solutions to the homogeneous wave equation, therefore, by Proposition 3.6, the problem reduces to the case when the $Y_{\mu,\theta}$ factor is in $\overset{\circ}{X}_\lambda$ supported in a θ sector, i.e. to the case when $\theta = 0$.

Finally, consider the case when $\alpha \geq \theta$. Then the output at distance $\alpha^2\mu$ from the cone comes from pairs of α-sectors at angle α. Hence by orthogonality it suffices to prove the desired dyadic estimate,

$$\|(\tilde{Q}_{\frac{1}{10}\alpha^2\lambda\mu} u) \cdot v\|_{L^2} \lesssim \mu^{\frac{n+1}{4}} d^{\frac{n-3}{4}} \|u\|_{Z_{\lambda,\alpha}} \|v\|_{Y_{\mu,\theta}} \tag{49}$$

in the case when v is a $Y_{\mu,\theta}$-atom and u, v are frequency localized in α-sectors $A^i_{\lambda,\alpha}$, $A^j_{\mu,\alpha}$ at angle α.

For this we use the characteristic energy estimates for u (which are part of the $Z_{\lambda,\alpha}$ norm) and the decomposition of v into truncated traveling waves in Proposition 6.4,

$$v = \int_D v_\Theta(t_\Theta, x_\Theta) d\Theta$$

where $D \subset K \cap S(0,1)$ is an $\frac{\theta}{10}$ enlargement of $C^j_\alpha \cap K \cap S(0,1)$. Then

$$\begin{aligned}
\|uv\|_{L^2} &\leq \| \int_D u v_\Theta d\Theta \|_{L^2} \\
&\leq \int_D \|u v_\Theta\|_{L^2} d\Theta \\
&\leq \int_D \|u\|_{L^\infty_{t_\Theta}(L^2_{x_\Theta})} \|v_\Theta\|_{L^2_{t_\Theta}(L^\infty_{x_\Theta})} d\Theta \\
&\leq \alpha^{-1} \|u\|_{Z_{\lambda,\alpha}} \int_D \|v_\Theta\|_{L^2_{t_\Theta}(L^\infty_{x_\Theta})} d\Theta \qquad \text{(energy estimates for } u\text{)} \\
&\lesssim \alpha^{-1} \|u\|_{Z_{\lambda,\alpha}} (\alpha\mu)^{\frac{n-1}{2}} \|v\|_{Y_{\mu,\theta}} \qquad \text{(use (33) for } v\text{)}
\end{aligned}$$

This gives (49) and concludes the proofs of (41), (42). △

Proof of (43). This proof proceeds in a similar fashion, with the (nonessential) difference that now we truncate both factors at distance $\frac{1}{10}\alpha^2\mu$ from the cone. The problem reduces to the counterpart of (49), namely

$$\|(\tilde{Q}_{\frac{1}{10}\alpha^2\lambda\mu} u) \cdot (\tilde{Q}_{\frac{1}{10}\alpha^2\mu^2} v)\|_{L^2} \lesssim \alpha^{-1} (\alpha\lambda)^{\frac{n-1}{2}} \|u\|_{Y_{\lambda,\theta}} \|v\|_{Z_{\mu,\alpha}} \tag{50}$$

in the case when $\theta \ll \alpha$, u is a $Y_{\lambda,\theta}$-atom and u, v are frequency localized in α-sectors $A^i_{\lambda,\alpha}$, $A^j_{\mu,\alpha}$ at angle α.

To prove (50) we use again the characteristic energy estimates for v and the decomposition of u in Proposition 6.4 as a superposition of truncated traveling waves. The constant we get is much worse because this time we use (33) for the frequency λ factor instead of the frequency μ factor. $\triangle$

