PHASE SPACE TRANSFORMS AND MICROLOCAL ANALYSIS

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1. Introduction

The aim of this notes is to introduce a phase space approach to microlocal analysis. This is just a beginning, and there are many directions one can take from here.

The main tool in our analysis is the Bargman transform, which is a phase space transform. In other words, it allows one to represent functions as smooth superpositions of elementary pieces, or coherent states. The coherent states are strongly localized both in position and in frequency, precisely on the scale of the uncertainty principle.

This type of analysis has its origins in physics. On the mathematical side, a close relative, namely the FBI transform, was successfully used in the study of partial differential operators with analytic coefficients, see for instance [10]. For additional information about the FBI transform we refer the reader to Delort's monograph [2], Folland's book [5] and to the author's article [15]. Also closely related related topics are discussed in [9], [8]. More recently, phase space transforms were used to construct parametrices for wave and Schrödinger operators with rough coefficients in [14], [12], [7].

We note that there is also an alternate approach to phase space analysis, namely to replace smooth decompositions with discrete decompositions. This was first outlined in Fefferman [4] and pursued later by a number of authors. Most notably, we should mention Smith [11]'s introduction of wave packets in the study of the wave equation. However, as the reader will see, there is a significant advantage in using smooth families of coherent states as opposed to any discrete methods.

The road map for this article is as follows. First we introduce the Bargman transform and some simple properties. Then we use it to give a simple characterization of S^0_{00} type pseudodifferential operators.

Next, following [14] and [15], we introduce a higher order calculus and use it to prove some classical estimates, namely the sharp Gårding and the Fefferman-Phong inequality.

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The last two sections are devoted to Fourier integral operators. First we introduce S^0_{00} type Fourier integral operators associated to bilipschitz canonical transformations. Then, following in part [7], we study a class of evolution equations and show that the evolution operators are Fourier integral operators associated to the Hamilton flow maps. Finally, an Egorov theorem is also given.

2. The Bargman transform

The Bargman transform of a temperate distribution f is a smooth function in $\mathbb{C}^n$ defined as[1]

$$(Tf)(x,\xi) = 2^{-\frac{n}{2}}\pi^{-\frac{3n}{4}} \int e^{-\frac{1}{2}(x-y)^2} e^{i\xi(x-y)} f(y)\, dy \tag{1}$$

We note some simple mapping properties of T,

$$T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{2n}), \qquad T : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^{2n})$$

To understand better how it works, consider the L^2 normalized function

$$f_{x_0,\xi_0}(y) = \pi^{-\frac{n}{4}} e^{-\frac{1}{2}(y-x_0)^2} e^{i(y-x_0)}$$

which is localized in a neighborhood of size 1 of x_0 and frequency localized in a neighborhood of size 1 of ξ_0. Due to the uncertainty principle this is the best one can do when trying to localize in both space and frequency. Such functions have been used by physicists, most notably in quantum mechanics, under the name of *coherent states*. Its Bargman transform

$$(Tf)(x,\xi) = \pi^{-\frac{n}{2}} e^{-\frac{1}{4}[(x-x_0)^2+(\xi-\xi_0)^2]} e^{-i\frac{1}{2}(x-x_0)(\xi+\xi_0)}\,.$$

is concentrated on the unit scale near (x_0, ξ_0).

In some ways the Bargman transform is similar to the Fourier transform. Most notably, using the Fourier inversion formula and some Gaussian integration one easily obtains

Proposition 2.1. *The Bargman transform T satisfies*

$$T^*T = I$$

A consequence of this is that T is an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$. An inversion formula also follows,

$$f(y) = 2^{-\frac{n}{2}}\pi^{-\frac{3n}{4}} \int \Phi(z) e^{-\frac{1}{2}(\bar{z}-y)^2} (Tf)(z)\, dx d\xi\,. \tag{2}$$

[1]An alternative definition uses an additional factor of $e^{\frac{1}{2}\xi^2}$ in the integral. This has the advantage that it makes Tf a holomorphic function of $z = x - i\xi$. It also accounts for small differences in formulas in various papers on the subject.

One might find this a more natural starting point for the Bargman transform; then the idea is to seek a representation of functions as superpositions of coherent states.

However this is where the similarity ends. A short computation yields the Cauchy-Riemann type relation

$$i\partial_\xi Tf = (\partial_x - i\xi)Tf \tag{3}$$

Thus T is not surjective. As it turns out, its range in L^2 consists exactly of those functions which satisfy (3). This also shows that (2) is not the only possible inversion formula.

3. The S^0_{00} calculus

Given a temperate distribution $a \in \mathcal{S}'(\mathbb{R}^{2n})$ we define the corresponding Weyl operator as

$$a^w(x,D)f(x) = \int a(\frac{x+y}{2},\xi)e^{i(x-y)\xi}f(y)dyd\xi$$

as an operator mapping

$$a^w : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$$

We consider a class of symbols which satisfy some additional conditions:

Definition 3.1. *We say that $a \in S^0_{00}$ if for all multiindices α and β it satisfies the bounds*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq c_{\alpha\beta}$$

We denote by OPS^0_{00} the corresponding class of symbols.

The main result of this section establishes the connection between OPS^0_{00} type operators and their phase space representation.

Theorem 1. *Let $A : S(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. Then $A \in OPS^0_{00}$ if and only if the kernel K of TAT^* satisfies the bounds*

$$|K(x_1,\xi_1,x_2,\xi_2)| \leq c_N(1+|x_1-x_2|+|\xi_1-\xi_2|)^{-N}, \qquad N \in \mathbb{N} \tag{4}$$

Proof. a) Let $a \in S^0_{00}$. Then the kernel of Ta^wT^* has the form

$$\begin{aligned} K(x_2,\xi_2,x_1,\xi_1) &= c_n \int e^{-\frac{1}{2}(x_2-y_2)^2} e^{i\xi_2(x_2-y_2)} a(\frac{y_1+y_2}{2},\eta) e^{i\eta(y_2-y_1)} \\ &\quad e^{-\frac{1}{2}(x_1-y_1)^2} e^{i\xi_1(y_1-x_1)} dy_1 d\eta dy_2 \end{aligned}$$

