# Carleman estimates and unique continuation for second order elliptic equations with nonsmooth coefficients

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## 1 Introduction

This work is devoted to the strong unique continuation problem for second order elliptic equations with nonsmooth coefficients. Consider the second order elliptic operator

$$P = \partial_i g^{ij}(x)\partial_j$$

in $\mathbb{R}^n$, the potential $V$ and the vector fields $W_1$ and $W_2$. To these we associate the differential equation

$$Pu = Vu + W_1\nabla u + \nabla(W_2 u) \tag{1}$$

Given a function $u \in L^2_{loc}$ and $x_0 \in \mathbb{R}^n$ we say that $u$ vanishes of infinite order at $x_0$ if there exists $R$ so that for each integer $N$ we have

$$\int_{B_r(x_0)} |u|^2\, dx \le c_N r^N, \qquad r < R \tag{2}$$

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We say that the problem (1) has the strong unique continuation property **(SUCP)** if for every $H^1$ function $u$ satisfying (1) in a ball $B_R(x_0)$ the following is true:

*If $u$ vanishes of infinite order at $x_0$ then $u = 0$ near $x_0$.*

Our aim is to prove that **(SUCP)** holds under sharp scale invariant assumptions on the metric $g$ and on the potentials $V$, $W_1$ and $W_2$. For simplicity we assume that $x_0 = 0$.

To state our assumptions on $g$, $V$, $W_1$ and $W_2$ we introduce the spaces $l^q(L^p)$ with norms

$$\|V\|^q_{l^q(L^p)} = \sum_{j\in\mathbb{Z}} \|V\|^q_{L^p(\{2^{j-1}\le|x|\le 2^j\})}, \qquad 1\le p\le\infty, \quad 1\le q<\infty \tag{3}$$

Here for the sake of uniformity in notation we let $j$ go over $\mathbb{Z}$. For the strong unique continuation property only sufficiently small $j$'s are relevant. In a similar manner we define the spaces $l^\infty(L^p)$, $c_0(L^p)$ and the weak $l^q$ spaces, $l^q_w(L^p)$.

Then we consider metrics $g$ uniformly bounded from above and below and satisfying

$$\||x|\nabla g\|_{l^1_w(L^\infty)} < \varepsilon, \qquad \varepsilon \text{ small} \tag{4}$$

This does not imply that $g$ is close to the Euclidean metric. However, in our estimates later on we use a perturbation argument starting from estimates for the Euclidean metric. This requires a stronger form of (4), namely

$$\|g - I_n\|_{l^1_w(L^\infty)} + \||x|\nabla g\|_{l^1_w(L^\infty)} < \varepsilon, \qquad \varepsilon \text{ small} \tag{5}$$

The reduction of (4) to (5) is carried out in the second section using a suitable change of coordinates.

For the potentials $V$, $W_1$ and $W_2$ we consider the following assumptions:

$$V\in l^\infty(L^{\frac{n}{2}}), \quad \limsup_{r\to 0}\|V\|_{L^{\frac{n}{2}}(\{r\le|x|\le 2r\})} \le \varepsilon, \quad \varepsilon \text{ small} \tag{6}$$

respectively

$$\|W_1\|_{l^1_w(L^n)} + \|W_2\|_{l^1_w(L^n)} < \varepsilon, \qquad \varepsilon \text{ small} \tag{7}$$

A simpler replacement of (4), (6) and (7) is

$$|x|\,|\nabla g| \in l^1(L^\infty), \quad V\in c_0(L^{\frac{n}{2}}), \quad W_1, W_2 \in l^1(L^n). \tag{8}$$

If this holds then the smallness condition in (4), (6) is satisfied in a small neighborhood of the origin.

Now we can state our main result.

**Theorem 1.** *Assume that (4), (6) and (7) hold. Then* **(SUCP)** *holds at* 0 *for* $H^1$ *solutions* $u$ *to (1).*

Our results are essentially sharp. On one hand there are counterexamples due to Miller [20] and Plis [22] involving metrics which are Hölder of all orders less than one. On the other hand, the functions $e^{-(-\ln|x|)^{1+\varepsilon}}$ provide a straightforward counterexample with $V \in L^p$, $p < \frac{n}{2}$, or with $W \in L^p$, $p < n$. The smallness assumption on $V$ in $l^\infty(L^{\frac{n}{2}})$ is necessary due to a counterexample of Wolff [30]. However, uniqueness holds for $V = C|x|^{-2}$ for large $C$, see Pan [21]. Wolff [30] constructs counterexamples to **(SUCP)** with $W_1 \in l_w^{\frac{2n}{n-2}}(L^n)$, $V = W_2 = 0$. The only gap which is left in our results is therefore the gap between $W_i \in l_w^1(L^n)$ and $W_i \in l_w^{\frac{2n}{n-2}}(L^n)$. This gap can be filled, at least to a certain extent, but only at the expense of making the proofs considerably more technical.