Proof of (44). If either factor is in $X_\lambda^{\frac{1}{2},1}$ then we can use the trace method to reduce the problem to the case when that factor is in $\overset{\circ}{X}_\lambda$. Hence, it remains to look at products of the form

$$Y_{\lambda,\theta_1} \cdot Y_{\lambda,\theta_2}$$

Estimate the output

$$Q_{\alpha^2\lambda^2}(Y_{\lambda,\theta_1} \cdot Y_{\lambda,\theta_2})$$

in the region

$$\Box(\Xi) \approx \alpha^2\lambda^2,$$

i) If $\theta_1 \geq \alpha, \theta_2$, by orthogonality it suffices to obtain the estimate in the case when both factors are frequency localized in θ_1 sectors. Then, by Corollary 3.4, the first factor is a superposition of truncated L^2 solutions to the homogeneous wave equation. Hence, by Theorem 3.6 we can assume that the first factor is a solution to the wave equation, i.e. that $\theta_1 = 0$. Using this argument once or twice the problem reduces to the second case:

ii) If $\alpha \geq \theta_1 \geq \theta_2$ then we reduce the problem first by orthogonality to α-sectors at angle α. Following the proof of Theorem 3, we would now like to split these sectors further in length up to a length of $\alpha\lambda$. However, we want to do that without loosing orthogonality, and this presents a difficulty.

First of all, the α-sectors consist of many θ_1-sectors, respectively θ_2-sectors, and each corresponds to a different characteristic direction. Hence, we need to partition these smaller sectors individually. Now suppose we take a θ_1-sector and we try to truncate it into pieces of length $\alpha\lambda$. The only way we can truncate $L^1_{t_\Theta}(L^2_{x_\Theta})$ and still retain the orthogonality is if the multipliers we use act only on the L^2 norm, i.e. if their symbol depends only on ξ_Θ. But the direction Θ could be as close as θ_1 to our θ_1 sector, therefore when we truncate using multipliers which are constant in the direction Θ we can still get pieces of length λ due to the $\theta_1^2\lambda$ thickness of θ_1-sectors. However, we can overcome this difficulty if we truncate first the thickness to $\alpha\theta_1^2\lambda$,

$$Y_{\lambda,\theta_1} = \tilde{Q}_{\alpha\theta_1^2\lambda^2} Y_{\lambda,\theta_1} + (1 - \tilde{Q}_{\alpha\theta_1^2\lambda^2}) Y_{\lambda,\theta_1}$$

Then the inner part can be orthogonally decomposed into $\alpha\lambda$ long pieces, while the outer part is in $X_\lambda^{\frac{1}{2},\infty}$ and further in[3] $|\ln\alpha| X_\lambda^{\frac{1}{2},1}$. The $X_\lambda^{\frac{1}{2},1}$ factor can be replaced with a corresponding solution to the homogeneous wave equation using the trace method. The $|\ln\alpha|$ factor can be absorbed into ϵ since the region where the Fourier transform of the product is supported, $\tilde{A}_\lambda \cap B_{\alpha^2\lambda^2}$, is at distance $d \geq \alpha^2\lambda$ from the cone; thus $|\alpha| \leq \sqrt{\frac{d}{\lambda}}$.

[3] Again we count the number of dyadic regions in the support of the outer part

It remains to prove the counterpart of (24), namely

$$\|u_1 u_2\|_{L^2} \leq \alpha^{-\frac{1}{2}}(\alpha\lambda)^{\frac{n-1}{2}} \|u_1\|_{Y_{\lambda,\theta_1}} \|u_2\|_{Y_{\lambda,\theta_2}} \tag{51}$$

in the case when u_i have Fourier transform supported in $(\alpha\lambda)^n \times \alpha\theta_i^2\lambda$ "parallelepipeds" on the cone, at angle α. Now we continue the argument as in the proof of (24). The Fourier transform of $u_1 u_2$ is supported in an $(\alpha\lambda)^n \times \alpha^2\lambda$ parallelepiped. We would like to reduce the problem to α^2 sectors, but here we do this in two steps. We reduce the problem first to θ sectors, where $\theta = \max\{\theta_1, \theta_2, \alpha^2\}$. Arguing as in the proof of (24), (see also Figures 2,3) (51) reduces to

$$\|u_1 u_2\|_{L^2} \leq \theta^{-\frac{1}{2}}(\theta\lambda)^{\frac{n-1}{2}} \|u_1\|_{Y_{\lambda,\theta_1}} \|u_2\|_{Y_{\lambda,\theta_2}} \tag{52}$$

provided that the functions u_i have Fourier transform supported in $\alpha\lambda \times (\theta\lambda)^{n-1} \times \alpha\theta_i^2\lambda$ sectors on the cone at angle α.