We change variables $z_j = y_j - x_j$ to obtain

$$\begin{aligned} K(x_2,\xi_2,x_1,\xi_1) &= c_n \int e^{-\frac{1}{2}z_2^2} e^{i(\eta-\xi_2)z_2} a(\frac{x_1+z_1+x_2+z_2}{2},\eta) e^{i\eta(x_2-x_1)} \\ & \qquad e^{-\frac{1}{2}z_1^2} e^{i(\xi_1-\eta)z_1} dz_1 d\eta dz_2 \end{aligned}$$

The integration with respect to z_1, z_2 yields

$$K(x_2,\xi_2,x_1,\xi_1) = \int b(\eta-\xi_2,\eta-\xi_1,x_1+x_2,\eta) e^{i\eta(x_2-x_1)} d\eta$$

where b is a Schwartz function in the first two variables and smooth and bounded in the last two. Then integration in η yields

$$K(x_2,\xi_2,x_1,\xi_1) = c(\xi_1-\xi_2,x_1-x_2,\xi_1,x_1) e^{i\xi_1(x_1-x_2)}$$

where c is a Schwartz function in the first two variables and smooth and bounded in the last two.

b) If K is the kernel of TAT^* then the kernel of A is

$$\begin{aligned} L(y_2,y_1) &= c_n \int e^{-\frac{1}{2}(x_2-y_2)^2} e^{i\xi_2(y_2-x_2)} K(x_2,\xi_2,x_1,\xi_1) \\ & \qquad e^{-\frac{1}{2}(x_1-y_1)^2} e^{i\xi_1(x_1-y_1)} dx_1 dx_2 d\xi_1 d\xi_2 \end{aligned}$$

On the other hand, the Weyl symbol of A is

$$a(y,\eta) = \int L(y+z,y-z) e^{-2i\eta z} dz$$

We substitute L from above,

$$\begin{aligned} a(y,\eta) &= c_n \int e^{-\frac{1}{2}(x_2-y-z)^2} e^{i\xi_2(y+z-x_2)} K(x_2,\xi_2,x_1,\xi_1) \\ & \qquad e^{-\frac{1}{2}(x_1-y+z)^2} e^{i\xi_1(x_1-y+z)} e^{-2i\eta z} dx_1 dx_2 d\xi_1 d\xi_2 \end{aligned}$$

and do the explicit z integration

$$\begin{aligned} a(y,\eta) &= c_n \int e^{-\frac{1}{4}(x_1+x_2-2y)^2} e^{-\frac{1}{4}(\xi_1+\xi_2-2\eta)^2} K(x_2,\xi_2,x_1,\xi_1) \\ & \qquad e^{\frac{i}{2}(\xi_1-\xi_2)(x_1+x_2)} e^{iy(\xi_2-\xi_1)} e^{i\eta(x_1-x_2)} dx_1 dx_2 d\xi_1 d\xi_2 \end{aligned}$$

To estimate the size of a we take absolute values and integrate directly using the bound on K. For the derivatives of a we first differentiate and then take absolute values and integrate.

$\square$

Corollary 3.2. *OPS^0_{00} is an algebra of bounded operators in $L^2(\mathbb{R}^n)$.*

Proof. Let $A_1, A_2 \in OPS^0_{00}$. Then
$$TA_1A_2T^* = (TA_1T^*)(TA_2T_*)$$
The two operators on the right are integral operators with kernels which decay rapidly away from the diagonal. Then a simple integration shows that $TA_1A_2T^*$ is also an integral operator with a similar kernel.

For the L^2 boundedness we begin with $A \in OPS^0_{00}$ and write
$$A = T^*(TAT^*)T$$
The left and right factors are L^2 bounded. The middle factor is an integral operators with a rapidly decaying kernel off the diagonal, so it is also L^2 bounded. □

4. A conjugation result

Given a symbol $a(x, \xi)$, in this section we seek to determine an approximate conjugate $\tilde{A}^w$ of a^w with respect to T,
$$TA^w \approx \tilde{A}^w T + \text{error}$$
Based on the classical vs. quantum correspondence, we expect the main term in $\tilde{A}^w$ to be exactly the multiplication by the symbol. Such an analysis is not so meaningful for S^0_{00} symbols because the error would in general have the same size as the principal part. Thus we introduce some more general classes of symbols:

Definition 4.1. *Let k be a nonnegative integer. We say that $a \in S^{0,(k)}_{00}$ if it satisfies the bounds*
$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq c_{\alpha\beta}, \qquad |\alpha| + |\beta| \geq k$$
We denote by $OPS^{0,(k)}_{00}$ the corresponding class of symbols.

These classes allow for instance symbols which are polynomials in x and ξ. Such symbols are our first candidates for conjugation.

To take advantage of the Cauchy-Riemann equations (3) it is convenient to introduce complex notations for the cotangent bundle. Thus we set
$$z = x - i\xi$$
The complex differentiation operators are
$$\partial = \frac{1}{2}(\partial_x + i\partial_\xi), \qquad \bar{\partial} = \frac{1}{2}(\partial_x + i\partial_\xi)$$
The Cauchy-Riemann equations (3) have the form
$$(\bar{\partial} - \frac{i}{2}\xi)T = 0 \tag{5}$$

For first order polynomials we easily conjugate

$$Ty = (x + (\partial - \frac{i}{2}\xi))T, \quad TD = (\xi + \frac{1}{i}(\partial - \frac{i}{2}\xi))T\,,$$

which can be rewritten in the form

$$T(y - iD) = zT, \quad T(y + iD) = [\bar{z} + 2(\partial - \frac{i}{2}\xi)]T\,.$$

Based on this, one can take a Taylor series expansion of a symbol a and obtain the formal asymptotics

$$Ta^w \approx \sum_{\alpha \leq \beta} 2^{|\beta|-|\alpha|} \frac{\partial^\alpha \bar{\partial}^\beta a(x,\xi)}{\alpha!(\beta-\alpha)!}(\partial - i\xi)^{\beta-\alpha} T\,.$$

which is an exact formula if a is a polynomial. To gain a better understanding of the size of the terms in the above series we note the following estimate proved in [13]:

Lemma 4.2. *The following estimate holds:*

$$\|(\partial - \frac{i}{2}\xi)^\alpha Tu\|_{L^2_\phi} = c_\alpha \|u\|_{L^2}\,. \tag{6}$$