A brief (and incomplete) history of the results on this topic is summarized in the following table:

| | | | | | |
|---|---|---|---|---|---|
| 1939 | Carleman [7] | $g^{ij} \in C^2$ | $V \in L^\infty$ | $W \in L^\infty$ | $n = 2$ |
| 1957 | Aronszajn [2] | $g^{ij} \in C^2$ | $V \in L^\infty$ | $W \in L^\infty$ | $n \geq 3$ |
| 1962 | Aronszajn & al. [3] | $g^{ij} \in C^1$ | $V \in L^\infty$ | $W \in L^\infty$ | $n \geq 2$ |
| 1985 | Jerison-Kenig [15] | $g^{ij} = \delta_{ij}$ | $V \in L^{\frac{n}{2}}$ | $W = 0$ | $n > 2$ |
| 1990 | Sogge [26] | $g^{ij} \in C^\infty$ | $V \in L^{\frac{n}{2}}$ | $W \in L^\infty$ | $n > 2$ |
| 1992 | Wolff [31] | $g^{ij} = \delta_{ij}$ | $V \in L^{\frac{n}{2}}$ | $W \in L^{\max\{n, \frac{3n-4}{2}\}}$ | $n > 2$ |
| 1999 | Regbaoui [23] | $g^{ij} = \delta_{ij}$ | $V = 0$ | $W \in L^{\frac{7n-2}{6}}$ | $n \geq 6$ |

Considerable work has also been done on the corresponding weak unique continuation problem, where what may be the sharp result is due to Wolff (see [31] and references therein) with the assumptions $g^{ij} \in C^1$, $V \in L^{\frac{n}{2}}$ and $W \in L^n$. In this case, however, the best counterexample is for $V \in L^1$ (see Kenig-Nadirashvili [16]) and thus far away from Wolff's positive result.

Other related articles on this problem are [1, 5, 8, 13, 14, 17, 18, 25, 27], on second order degenerate elliptic operators [4, 12], on elliptic systems [10, 11, 24], on the Dirac operator [6, 19], on higher order elliptic operators [9, 28].

The layout of the proof is as follows. In Section 2 we describe the change of coordinates which achieves the reduction of (4) to (5). The strong unique continuation result is a standard consequence of certain estimates of Carleman type, which we state in Section 3. More precisely, Theorem 1 follows from Corollary 3.1 in the same way as in [15]. The rest of the article is devoted to the proof of the Carleman estimates. We start in Section 4 with local estimates in the special case $P = \Delta$ with a radially symmetric exponential weight. Then in Section 5 we use a perturbation argument to transfer these estimates to variable coefficient operators and more general exponential weights. The

global construction of the weights, as well as the global Carleman estimates, are explained in Section 6. Finally, in Section 7 we use Wolff's weight osculation argument to select weights for which the desired gradient estimates hold.

# 2 Changes of coordinates

The main result of this section asserts that we can find a change of coordinates which allows us to replace (4) by (5) in the proof of Theorem 1.

**Theorem 2.** *Let $g$ satisfy (4). Then there exists a locally $C^2$ change of coordinates $\chi$ with $\chi(0) = 0$ and locally Lipschitz function $\bar{g}$ which satisfy*

$$\||x|\nabla^2\chi\|_{l^1_w(L^\infty)} + \||x|\nabla\bar{g}\|_{l^1_w(L^\infty)} < C\varepsilon, \qquad \varepsilon \text{ small} \tag{9}$$

*so that $(\bar{g})^{-1}\chi^* g$ satisfies (5). Here $\chi^* g$ is the metric in the new coordinates.*

**Proof:** If (9) holds then it is easy to see that $\chi^* g$ satisfies (4). Observe also that for any $r \leq |x_0| \leq 2r$ we have

$$\sup_{r\leq|x|\leq 2r} |\chi^* g - I_n| \lesssim |\chi^* g(x_0) - I_n| + \sup_{r\leq|x|\leq 2r} |x||\nabla(\chi^* g)|$$