If $\theta = \theta_1$ then we can use Propositions 3.4, 3.6 to substitute the first factor with a solution to the homogeneous wave equation. If necessary we do the same for the second factor. Eventually we arrive at the case when $\theta = \alpha^2 > \theta_1, \theta_2$, and the estimate to prove is the counterpart of (23),

$$\|u_1 u_2\|_{L^2} \leq \alpha^{-1}(\alpha^2\lambda)^{\frac{n-1}{2}} \|u_1\|_{Y_{\lambda,\theta_1}} \|u_2\|_{Y_{\lambda,\theta_2}} \tag{53}$$

for $\hat{u}_1, \hat{u}_2$ supported in α^2 sectors A^i_{λ,α^2}, A^j_{λ,α^2} at angle α.

This follows as before from the characteristic energy estimates for u_1 (which are a consequence of the embedding $F_\lambda \subset G_\lambda$) and the decomposition of u_2 into truncated traveling waves in Proposition 6.4. $\triangle$

Proof of (45). Observe that (45) is equivalent to

$$F_\lambda \cdot G_\lambda \subset \lambda(\mu^\epsilon X^{\frac{3-n}{4},\frac{3-n}{4}+\epsilon} + \mu^{-\epsilon} X^{\frac{3-n}{4}-2\epsilon,\frac{3-n}{4}+\epsilon}) \tag{54}$$

(the equivalence holds only when we consider it for all dyadic values of $\mu \ll \lambda$). Then the trace method takes care of the case when the first factor is in $X_\lambda^{\frac{1}{2},1}$. Thus it suffices to consider products of the form

$$S_\mu(Y_{\lambda,\theta} \cdot G_\lambda)$$

We truncate G_λ at distance $\frac{1}{10}\theta^2\lambda$ from the cone,

$$G_\lambda = \tilde{Q}_{\frac{1}{10}\theta^2\lambda} G_\lambda + (1 - \tilde{Q}_{\frac{1}{10}\theta^2\lambda}) G_\lambda$$

For the second term we use the embedding into $X_\lambda^{\frac{1}{2},\infty}$ and count the dyadic regions to get

$$(1 - \tilde{Q}_{\frac{1}{10}\theta^2\lambda}) G_\lambda \subset |\ln\theta| X_\lambda^{\frac{1}{2},1}$$

Then the estimate for

$$Y_{\lambda,\theta} \cdot (1 - \tilde{Q}_{\frac{1}{10}\theta^2\lambda}) G_\lambda$$

follows from (44). It remains to analyze the product

$$Y_{\lambda,\theta} \cdot \tilde{Q}_{\frac{1}{10}\theta^2\lambda} G_\lambda \tag{55}$$

i) First we estimate the Fourier transform of the product (55) in the dyadic region $\tilde{B}_{\theta^2\lambda^2}$. Now the output within distance $\theta^2\lambda$ from the cone is generated by θ-sectors at angle at most $O(\theta)$. By orthogonality the problem reduces to the case when both factors are frequency localized in such regions. But then, by Proposition 3.4, the first factor is a superposition of truncated solutions for the homogeneous wave equation, therefore, by Proposition 3.6, the problem reduces to the case when the first factor is in $\overset{\circ}{X}_\lambda$, which corresponds to $\theta = 0$. Hence the problem is reduced to the next case.

ii) For $\alpha \gg \theta$ estimate the product in (55) in the dyadic region $B_{\alpha^2\lambda^2}$. It suffices to show that

$$Q_{\alpha^2\lambda^2}(Y_{\lambda,\theta} \cdot \tilde{Q}_{\frac{1}{10}\theta^2\lambda} G_\lambda) \subset X^{\frac{3-n}{4},\frac{3-n}{4}}$$

uniformly in α. The summation with respect to α is taken care of by the ϵ in (45).