For $k > 0$ define the partial sum

$$\tilde{A}_k^w = \sum_{\alpha \leq \beta}^{|\alpha|+|\beta|<k} 2^{|\beta|-|\alpha|} \frac{\partial^\alpha \bar{\partial}^\beta a(x,\xi)}{\alpha!(\beta-\alpha)!}(\partial - \frac{i}{2}\xi)^{\beta-\alpha} T$$

Then the main conjugation result is

Theorem 2. *Let $a \in S^{0,(k)}_{0,0}$. Then*

$$\|Ta^w - \tilde{A}_k^w T\|_{L^2 \to L^2} \lesssim 1 \tag{7}$$

We note some special cases. If $k = 1$ then the approximate conjugate operator is simply the multiplication by the symbol,

$$\tilde{A}_1^w = a$$

For $k = 2$ we get

$$\tilde{A}_2^w = a + 2\bar{\partial}a(\partial - \frac{i}{2}\xi)\,. \tag{8}$$

Using the Cauchy-Riemann equations (3) this can also be replaced by the more revealing formula

$$\tilde{A}_2^w = a + i(a_x \partial_\xi - a_\xi(\partial_x - i\xi))$$

where we see that the second term is nothing but the Hamilton flow for the symbol a. For later use we also record the $k=4$ case,

$$\tilde{A}^w_4 = a + 2\bar{\partial}a(\partial - \frac{i}{2}\xi) + 2\bar{\partial}^2 a(\partial - \frac{i}{2}\xi)^2 + \frac{4}{3}\bar{\partial}^3 a(\partial - \frac{i}{2}\xi)^3$$
$$+ \partial\bar{\partial}a + 2\partial\bar{\partial}^2 a(\partial - \frac{i}{2}\xi)\,. \tag{9}$$

Proof. The operator TA^w has the form

$$(TA^w u)(x,\xi) = c_n \int e^{-\frac{1}{2}(x-\tilde{y})^2} e^{i\xi(x-\tilde{y})} a(\frac{y+\tilde{y}}{2},\eta) e^{i(\tilde{y}-y)\eta} u(y)dyd\eta\,.$$

Introduce the new complex variable

$$w = \frac{y+\tilde{y}}{2} - i\eta\,.$$

Also use the notation $z = x - i\xi$. Then

$$(TA^w u)(x,\xi) = c_n \int a(w) e^{\phi(z,w,y)} u(y) dw d\bar{w} dy\,.$$

where

$$\begin{aligned}
\phi(z,w,y) &= -\frac{1}{2}(z+y-w-\bar{w})^2 + \frac{1}{2}(\bar{w}-w)(\bar{w}+w-2y) - \frac{1}{2}\xi^2 \\
&= \bar{w}(z-w) - \frac{1}{2}[(z+y-w)^2 + w(w-2y)] - \frac{1}{2}\xi^2\,.
\end{aligned}$$

We claim that $\tilde{A}^w_k$ is the operator obtained from this by replacing $a(w)$ by its k-th Taylor polynomial at z,

$$(\tilde{A}^w_k T u)(z) = \sum_{|\alpha|+|\beta|<k} \frac{\partial^\alpha \bar{\partial}^\beta a(z)}{\alpha!\beta!} \int (w-z)^\alpha (\bar{w}-\bar{z})^\beta e^{\phi(z,w,y)} u(y) dw d\bar{w} dy\,. \tag{10}$$

To prove this assertion we integrate by parts. Observe that

$$\frac{\partial\phi}{\partial\bar{w}} = (z-w), \qquad 2(\frac{\partial\phi}{\partial z} - \frac{i}{2}\xi) - \frac{\partial\phi}{\partial w} = (\bar{w}-\bar{z})\,.$$

Then we use the first relation to eliminate all the $(w-z)$ factors and the second relation to deal with the $(\bar{w}-\bar{z})$ factors. We get

$$\begin{aligned}
\int (w-z)^\alpha (\bar{w}-\bar{z})^\beta e^{\phi(z,w,y)} dw d\bar{w} &= \frac{\beta!}{(\beta-\alpha)!} \int (\bar{w}-\bar{z})^{\beta-\alpha} e^{\phi(z,w,y)} dw d\bar{w} \\
&= \frac{\beta!}{(\beta-\alpha)!} [2(\frac{\partial}{\partial z} - \frac{i}{2}\xi)]^{\beta-\alpha} \int e^{\phi(z,w,y)} dw d\bar{w}\,.
\end{aligned}$$

This proves (10). Note that the above computations are rigorous since the phase function ϕ is non-positive definite.

It remains to prove the $L^2 \to L^2_\phi$ remainder bound. It is easier to visualize the analysis in the real setting, where R_k has the form

$$(R_k u)(x,\xi) = \int b(x,\xi,\frac{y+\tilde{y}}{2},\eta) e^{i\xi(x-\tilde{y})} e^{-\frac{1}{2}(x-\tilde{y})^2} e^{i(\tilde{y}-y)\eta} d\tilde{y} d\eta \,.$$

and b is the order k remainder in the Taylor series for the symbol a,

$$b(x,\xi,y,\eta) = a(y,\eta) - \sum_{|\alpha|+|\beta|<k} \frac{\partial_x^\alpha \partial_\xi^\beta a(x,\xi)}{\alpha!\beta!} (y-x)^\alpha (\xi-\eta)^\beta \,.$$

Note that b satisfies the bounds

(11) $$|\partial_y^\alpha \partial_\eta^\beta b(x,\xi,y,\eta)| \leq c_{\alpha,\beta} (|x-y| + |\xi-\eta|)^{\max\{k-|\alpha|-|\beta|,0\}} \,.$$

We can eliminate part of the exponential weight if we observe that

$$\|R_k\|_{L^2 \to L^2_\phi} = \|e^{-ix\xi} e^{-\frac{1}{2}\xi^2} R_k\|_{L^2 \to L^2} \,.$$

We write the kernel $H(x,\xi,y)$ of R_k in the form

$$H(x,\xi,y) = e^{i\xi(x-y)} c(x,\xi,y-x)$$

where

$$c(x,\xi,y-x) = \int b(x,\xi,\frac{y+\tilde{y}}{2},\eta) e^{-\frac{1}{2}(x-\tilde{y})^2} e^{i(\tilde{y}-y)(\eta-\xi)} d\tilde{y} d\eta \,.$$