This shows that, in order to get the remaining part of (5) for $\chi^* g$, it suffices to insure that it equals $I_n$ at least once in each dyadic region. Consequently, set $x_k = 2^k e_1$ and try to find $\chi$ satisfying

$$\chi(x_k) = x_k, \quad \nabla\chi(x_k)e_1 = e_1, \quad \chi^* g(x_k) = I_n \tag{10}$$

The last relation is equivalent to

$$\nabla^T\chi(x_k)g(x_k)\nabla\chi(x_k) = I_n$$

or to

$$g(x_k) = (\nabla^T\chi(x_k))^{-1}(\nabla\chi(x_k))^{-1} \tag{11}$$

The second part of (10) requires $g^{11}(x_k) = 1$, but this can be easily achieved multiplying $g$ by $(g^{11})^{-1}$. Then we represent $g(x_k)$ in the form

$$g(x_k) = \begin{pmatrix} 1 & B_k^T \\ B_k & A_k \end{pmatrix}$$

Since $g(x_k)$ are uniformly positive definite, it follows that the $(n-1)\times(n-1)$ matrices $A_k - B_k B_k^T$ are uniformly positive definite. Now set

$$(\nabla\chi(x_k))^{-1} = \begin{pmatrix} 1 & B_k^T \\ 0 & (A_k - B_k B_k^T)^{\frac{1}{2}} \end{pmatrix} \tag{12}$$

Then (11) holds, which in turn implies that (10) holds.

It remains to extend $\chi$ to the whole $\mathbb{R}^n$ so that (9) is satisfied. This is possible since, by (12),

$$\begin{aligned} |\nabla\chi(x_k) - \nabla\chi(x_{k+1})| &\lesssim |g(x_k) - g(x_{k+1})| \\ &\lesssim \sup_{2^k \le |x| \le 2^{k+1}} |x||\nabla g(x)| \end{aligned}$$

Note that the extension can be done trivially by the identity on the line $\mathbb{R}e_1$.

# 3 Carleman estimates

We first recall the estimate of Jerison and Kenig [15],

$$\||x|^{-\tau}u\|_{L^p} \lesssim \||x|^{-\tau}\Delta u\|_{L^{p'}} \tag{13}$$

where $p$ and $p'$ are dual exponents satisfying the gap condition

$$\frac{1}{p'} - \frac{1}{p} = \frac{2}{n}$$

This holds for all $u$ vanishing of infinite order at 0 uniformly with respect to $\tau$ away from $\pm(\frac{n-2}{2} + \mathbb{N})$. This implies that **(SUCP)** holds in the case when $g = I_n$, $V \in L^n$ and $W = 0$.

Here and in the sequel the notation $X \lesssim Y$ means $X \le cY$ with some constant $c$ which only depends on the space dimension $n$. The notation $X \ll Y$, on the other hand, stands for $X \le \varepsilon Y$ for some sufficiently small constant $\varepsilon$, again depending only on $n$.

The estimate (13) corresponds to the weight function

$$\varphi(x) = -\tau \ln |x|$$

which satisfies a degenerate pseudoconvexity condition. In the variable coefficient case we can no longer use this weight because the pseudoconvexity condition may fail. Instead we need to use weight functions of the form

$$\varphi(x) = h(-ln|x|)$$

where $h$ is convex. Indeed if we do this we obtain the following result

**Theorem 3.** *Assume that (5) holds. Then for each $\tau > 0$ there exists a convex function $h$ satisfying $h' \in [\tau, \tau^2]$ so that*

$$\|e^{\varphi(x)}u\|_{l^{p'}(L^p)} \lesssim \|e^{\varphi(x)}P(x,\partial)u\|_{L^{p'}} \tag{14}$$

*for all $u$ vanishing of infinite order at 0 and $\infty$.*

By a standard argument this implies Theorem 1 in the case when $W_1 = W_2 = 0$. To deal with $W_1$ and $W_2$ we need a modification of this result where we allow $\varphi$ to depend also on the angular variable,

$$\varphi(x) = h(-\ln(x)) + k(-\ln(x), \theta)$$

In addition, we need to obtain bounds for $\nabla u$ and to allow gradients of integrable functions in $Pu$. Thus we define the space $X_\varphi$ with norm

$$\|v\|_{X_\varphi} = \|v\|_{L^p} + \|\nabla v\|_{L^2+|\nabla\varphi|L^p}$$

where

$$\|w\|_{L^2+|\nabla\varphi|L^p} = \inf_{w=w_1+w_2} \|w_1\|_{L^2} + \| \, |\nabla\varphi|^{-1} w_2\|_{L^p}$$