For the G_λ factor it is enough to use the $Z_{\lambda,\alpha}$ norm. The output in $B_{\alpha^2\lambda^2}$ is generated by α-sectors at angle α. Hence by orthogonality the problem reduces to the case when both factors are frequency localized in such regions. It suffices to prove the inequality

$$\|uv\|_{L^2} \leq \lambda(\alpha^2\lambda^2)^{\frac{n-3}{4}} \|u\|_{Y_{\lambda,\theta}} \|v\|_{Z_{\lambda,\alpha}} \tag{56}$$

in the case when $\hat{u}$, $\hat{v}$ are supported in α-sectors $Q^i_{\lambda,\alpha}$, $Q^j_{\lambda,\alpha}$ at angle α. But

$$\lambda(\alpha^2\lambda^2)^{\frac{n-3}{4}} = \alpha^{-1}(\alpha\lambda)^{\frac{n-1}{2}}$$

Then (56) follows as before from the characteristic energy estimates for v and the decomposition of u in Proposition 6.4 as a superposition of truncated traveling waves.

7 Paley-Littlewood decompositions and multiplicative estimates

Define the spaces F, F^s as follows:

$$\|u\|_F = \sum_{\lambda=2^j} \lambda^{\frac{n}{2}} \|S_\lambda u\|_{F_\lambda}$$

$$\|u\|^2_{F^s} = \sum_{\lambda=2^j} \lambda^{2s} \|S_\lambda F\|^2_{F_\lambda}$$

Then (iii) requires the dyadic estimates

$$\|S_\lambda u \cdot S_\mu v\|_{F_\lambda} \lesssim \mu^{\frac{n}{2}} \|S_\lambda u\|_{F_\lambda} \|S_\mu v\|_{F_\mu} \qquad \mu \ll \lambda$$

respectively

$$\sum_{\mu \leq \lambda} \mu^{\frac{n}{2}} \|S_\mu(S_\lambda u \cdot S_\lambda v)\|_{F_\mu} \lesssim \lambda^n \|S_\lambda u\|_{F_\lambda} \|S_\lambda v\|_{F_\lambda} \qquad \mu \leq \lambda$$

which follow from (41), (44) in Theorem 5.

Similarly, (iv) requires the dyadic estimates

$$\|S_\lambda u \cdot S_\mu v\|_{\Box F_\lambda} \lesssim \mu^{\frac{n}{2}} \|S_\lambda u\|_{F_\lambda} \|S_\mu v\|_{\Box F_\mu} \qquad \mu \ll \lambda$$

$$\|S_\lambda u S_\mu v\|_{\Box F_\lambda} \lesssim \mu^{\frac{n}{2}} \|S_\lambda u\|_{\Box F_\lambda} \|S_\mu v\|_{F_\mu} \qquad \mu \ll \lambda$$

respectively

$$\sum_{\mu \leq \lambda} \mu^{\frac{n}{2}} \|S_\mu(S_\lambda u S_\lambda v)\|_{\Box F_\mu} \lesssim \lambda^n \|S_\lambda u\|_{F_\lambda} \|S_\lambda v\|_{\Box F_\lambda} \qquad \mu \leq \lambda$$

By duality these estimates are equivalent to

$$\|S_\mu(S_\lambda u S_\lambda v)\|_{G_\mu} \leq c\lambda \mu^{\frac{n}{2}-1} \|S_\lambda u\|_{F_\lambda} \|S_\lambda v\|_{G_\lambda} \qquad \mu \leq \lambda$$

$$\|S_\lambda u S_\mu v\|_{G_\lambda} \leq c\mu^{\frac{n}{2}} \|S_\lambda u\|_{G_\lambda} \|S_\mu v\|_{F_\mu} \qquad \mu \ll \lambda$$

respectively

$$\|\sum_{\mu \leq \lambda} S_\mu u \cdot S_\lambda v\|_{G_\lambda} \leq c\lambda^{n-1} \|S_\lambda v\|_{F_\lambda} \sup_{\mu \leq \lambda} \mu^{-\frac{n}{2}+1} \|S_\mu u\|_{G_\mu}$$

which in turn follow from (45), (42) and (43) in Theorem 5.

Finally, (iii)', (iv)' reduce in a similar manner to the dyadic estimates in Theorem 5. △

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