After the change of variable $\eta := \eta - \xi$, $\tilde{y} := \tilde{y} - x$ this becomes

$$c(x,\xi,y) = \int b(x,\xi,x+\frac{y+\tilde{y}}{2},\xi+\eta) e^{-\frac{1}{2}\tilde{y}^2} e^{i(\tilde{y}-y)\eta} d\tilde{y} d\eta \,.$$

We claim that c is a Schwartz function in y, uniformly in x, ξ. Given the bound (11) on b, it suffices to show that the function

$$c(y) = \int b(y+\tilde{y},\eta) e^{-\frac{1}{2}\tilde{y}^2} e^{i(\tilde{y}-y)\eta} d\tilde{y} d\eta$$

is a Schwartz function provided that b satisfies

$$|\partial_y^\alpha \partial_\eta^\beta b(y,\eta)| \leq c_{\alpha,\beta} (|y| + |\eta|)^{\max\{k-|\alpha|-|\beta|,0\}} \,.$$

Indeed, the function $b(y+\tilde{y},\eta) e^{-\frac{1}{2}\tilde{y}^2}$ is a Schwartz function in $\tilde{y}$ of size at most $(|y|+|\eta|)^k$. Then integrating with respect to $\tilde{y}$ we get a Schwartz function in η of size at most $|y|^k$. Finally, integrating with respect to η we get a Schwartz function in y.

The $L^2 \to L^2$ boundedness of R_k is equivalent to the L^2 boundedness of $R_k R_k^*$, whose kernel is

$$K(x,\xi,\tilde{x},\tilde{\xi}) = \int c(x,\xi,x-y) c(\tilde{x},\tilde{\xi},\tilde{x}-y) e^{iy(\tilde{\xi}-\xi)} d\eta \,.$$

Integrating with respect to y we get

$$|K(x,\xi,\tilde{x},\tilde{\xi})| \leq c_N(1+|x-\tilde{x}|+|\xi-\tilde{\xi}|)^{-N}$$

and the L^2 boundedness follows. $\square$

5. The Gårding and Fefferman-Phong inequalities

As an immediate application of the phase space representations of pseudodifferential operators obtained in the previous section one can provide alternate proofs for the classical Gårding and Fefferman-Phong inequalities, see [6], [3], [1]. For this we choose a simple setup which is adapted to our symbol classes. We begin with

Definition 5.1. *An operator $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is semipositive if there is $C > 0$ so that*

$$\langle A\phi, \phi\rangle \geq -C\|\phi\|_{L^2}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Then the main result we prove, following [15], is

Theorem 3. *a) (The sharp Gårding's inequality) Let $a \in S^{0,(2)}_{0,0}$ be a real $N \times N$ nonnegative symbol. Then $a^w(\frac{x+y}{2}, D)$ is semipositive.*

b) (The Fefferman-Phong inequality) Let $a \in S^{0,(4)}_{0,0}$ be a real scalar nonnegative symbol in $T^*\mathbb{R}^n$. Then $a^w(\frac{x+y}{2}, D)$ is semipositive.

Proof. a) By Theorem 2 with $k = 2$ we have

$$\langle a^w u, u\rangle = \langle Ta^w u, Tu\rangle = \langle \tilde{A}_2^w Tu, Tu\rangle + O(\|u\|_{L^2})$$

It remains to prove that $\tilde{A}_2^w$ is semipositive on the space of $L^2(\mathbb{R}^{2n})$ functions which satisfy the Cauchy-Riemann equations (5). For such functions v we have

$$\begin{aligned}
\langle \tilde{A}_2^w v, v\rangle &= \langle av, v\rangle + \langle 2\bar{\partial}a(\partial - \frac{i}{2}\xi)v, v\rangle \\
&= \langle (a - 2\partial\bar{\partial}a)v, v\rangle - \langle 2\bar{\partial}av, (\bar{\partial} - \frac{i}{2}\xi)v\rangle \\
&= \langle (a - 2\partial\bar{\partial}a)v, v\rangle
\end{aligned}$$

By hypothesis a is nonnegative and $\partial\bar{\partial}a$ is bounded, so this concludes the proof.

b) The classical proof [3] of the Fefferman-Phong inequality, refined in [1], is based on successive localizations combined on an inductive argument with respect to the dimension. All this is done at the operator level. The idea is to successively peel off squares of operators in a sufficiently localized setting, all while retaining sufficient orthogonality to be able to assemble together the localized results. Instead, a simpler idea is to reduce the problem to a statement about decompositions of $\dot{C}^{3,1}$ nonnegative functions as sums of squares of $\dot{C}^{1,1}$ functions:

Proposition 5.2. [15] *There exist $K, M > 0$ depending only on the dimension n so that for any nonnegative function ϕ in $\mathbb{R}^n$ satisfying*

$$|\nabla^4 \phi| \leq 1$$

there exist functions $\{\phi_k\}_{k=1\ldots K}$ so that

$$\phi = \sum_{k=1}^{K} \phi_k^2$$

and

$$\sum_k |\nabla^2 \phi_k|^2 + |(\nabla^3 \phi_k)(\nabla \phi_k)| \leq M\,. \tag{12}$$

We rely on [15] for the proof of the Proposition, but show here how it implies the Fefferman-Phong inequality.

By Theorem 2 with $k = 4$, the semipositivity of a^w is equivalent to the semipositivity of $\tilde{A}_4^w$ on the space of functions which satisfy (5),

$$\langle \tilde{A}_4^w v, v\rangle \geq -c\|v\|_{L^2}^2$$

At this point we discard all the information about the higher order derivatives of the symbol a and only retain the bound on the fourth order derivatives. The operator $\tilde{A}_4^w$ is given by (9). Then, as before, we integrate by parts all the terms in $\tilde{A}_{l,4}^w$ which contain $(\partial - \frac{i}{2}\xi)$,

$$\langle 2\bar{\partial}a(\partial - \frac{i}{2}\xi)v, v\rangle = -2\langle(\partial\bar{\partial}a)v, v\rangle\,.$$

This term in not necessarily bounded in L^2_ϕ since we have no information about the second derivatives of a.