The dual space $X'_\varphi$ is

$$X'_\varphi = L^{p'} + \nabla(L^2 \cap |\nabla\varphi|^{-1} L^{p'})$$

with norm

$$\|f\|_{X'_\varphi} = \inf_{f=f_1+\nabla f_2} \|f_1\|_{L^{p'}} + \|f_2\|_{L^2} + \| \, |\nabla\varphi| f_2\|_{L^{p'}}$$

Then the following generalization of (14) holds,

$$\|e^{\varphi(x)} u\|_{l^{p'}(X_\varphi)} \lesssim \|e^{\varphi(x)} Pu\|_{l^{p'}(X'_\varphi)} \tag{15}$$

Here $l^{p'}(X_\varphi)$ has the same meaning as in (3), i.e. first we evaluate the $X_\varphi$ norm on dyadic annuli and then we use the dyadic $l^{p'}$ summation.

Naively (15) differs from (14) by an elliptic estimate. This is because the symbol of $e^{\varphi(x)} P(x, \partial) e^{-\varphi(x)}$ can only vanish in the region where the frequency is of the order $|\nabla\varphi|$. Away from this region both (14) and (15) are (microlocally) elliptic estimates. This is explained in detail in the proof of Propositions 4.1, 4.2. In effect (see Theorem 5) we show that such an estimate holds uniformly for all $u$ and a large class of weight functions $\varphi$ which are also allowed to depend on the angular variable.

In order to produce a direct argument for the unique continuation result when $W_1$ and/or $W_2$ are nonzero one would need to be able to add to (15) an $L^2$ bound for $e^{\varphi(x)} \nabla u$, and/or to add an $\nabla L^2$ term to $Pu$. In other words, this would mean replacing $X_\varphi$ by

$$\|v\|_{X_\varphi} = \|v\|_{L^p} + \|\nabla v\|_{L^2}$$

Unfortunately, it is known that such an estimate cannot hold uniformly in $u$ and $\tau$ so the situation appears hopeless. To overcome this difficulty we adapt an idea of Wolff which takes advantage of the fact that we know (15) to be true

for a large range of functions $\varphi$. Roughly speaking, the idea is to show that one can choose the function $\varphi$ in such a way so that the terms $e^{\varphi}W_1\nabla\varphi$ and $e^{\varphi}W_2\varphi$ are concentrated on small sets; then, on such sets, to use the estimate (15). The main result is summarized in the following theorem:

**Theorem 4.** *Assume that* (5) *holds. Then for each* $\tau > 0$, $W_1, W_2 \in l^1_w(L^n)$ *and each function* $u$ *vanishing of infinite order at* $0$ *and* $\infty$ *there exists a function* $\varphi$ *satisfying*

$$\tau \leq -r\partial_r\varphi \leq \tau^2, \qquad |\partial_\theta\varphi| \leq |r\partial_r\varphi| \tag{16}$$

*so that*

$$\|e^{\varphi(x)}u\|_{l^{p'}(X_\varphi)} + \frac{\|e^{\varphi(x)}W_1\nabla u\|_{L^{p'}}}{\|W_1\|_{l^1_w(L^n)}} + \frac{\|e^{\varphi(x)}W_2 u\|_{|\nabla\varphi|^{-1}L^{p'}}}{\|W_2\|_{l^1_w(L^n)}} \lesssim \|e^{\varphi(x)}P(x,\partial)u\|_{l^{p'}(X'_\varphi)}, \tag{17}$$

The first part of (16) insures that $\varphi$ blows up polynomially at 0, while the second part implies that its level sets are not two far from circles, more precisely they are contained between two circles whose radiuses have a fixed ratio.

The estimates of Theorem 4 deviate from classical Carleman estimates: Instead of $e^{\tau\varphi(x)}$ we have to use different functions $\varphi$ for each $\tau$. It is an essential feature of our estimates that the choice of the weight function $\varphi$ depends on $u$, $W_1$ and $W_2$. It is known that such an estimate cannot be true for all $\tau$ with $\varphi$ independent of $u$. Nevertheless we easily obtain the following corollary as a consequence of (17):

**Corollary 3.1.** *Assume that (4), (6) and (7) hold. Then for each* $\tau > 0$ *and each function* $u$ *vanishing of infinite order at* $0$ *and* $\infty$ *which solves*

$$P(x,\partial)u - (Vu + W_1\nabla u + \nabla W_2 u) = f$$

*there exists a function* $\varphi$ *satisfying (16) so that*

$$\|e^{\varphi(x)}u\|_{L^p} \lesssim \|e^{\varphi(x)}f\|_{L^{p'}} \tag{18}$$

This implies the desired unique continuation result.