$$\langle 2\bar{\partial}^2 a(\partial - \frac{i}{2}\xi)^2 v, v\rangle = 2\langle(\partial^2\bar{\partial}^2 a)v, v\rangle\,.$$

This is bounded since the fourth derivatives of a are bounded.

$$\langle \frac{4}{3}\bar{\partial}^3 a(\partial - \frac{i}{2}\xi)^3 v, v\rangle = -\frac{4}{3}\langle(\partial\bar{\partial}^3 a)(\partial - \frac{i}{2}\xi)^2 v, v\rangle\,.$$

This is also bounded by Lemma 4.2.

$$\langle 2\partial\bar{\partial}^2 a(\partial - \frac{i}{2}\xi)v, v\rangle = -2\langle(\partial^2\bar{\partial}^2 a)v, v\rangle$$

which is also bounded. Summing up all these estimates we obtain the following simple formula:

$$\langle \tilde{A}_4^w v, v\rangle = \langle (a - \partial\bar{\partial}a)v, v\rangle + O(\|v\|^2_{L^2})\,. \tag{13}$$

By Proposition 5.2 we can assume without any restriction in generality that $a = \phi^2$ where ϕ satisfies the bounds in (12). Then

$$a - \partial\bar{\partial}a = \phi^2 - 2(\partial\phi)(\bar{\partial}\phi) - 2\phi(\partial\bar{\partial}\phi)\,.$$

On the other hand we use the same integration by parts procedure to evaluate the nonnegative quadratic form

$$\begin{aligned}
Q(v,v) &= \|(\phi + 2\bar{\partial}\phi(\partial - \frac{i}{2}\xi) + \partial\bar{\partial}\phi)v\|^2_{L^2_\Psi} \\
&= \langle (\phi + \partial\bar{\partial}\phi)^2 v, v\rangle + 4\Re\langle (\partial\phi)(\phi+\partial\bar{\partial}\phi)v, (\partial - \frac{i}{2}\xi))v\rangle \\
&+ 4\langle \bar{\partial}\phi(\partial - \frac{i}{2}\xi))v, \bar{\partial}\phi(\partial - \frac{i}{2}\xi))v\rangle \\
&= \langle (\phi + \partial\bar{\partial}\phi)^2 v, v\rangle - 4\Re\langle [\bar{\partial}((\partial\phi)(\phi+\partial\bar{\partial}\phi))]v, v\rangle \\
&+ 4\langle \frac{1}{2}(\partial\phi)(\bar{\partial}\phi) + \partial\bar{\partial}(\partial\phi)(\bar{\partial}\phi)v, v\rangle \\
&= \langle [\phi^2 - 2(\partial\phi)(\bar{\partial}\phi) - 2\phi(\partial\bar{\partial}\phi)]v, v\rangle \\
&+ \langle [(\partial\bar{\partial}\phi)^2 - 4\Re\bar{\partial}(\partial\phi)(\partial\bar{\partial}\phi) + 4\partial\bar{\partial}(\partial\phi)(\bar{\partial}\phi)]v, v\rangle^2\,.
\end{aligned}$$

By (12) the second term is bounded, therefore

$$\langle \tilde{A}_4^w v, v\rangle = Q(v,v) + O(\|v\|^2_{L^2})\,.$$

Hence the semipositivity of $\tilde{A}_{l,4}^w$ follows. ☐

6. Canonical transformations and S^0_{00} type Fourier integral operators

Here we define Fourier integral operators associated to canonical transformations and study their properties. To keep things simple we considerably restrict the class of canonical transformations we work with.

Definition 6.1. *$\chi : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ is a smooth bilipschitz canonical transformation if*

(i) χ is smooth,

$$|\partial_x^\alpha \partial_\xi^\beta \chi(x,\xi)| \le c_{\alpha,\beta}, \qquad |\alpha| + |\beta| > 0$$

(ii) χ is a diffeomorphism.

(iii) Both χ and χ^{-1} are Lipschitz continuous.

(iv) χ preserves the symplectic form, i.e. if $(y,\eta) = \chi(x,\xi)$ then $dy \wedge d\eta = dx \wedge d\xi$.

To such canonical transformations we want to associate Fourier integral operators. However, there is no well-defined notion of a symbol for an S^0_{00} type Fourier integral operator. Hence it is more efficient to begin with the phase space representation in the first place.

Definition 6.2. *Let $\chi : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ be a smooth bilipschitz canonical transformation. An operator $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is an S^0_{00} type Fourier integral operator associated to χ if the kernel K of TAT^* satisfies*

$$|K(y,\eta,x,\xi)| \leq c_N(1 + |(y,\eta) - \chi(x,\xi)|)^{-N}$$

By Theorem 1 this definition is consistent with the corresponding class of pseudodifferential operators.

Proposition 6.3. *The S^0_{00} type Fourier integral operator associated to the identity are exactly the S^0_{00} type pseudodifferential operators.*

Another easy consequence of this definition is

Proposition 6.4. *S^0_{00} type Fourier integral operators are L^2 bounded.*

Also the composition result requires hardly any work at all:

Proposition 6.5. *Let $\chi_1, \chi_2 : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ be smooth bilipschitz canonical transformations. Suppose A_1, A_2 are S^0_{00} type Fourier integral operators associated to χ_1, respectively χ_2. Then A_1A_2 is an S^0_{00} type Fourier integral operator associated to $\chi_1 \circ \chi_2$.*

The interesting part is to return closer to a classical representation of the Fourier integral operators. The classical representations of Fourier integral operators rely on parametrizations of the canonical transformations which are in general not global. Instead here we obtain a phase space representation. This involves one extra integration, but on the other hand it is global, has more symmetry and closely mirrors the canonical transformation.

Theorem 4. *Let $\chi : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ be a smooth bilipschitz canonical transformation. An operator $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is an S^0_{00} type Fourier integral operator associated to χ if and only if it can be represented in the form*

$$Au(\tilde{y}) = \int e^{-\frac{1}{2}(x-y)^2} e^{i\xi(x-y)} e^{i\psi(x,\xi)} e^{-i\tilde{\xi}(\tilde{x}-\tilde{y})} G(\tilde{y},\tilde{x},\tilde{\xi}) u(y)\, dx d\xi dy \tag{14}$$

where $(\tilde{x},\tilde{\xi}) = \chi(x,\xi)$, G satisfies

$$|(y-x)^\gamma \partial_x^\alpha \partial_\xi^\beta \partial_y^\nu G(y,x,\xi)| \leq c_{\gamma\alpha\beta\nu}$$

and the phase function ψ is determined modulo constants by

$$d\psi = \tilde{\xi} d\tilde{x} - \xi dx \tag{15}$$

Proof. It is fairly easy to see that any operator as in (14) is an S^0_{00} type Fourier integral operator associated to χ; this only involves testing the expression $e^{-i\tilde{\xi}(\tilde{x}-\tilde{y})}G(\tilde{y},\tilde{x},\tilde{\xi})$ against coherent states.

It is somewhat more difficult to prove the converse. We rewrite (14) in the form

$$Au(\tilde{y}) = \int e^{i\psi(x,\xi)} e^{-i\tilde{\xi}(\tilde{x}-\tilde{y})} G(\tilde{y},\tilde{x},\tilde{\xi}) Tu(x,\xi) dx d\xi$$

On the other hand,

$$Au = T^*TAT^*TT^*T$$

Then we can choose $e^{i\phi(x,\xi)}e^{-i\tilde{\xi}(\tilde{x}-\tilde{y})}G(\tilde{y},\tilde{x},\tilde{\xi})$ to be the kernel for the operator TAT^*TT^*. It remains to prove the desired bounds for G.

The kernel for TT^* is

$$\begin{aligned} L(x_1,\xi_1,x,\xi) &= c_n \int e^{-\frac{1}{2}(x_1-y)^2} e^{i(x_1-y)\xi_1} e^{-\frac{1}{2}(x-y)^2} e^{i(y-x)\xi} dy \\ &= c_n e^{-\frac{1}{4}(x-x_1)^2} e^{-\frac{1}{4}(\xi-\xi_1)^2} e^{\frac{i}{2}(x_1-x)(\xi_1+\xi)} \end{aligned}$$

Hence if K is the kernel of TAT^* then we must have

$$\begin{aligned} e^{i\psi(x,\xi)} e^{-i\tilde{\xi}(\tilde{x}-\tilde{y})} G(\tilde{y},\tilde{x},\tilde{\xi}) &= \int e^{-\frac{1}{2}(x_2-\tilde{y})^2} e^{i(x_2-\tilde{y})\tilde{\xi}_2} K(x_2,\xi_2,x_1,\xi_1) \\ & \quad e^{-\frac{1}{4}(x-x_1)^2} e^{-\frac{1}{4}(\xi-\xi_1)^2} e^{\frac{i}{2}(x_1-x)(\xi_1+\xi)} dx_1 d\xi_1 dx_2 d\xi_2 \end{aligned}$$

In the integral on the right we expect no cancellations because we only have pointwise bounds on K. The decay of G is easy to obtain:

$$\begin{aligned} |G(\tilde{y},\tilde{x},\tilde{\xi})| &\lesssim \int e^{-\frac{1}{2}(x_2-\tilde{y})^2} (1+|(x_2,\xi_2)-\chi(x_1,\xi_1)|)^{-N} \\ & \quad e^{-\frac{1}{4}(x-x_1)^2} e^{-\frac{1}{4}(\xi-\xi_1)^2} dx_1 d\xi_1 dx_2 d\xi_2 \\ &\lesssim \int e^{-\frac{1}{2}(x_2-\tilde{y})^2} (1+|(x_2,\xi_2)-\chi(x,\xi)|)^{-N} dx_2 d\xi_2 \\ &= \int e^{-\frac{1}{2}(x_2-\tilde{y})^2} (1+|(x_2,\xi_2)-(\tilde{x},\tilde{\xi})|)^{-N} dx_2 d\xi_2 \\ &\lesssim (1+|y-\tilde{x}|)^{-N} \end{aligned}$$

To also bound the derivatives of G we need in addition to control the regularity of the phase function,

$$|\partial_{\tilde{y}}^{\gamma} \partial_x^{\alpha} \partial_\xi^{\beta} \phi| \leq c_{\alpha\beta\gamma}(1+|\tilde{x}-\tilde{y}|+|x_2-\tilde{y}|+|\tilde{\xi}-\xi_2|+|x_1-x|+|\xi_1-\xi|)$$

where

$$\phi = -\psi(x,\xi) + \tilde{\xi}(\tilde{x} - \tilde{y}) + (x_2 - \tilde{y})\tilde{\xi}_2 + \frac{1}{2}(x_1 - x)(\xi_1 + \xi)$$

We split

$$\phi = \phi_1 + \phi_2,$$

where

$$\phi_1 = (\tilde{\xi} - \tilde{\xi}_2)(\tilde{y} - x_2) + \frac{1}{2}(x_1 - x)(\xi_1 - \xi)$$

$$\phi_2 = \tilde{\xi}(\tilde{x} - x_2) - \xi(x - x_1) - \psi(x,\xi)$$

The regularity of ϕ_1 follows directly from the regularity of the canonical transformation χ. This is also the case with ϕ_2 provided at least one derivative falls on the $\tilde{\xi}$ in the first term or on the ξ in the last term. But everything else cancels due to (15). □

7. Evolution equations

A main source of Fourier integral operators in analysis is provided by evolution operators associated to various evolution equations. The wave equation and the Schrodinger equation are good examples.

The classical dynamics is described by the Hamilton flow, which generates the canonical transformations governing the evolution. On the quantum side the evolution is governed by Fourier integral integral operators which correspond to these canonical transformations.

The context we choose here is fairly simple, yet it contains many interesting examples. We study evolution equations of the form

$$(D_t + a^w(t,x,D))u = f \qquad u(0) = u_0 \tag{16}$$

where $t \in [0,1]$ and $x \in \mathbb{R}^n$. We assume that

(i) The symbol $a(t,x,\xi)$ is real.

(ii) $a(t,\cdot,\cdot) \in S^{0,(2)}_{00}$ uniformly in t.

(iii) $a(t,x,\xi)$ is continuous in t.

If the symbol a is real then a^w is selfadjoint. Using this one can easily prove

Proposition 7.1. *Under the assumptions (i), (ii), (iii) above the equation* (16) *is forward and backward well posed in $L^2(\mathbb{R}^n)$. Furthermore, the evolution operators $S(t,s)$ defined by*

$$S(t,s)u(s) = u(t), \qquad t,s \in [0,1]$$

are L^2 isometries.