# 4 Polar coordinates and estimates for the flat case

Introduce the polar coordinates

$$x = e^{-s}\theta, \qquad (s,\theta) \in \mathbb{R} \times \mathbb{S}^{n-1} := \mathcal{C}.$$

Denote $\tilde{\nabla} = (\partial_s, \partial_\theta)$. Then

$$\frac{dx}{|x|^n} = dsd\theta.$$

On the other hand, one can compute the form of the Laplace operator in the new coordinates,

$$|x|^2\Delta = \partial_s^2 - (n-2)\partial_s + \Delta_\theta$$

or, after further conjugation by $|x|^{-\frac{n-2}{2}} = e^{\frac{n-2}{2}s}$,

$$|x|^{\frac{n+2}{2}}\Delta|x|^{-\frac{n-2}{2}} = \partial_s^2 + \Delta_\theta - \left(\frac{n-2}{2}\right)^2.$$

Using the transformation $v = |x|^{\frac{n-2}{2}}u$, Jerison and Kenig's result (13), for instance, becomes

$$\|e^{\tau s}v\|_{L^p(\mathcal{C})} \lesssim \|e^{\tau s}\tilde{\Delta}v\|_{L^{p'}(\mathcal{C})} \tag{19}$$

where

$$\tilde{\Delta} = \partial_s^2 + \Delta_\theta - (\frac{n-2}{2})^2.$$

The assumption that $u$ vanishes of infinite order at 0 and $\infty$ translates into a faster than exponential decay for $v$ when $s$ approaches $\pm\infty$.

The spectrum of $-\Delta_\theta + \frac{(n-2)^2}{4}$ is $(\frac{n-2}{2} + \mathbb{N})^2$. Therefore (19) cannot hold if

$$\tau = \pm\Big(\frac{n-2}{2} + \lambda\Big).$$

This accounts for the restriction that $\tau$ should stay away from $\pm(\frac{n-2}{2} + \mathbb{N})$.

Unfortunately (19) is not stable with respect to "small" perturbations of the operator or of the exponential weight since the exponential weight satisfies only a degenerate pseudoconvexity condition, which can be easily broken with arbitrarily small perturbations. To avoid this, we modify the estimates in two directions:

1. Instead of exponential weights $e^{\tau s}$ we use weights $e^{h(s)}$ where $h$ is a convex function.

2. We complement the estimates by $L^2$ estimates to handle perturbations.

Following the same reasoning as above one sees that $h'$ may not be close to an half integer for a long time. Hence we require

$$|h''| + dist(2h', \mathbb{Z})| \geq \frac{1}{4} \tag{20}$$

Note, however, that we want to obtain estimates which are valid for all functions $u$ vanishing of infinite order at 0 and $\infty$. This limits further our choices for $h$ to functions which have at most linear growth at infinity.

The function spaces we use in the sequel are based on the $X_\varphi$ spaces introduced in the previous section. To these we need to add some better $L^2$ estimates which are essential in the localization arguments. For $\tau > 0$ and $0 \leq \varepsilon \leq 1$ we introduce the spaces $\tilde{X}_{\tau,\varepsilon}$ of functions defined on the cylinder $\mathcal{C}$, which are to be used for $v$:

$$\tilde{X}_{\tau,\varepsilon} = \{v \in L^p \cap \tau^{-\frac{1}{2}}(1+\varepsilon\tau)^{-\frac{1}{4}}L^2, \quad \nabla v \in (L^2 + \tau L^p) \ \cap \tau^{\frac{1}{2}}(1+\varepsilon\tau)^{-\frac{1}{4}}L^2\}$$

For the right hand side of the equation we use the dual space,

$$\tilde{X}'_{\tau,\varepsilon} = L^{p'} + \tau^{\frac{1}{2}}(1+\varepsilon\tau)^{\frac{1}{4}}L^2 + \nabla(L^2 \cap \tau^{-1}L^{p'}) + \tau^{-\frac{1}{2}}(1+\varepsilon\tau)^{\frac{1}{4}}\nabla L^2$$

In $\mathbb{R}^n$ we introduce corresponding norms by reverting the transformation we described earlier. Thus, we set

$$\|u\|_{X_{\tau,\varepsilon}} = \||x|^{\frac{n-2}{2}}u\|_{\tilde{X}_{\tau,\varepsilon}}$$

Then we also have

$$\|g\|_{X'_{\tau,\varepsilon}} = \||x|^{\frac{n+2}{2}}g\|_{\tilde{X}'_{\tau,\varepsilon}}$$

Observe that if $|\nabla\varphi| = O(\tau)$ then

$$X_{\tau,\varepsilon} \subset X_\varphi$$

The only difference between the two spaces is in the additional $L^2$ norms involving $\varepsilon$ which are part of the $X_{\tau,\varepsilon}$ norm. The $L^2$ norms are essential in order to localize estimates to a fixed dyadic scale in $|x|$ (which corresponds to intervals of fixed length in $s$). The $\varepsilon$ terms are used to describe the improvement in the estimates when our exponential weight has at least "$\varepsilon$" convexity; they are essential in the localization argument which makes the transition from constant to variable coefficients.