The Hamilton flow associated to this evolution is

$$\dot{x} = a_\xi(t,x,\xi), \qquad \dot{\xi} = -a_x(t,x,\xi)$$

We denote the trajectories of the flow by

$$[0,1] \ni t \to (x^t, \xi^t)$$

and the fixed time maps by $\chi(t,s) : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$,

$$\chi(t,s)(x^s,\xi^s) = (x^t,\xi^t)$$

Then we have

Proposition 7.2. *Under the assumptions (i), (ii), (iii) above the fixed time maps $\chi(t,s)$ are smooth bilipschitz canonical transformations.*

The fact that $\chi(t,s)$ is a canonical transformation is a property shared by all Hamiltonian flows. The smoothness is obtained by studying the linearized flow. We note that the linearization of the flow involves the second derivatives of the symbol a. This justifies our choice of the symbol class in (ii).

7.1. **A parametrix construction.** We can take advantage of Theorem 2 with $k = 2$ to construct a parametrix for the equation (16). The approximate conjugated operator is

$$\tilde{A} = a(x,\xi) + a_\xi(\frac{1}{i}\partial_x - \xi) - \frac{1}{i}a_x\partial_\xi$$

and we the error estimate has the form

$$\|Ta^w - \tilde{A}T\|_{L^2\to L^2} \lesssim 1$$

We also note the dual bound

$$\|T^*\tilde{A} - a^wT^*\|_{L^2\to L^2} \lesssim 1$$

The operator $\tilde{A}$ is selfadjoint, therefore it generates an isometric evolution operator $\tilde{S}(t,s)$ in $L^2(\mathbb{R}^{2n})$. Then a natural choice for a forward parametrix is the operator

$$K(t,s) = 1_{t\geq s}T^*\tilde{S}(t,s)T$$

Given the above error estimates, it is straightforward to prove that this provides a good approximate solution in the L^2 sense:

Proposition 7.3. *Under the assumptions (i), (ii), (iii) above the operator $K(t,s)$ satisfies*

$$\|K(t,s)\|_{L^2\to L^2} \leq 1$$
$$\|(D_t + a^w)K(t,s)\|_{L^2\to L^2} \lesssim 1$$
$$\lim_{t\to s+} K(t,s) = 1_{L^2}$$

The kernel of the parametrix K is easy to describe. We begin with the evolution operator $\tilde{S}(t,s)$ in the phase space. It corresponds to the transport type operator

$$D_t + \tilde{A} = -i(\partial_t + a_\xi \partial_x - a_x \partial_\xi) + a(x,\xi) - \xi a_\xi$$

Solutions are transported along the Hamilton flow for $D_t + a^w$. There is also a phase shift. We define the real phase function ψ by

$$\dot{\psi} = -a + \xi a_\xi, \quad \psi(0,\bar{x},\bar{\xi}) = 0 \tag{17}$$

where $\dot{\psi}$ denotes the differentiation along the flow. Then $\tilde{S}(t,s)$ is given by

$$(\tilde{S}(t,s)u)(x^t,\xi^t) = u(x^s,\xi^s)e^{i(\psi(t,x,\xi)-\psi(s,x,\xi))}$$

and the parametrix K has the kernel

$$K(t,y,s,\tilde{y}) = 2^{-d}\pi^{-\frac{3d}{2}} \int_{\mathbb{R}^{2n}} e^{-\frac{1}{2}(y-x^t)^2 - \frac{1}{2}(\tilde{y}-x^s)^2 + i\xi^t(y-x^t) - i\xi^s(\tilde{y}-x^s)} e^{i(\psi(t,x,\xi)-\psi(s,x,\xi))} dx\, d\xi.$$

We note that the phase ψ satisfies the relation

$$d\psi(t) = \xi^t dx^t - \xi dx \tag{18}$$

This is easily verified by writing the transport equations for both sides. By Theorem 4, it follows that the above parametrix is an S^0_{00} type Fourier integral operator associated to $\chi(t,s)$.

7.2. **The exact solution.** Inspired by the above parametrix, here we obtain a similar representation for the solution.

Theorem 5. *The kernel K of the fundamental solution operator $D_t + a^w$ can be represented in the form*

$$K(t,y,s,\tilde{y}) = \int_{\mathbb{R}^{2n}} e^{-\frac{1}{2}(\tilde{y}-x^s)^2} e^{-i\xi^s(\tilde{y}-x^s)} e^{i(\psi(t,x,\xi)-\psi(s,x,\xi))} e^{i\xi^t(y-x^t)} G(t,s,x,\xi,y)dx\, d\xi \tag{19}$$

where the function G satisfies

$$|(x^t-y)^\gamma \partial_x^\alpha \partial_\xi^\beta \partial_y^\nu G(t,s,x,\xi,y)| \lesssim c_{\gamma,\alpha,\beta,\nu} \tag{20}$$

By Theorem 4 we can also conclude that

Corollary 7.4. *The evolution operators $S(t,s)$ are S^0_{00} type Fourier integral operators associated to the canonical transformation $\chi(t,s)$.*

Proof. Without any restriction in generality take $s = 0$. We use the FBI transform to decompose the initial data u_0 into coherent states, and write

$$u = S(t,0)T^*Tu_0 = \int S(t,s)\phi_{x,\xi}Tu(x,\xi)dxd\xi$$

Then we can define the function G by

$$G(t,x,\xi,y) = 2^{-\frac{d}{2}}\pi^{-\frac{3d}{4}}e^{-i\xi^t(y-x^t)}e^{-i\psi(t,x,\xi)}(S(t,0)\phi_{x,\xi})(y)$$

so that (19) holds. It remains to prove that G satisfies the bounds (20).