Our first result is a global one:

**Proposition 4.1.** *Let $\tau \gg 1$. Consider a convex function $h$ satisfying (20) for which $|h'| \in [\tau, 2\tau]$. Then*

$$\|e^{h(-\ln(|x|)}u\|_{X_{\tau,0}} \lesssim \|e^{h(-\ln(|x|))}\Delta u\|_{X'_{\tau,0}} \tag{21}$$

*for all functions $u$ vanishing of infinite order at 0 and $\infty$.*

Next we consider in more detail the case when $h$ is uniformly convex in some region:

**Proposition 4.2.** *Let $\tau^{-1} < \varepsilon < 1$. We consider a convex function $h$ satisfying*

$$|h'| \in [\tau, 2\tau], \qquad h'' \in [\varepsilon\tau, \tau]$$

*in some interval $I$. Then*

$$\|e^{h(-\ln(|x|))}v\|_{X_{\tau,\varepsilon}} \lesssim \|e^{h(-\ln(|x|))}\Delta v\|_{X'_{\tau,\varepsilon}} \tag{22}$$

*for all functions $v$ supported in $\{x : |x| \in e^{-I}\}$.*

**Proof of Proposition 4.1:** In the $(s, \theta)$ coordinates (21) becomes

$$\sum_{j=0,1} \|e^{h(s)}v\|_{\tilde{X}_{\tau,0}} \lesssim \|e^{h(s)}\tilde{\Delta} v\|_{\tilde{X}'_{\tau,0}} \tag{23}$$

Then the first step is to find a left inverse for

$$e^{h(s)}\Big[\partial_s^2 + \Big(\Delta_\theta - \frac{(n-2)^2}{4}\Big)\Big]e^{-h(s)}$$

The spectrum of $-\Delta_\theta + \frac{(n-2)^2}{4}$ is $(\frac{n-2}{2} + \mathbb{N})^2$ therefore corresponding to each eigenvalue

$$\Big(\frac{n-2}{2} + \lambda\Big)^2$$

we need to find a "bounded" left inverse for

$$e^{h(s)}\Big[\partial_s^2 - \Big(\frac{n-2}{2} + \lambda\Big)^2\Big]e^{-h(s)}$$

For each integer $\lambda$ let $s(\lambda)$ be a solution for

$$h'(s) = \frac{n-2}{2} + \lambda$$

If $\lambda + \frac{n-2}{2}$ is outside the range of $h'$ (which is an interval) and no such solution exists then we set

$$s(\lambda) = \begin{cases} -\infty & \text{if } \lambda + \frac{n-2}{2} < h' \\ +\infty & \text{if } \lambda + \frac{n-2}{2} > h' \end{cases}$$

Then one can construct a left inverse by

$$\tilde{K}_\lambda(t,s)) = e^{h(s)-h(t)} \begin{cases} -\frac{1}{2}(\lambda + \frac{n-2}{2})^{-1}e^{-(\lambda+\frac{n-2}{2})|t-s|} & s < s(\lambda) \\ (\lambda + \frac{n-2}{2})^{-1}\sinh(\lambda + \frac{n-2}{2})(t-s)) & s(\lambda) < s < t \\ 0 & s(\lambda), t < s \end{cases}$$

This is essentially the kernel we will use in the sequel. It has however a jump discontinuities in $s$. This can be avoided by the following modification. We make a smooth transition between the two regimes in $s$ on the scale of $\tau^{-1}$. If $\chi$ is a smooth function which equals 0 in $(-\infty,-1]$ and 1 in $[1,\infty)$ then we shall work with the left inverse

$$\begin{aligned} K_\lambda(t,s)) &= e^{h(s)-h(t)}(\lambda+\frac{n-2}{2})^{-1}(\chi(\tau(s-s(\lambda)))\sinh(\lambda+\frac{n-2}{2})(t-s)_+ \\ &- \frac{1}{2}(1-\chi(\tau(s-s(\lambda)))e^{-(\lambda+\frac{n-2}{2})|t-s|}) \end{aligned} \tag{24}$$

Since $h'=O(\tau)$, the function $h$ does not vary much on the $\tau^{-1}$ scale therefore such a modification does not significantly alter the size of the kernel $K_\lambda$.

The left inverse for the full operator has kernel

$$K(t,s)=\sum_{\lambda\in\mathbb{N}} K_\lambda(s,t)E_\lambda$$

where $E_\lambda$ denotes the projection onto the corresponding eigenspace of the spherical Laplacian. By an abuse of notation we sometimes identify operators and their kernels and denote them by the same symbol.

To prove the desired estimates for $K$ we use the bounds for the projection operators (see Sogge [26])

$$\|E_\lambda\|_{L^{p'}\to L^p}\lesssim\lambda^{\frac{n-2}{n}},\quad \|E_\lambda\|_{L^{p'}\to L^2}\lesssim\lambda^{\frac{n-2}{2n}},\quad \|E_\lambda\|_{L^2\to L^p}\lesssim\lambda^{\frac{n-2}{2n}} \tag{25}$$

We decompose $K$ into three components, a low frequency elliptic part $K^e$, a high frequency elliptic part $K^h$ and a frequency $O(\tau)$ part, $K^\tau$, with

$$K^l(t,s)=\sum_{\lambda=0}^{[\tau/2]} K_\lambda(s,t)E_\lambda$$

$$K^h(t,s)=\sum_{\lambda=[4\tau]+1}^{\infty} K_\lambda(s,t)E_\lambda$$

$$K^\tau(t,s)=\sum_{\lambda=[\tau/2]+1}^{[4\tau]} K_\lambda(s,t)E_\lambda$$

The kernels of $K^l$ and $K^h$ are important for the elliptic regime. We derive the same estimate (26) and (28) below for both of them, which is stronger than the desired estimate (21) . This is a simple but maybe somewhat technical exercise which we do first. The crucial part is the estimate of $K^\tau$.

**The estimate for $K^l$:** In this case we prove the elliptic estimate

$$K^l : L^{p'} + \tau L^2 + H^{-1} \to L^p \cap \tau^{-1} L^2 \cap H^1. \tag{26}$$

If $\lambda < \tau/2$ then

$$|K_\lambda(t,s)| \lesssim \tau^{-1} e^{-c\tau|t-s|}, \quad \lambda < \frac{\tau}{2}. \tag{27}$$

To obtain the $L^{p'} \to L^p$ bound we use (27) to estimate

$$\begin{aligned} \|K^l(t,s)\|_{L^{p'}\to L^p} &\lesssim \tau^{-1} e^{-c\tau|t-s|} \sum_{\lambda=1}^{[\tau/2]} \lambda^{\frac{n-2}{n}} \\ &\lesssim \tau^{\frac{n-2}{n}} e^{-c\tau|t-s|} \\ &\lesssim |t-s|^{-\frac{n-2}{n}}. \end{aligned}$$

Then the global $L^{p'} \to L^p$ estimate follows using the Hardy-Littlewood-Sobolev inequality.

For the $L^{p'} \to L^2$ bound we proceed in a similar fashion. For fixed $t$, $s$ by (27) and (25) we get

$$\begin{aligned} \|K^l(t,s)\|_{L^{p'}\to L^2} &\lesssim \tau^{-1} e^{-c\tau|t-s|} \Big( \sum_{\lambda=1}^{[\tau/2]} \lambda^{\frac{n-2}{n}} \Big)^{\frac{1}{2}} \\ &\lesssim \tau^{-\frac{1}{n}} e^{-c\tau|t-s|}. \end{aligned}$$

If we bound the last expression by $\tau^{-1}|t-s|^{-\frac{n-1}{n}}$ then we obtain the $L^{p'} \to L^2$ estimate. The $L^2 \to L^p$ bound follows in an identical manner.

The $L^2 \to L^2$ bound is even simpler. By (27),

$$\|K^l(t,s)\|_{L^2\to L^2} \lesssim \tau^{-1} e^{-c\tau|t-s|}$$

and the kernel in the right hand side has an $L^1$ norm of $c\tau^{-2}$.

To obtain the desired $H^1$ bounds we need to perform a similar analysis with $K_\lambda(t,s)$ replaced with the kernels $\lambda K_\lambda(t,s)$, respectively $\partial_t K_\lambda(t,s)$. These satisfy the bound

$$|\lambda K_\lambda(t,s)| + |\partial_t K_\lambda(t,s)| \le e^{-c\tau|t-s|}.$$

i.e. with an additional $\tau$ factor. Then the estimates are identical to the $L^2$ estimates.