Suppose that we want to prove (20) at (x_0,ξ_0). We first note that the phase of G is essentially linear. Precisely, we have the relation

$$\left|\partial_x^\alpha\partial_\xi^\beta[\xi^t(y-x^t)+\psi(t,x,\xi)+\xi_0(x-x_0)]\right|_{x=x_0,\xi=\xi_0} \leq c_{\alpha,\beta}(1+|y-x_0^t|)$$

which follows from the regularity of the Hamilton flow and the relation (18). This allows us to replace the function G with

$$G_1(t,x,\xi,y) = e^{-i\xi_0^t(y-x_0^t)}e^{-i\psi(t,x_0,\xi_0)}\left(S(t,0)(e^{i\xi_0(x-x_0)}\phi_{x,\xi})\right)(y)$$

Then we translate G_1 to the origin by setting

$$G_2(t,x,\xi,y) = G_1(t,x_0+x,\xi_0+\xi,x_0^t+y)$$

The x and ξ variables are translated so that they are now centered at the origin. A routine computation shows that the function G_2 solves the modified equation

$$(D_t + a_2^w(t,y,D_y))G_2 = 0, \qquad G_2(0) = \phi_{x,\xi}$$

where

$$a_2(t,y,\eta) = a(t,x_0^t+y,\xi_0^t+\eta)-a(t,x_0^t,\xi_0^t)-ya_x(t,x_0^t,\xi_0^t)-\eta a_\xi(t,x_0^t,\xi_0^t)$$

still in $S_{00}^{0,(2)}$ but in addition vanishes of second order at $0\in\mathbb{R}^{2d}$. To differentiate it with respect to x,ξ it suffices to differentiate the initial data. But the functions

$$\partial_x^\alpha\partial_\xi^\beta\phi_{x,\xi}(y)|_{x=0,\xi=0}$$

are Schwartz functions in y. Hence it suffices to consider the problem

$$(D_t + a_2^w(t,y,D))v = 0, \qquad v(0) = v_0$$

where the initial data v_0 is a Schwartz function, and prove that the solution $v(t)$ is also a Schwartz function. This follows if we can prove energy estimates for the functions $y^\alpha\partial^\beta v$, which we do by induction over $k = |\alpha|+|\beta|$. If $k=0$ then we trivially have

$$\|v(t)\|_{L^2} = \|v(0)\|_{L^2}$$

For $k=1$ we compute the equations for yv and ∂v:

$$(D_t + a_2^w(t,y,D))(yv) = -i(\partial_\eta a)^w(t,y,D)v$$
$$(D_t + a_2^w(t,y,D))(\partial_y v) = i(\partial_y a)^w(t,y,D)v$$

To bound the right hand side we need the next lemma for the symbol $b = \partial_y a_2$ and $b = \partial_\xi a_2$. This is a special case of Theorem 3 in [15].

Lemma 7.5. *Suppose that the symbol $b(x,\xi)$ satisfies*

$$\|\partial_y^\alpha \partial_\eta^\beta b(y,\eta)\| \leq c_{\alpha,\beta} \qquad |\alpha| + |\beta| \geq 1$$

and also $b(0,0)=0$. Then

$$\|b^w(y,D)u\|_{L^2} \lesssim \|yu\|_{L^2} + \|\partial u\|_{L^2} + \|u\|_{L^2}$$

Using Lemma 7.5 and the Gronwall inequality we conclude that

$$\|yv(t)\|_{L^2} + \|\partial v(t)\|_{L^2} \lesssim \|yv_0\|_{L^2} + \|\partial v_0\|_{L^2} + \|v_0\|_{L^2}$$

It remains to do the induction step. We denote by L_k all operators of the form $x^\alpha \partial^\beta$ with $|\alpha| + |\beta| = k$. Suppose that

$$\sum_{j \leq k} \|L_j v(t)\|_{L^2} \leq c_k \sum_{j \leq k} \|L_j v_0\|_{L^2}$$

The functions $L_{k+1}v$ solve a weakly coupled system of the form

$$(D_t - a_2^w) L_{k+1} v = (\partial_{y,\eta} a_2)^w L_k v + \sum_{i \geq 2}^{i+j \leq k+1} (\partial_{y,\eta}^i a_2)^w L_j v$$

For this we use energy estimates and Gronwall's inequality. The first right hand side term is estimated using Lemma 7.5 and the second using the induction hypothesis.

□

7.3. **Egorov's Theorem.** Given a pseudodifferential operator B, we define the family of its conjugates along the flow by

$$B(t) = S(t,0)BS(0,t)$$

Egorov's theorem states that under reasonable assumptions, $B(t)$ is also a pseudodifferential operator whose principal symbol is obtained from the principal symbol of B by transport along the Hamilton flow.

A straightforward consequence of our composition result for Fourier integral operators is

Proposition 7.6. *Assume that $B \in OPS_{00}^0$. Then for each t we have $B(t) \in OPS_{00}^0$.*

This is not so satisfactory because in this context it is not meaningful to talk about the principal symbol of $B(t)$. However, the following result is considerably more interesting.

Proposition 7.7. *Assume that $B = b^w \in OPS_{00}^{0,(1)}$. Then for each t we have $B(t) = b^w(t) \in OPS_{00}^{0,(1)}$. In addition,*

$$b(t) - b \circ \chi(0,t) \in S_{00}^0$$

Proof. We transport the symbol b along the flow,

$$\tilde{b}(t) = b \circ \chi(0,t)$$

It is easy to see that $\tilde{b} \in S_{00}^{0,(1)}$. It remains to verify that

$$b(t) - \underline{(}t) \in S_{00}^0$$

We seek to obtain some convenient representation for the above difference.

For a time dependent family of operators $C(t)$ we compute

$$D_t S(0,t)C(t)S(t,0) = S(0,t)[D_t + a^w(t), C(t)]S(t,0)$$

Applying this to $B(t)$ the left hand side is zero so we obtain

$$[D_t + a^w(t), B(t)] = 0$$

Also by integration we have
(21)

$$S(0,t)C(t)S(t,0) = C(0) + \int_0^t S(0,s)(D_t C + [a^w(s), C(s)])S(s,0)ds$$

We apply (21) to $C(t) = b^w(t) - \tilde{b}^w(t)$. This yields

$$b^w(t) - \tilde{b}^w(t)) = \int_0^t S(t,s)[D_t + a^w(s), \tilde{b}^w(s)]S(s,t)ds$$

By Proposition 7.6, to conclude we need to show that

$$[D_t + a^w(s), \tilde{b}^w(s)] \in OPS_{0,0}^0$$

But this is a straightforward commutator estimate since the transport equation for $\tilde{b}$ yields

$$\tilde{b}_t + \{a, \tilde{b}\} = 0.$$

□

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