To obtain the bounds from $H^{-1}$, on the other hand, we need to work with the kernels $\lambda K_\lambda(t,s)$, respectively $\partial_s K_\lambda(t,s)$. But they satisfy the same bound as above.

Finally, for the bound from $H^{-1}$ into $H^1$ we should consider the kernels $\lambda^2 K_\lambda(t,s)$, $\lambda \partial_t K_\lambda(t,s)$ $\lambda \partial_s K_\lambda(t,s)$, respectively $\partial_t \partial_s K_\lambda(t,s)$. The first three are bound by $\tau e^{-c\tau|t-s|}$, while the fourth equals the $\delta_{t=s}$ plus a component satisfying the same bound.

**The estimate for $K^h$:** We again prove a stronger elliptic estimate:

$$K^h : L^{p'} + \tau L^2 + H^{-1} \to L^p \cap \tau^{-1} L^2 \cap H^1. \tag{28}$$

The proof is very similar to the previous one. For $\lambda > 4\tau$ it is easy to see that

$$|K_\lambda(t,s)| \lesssim \lambda^{-1} e^{-c\lambda|t-s|}. \tag{29}$$

To obtain the $L^{p'} \to L^p$ bound we use (29) to estimate

$$\begin{aligned} \|K^h(t,s)\|_{L^{p'}\to L^p} &\lesssim \sum_{\lambda=[4\tau]+1}^{\infty} \lambda^{-1}\lambda^{\frac{n-2}{n}} e^{-c\lambda|t-s|} \\ &\lesssim \int_0^\infty \lambda^{-\frac{2}{n}} e^{-c\lambda|t-s|} \\ &= c_n |t-s|^{-\frac{n-2}{n}}. \end{aligned}$$

Then the global $L^{p'} \to L^p$ estimate follows using the Hardy-Littlewood-Sobolev inequality.

For the $L^{p'} \to L^2$ bound we proceed in a similar fashion. For fixed $t$, $s$ by (29) and (25) we get

$$\begin{aligned} \|K^h(t,s)\|_{L^{p'}\to L^2} &\lesssim \Big( \sum_{\lambda=[4\tau]+1}^{\infty} \lambda^{-2}\lambda^{\frac{n-2}{n}} e^{-c\lambda|t-s|} \Big)^{\frac{1}{2}} \\ &\lesssim \tau^{-\frac{1}{n}} e^{-c\tau|t-s|}. \end{aligned}$$

and we conclude as in the low frequency case for this and for the $L^2 \to L^p$ bound. The $L^2 \to L^2$ bound is straightforward.

For the $H^1$ bounds we observe that

$$|\lambda K_\lambda(t,s)| + |\partial_t K_\lambda(t,s)| \le e^{-c\tau|t-s|}.$$

Then the corresponding $L^{p'} \to L^2$ bound, for instance, is a consequence of the estimate

$$\begin{aligned} \|\partial_t K^h(t,s)\|_{L^{p'}\to L^2} &\lesssim \Big( \sum_{\lambda=4\tau}^{\infty} \lambda^{\frac{n-2}{n}} e^{-c\lambda|t-s|} \Big)^{\frac{1}{2}} \\ &\lesssim \tau^{\frac{n-1}{n}} e^{-c\tau|t-s|} \end{aligned}$$

and the Hardy-Littlewood-Sobolev inequality. For the bounds from $H^{-1}$, the kernels $\lambda K_\lambda(t,s)$, respectively $\partial_s K_\lambda(t,s)$ satisfy the same bounds.

Finally, the bound from $H^{-1}$ into $H^1$ reduces to $L^2$ bounds for the kernels $\lambda^2 K_\lambda(t,s)$, $\lambda \partial_t K_\lambda(t,s)$ $\lambda \partial_s K_\lambda(t,s)$, respectively $\partial_t \partial_s K_\lambda(t,s)$. The first three are bound by $\lambda e^{-c\lambda|t-s|}$, while the fourth equals $\delta_{t=s}$ plus a component satisfying the again the same bound.

**The bounds for $K^\tau$:** This is the interesting case, where elliptic estimates of the type (26) and (28) fail and where we have to work with the more complicated norms $\tilde{X}_{\tau,0}$ and $\tilde{X}'_{\tau,0}$. Even if the arguments are similar to the previous ones the reader should keep in mind that here the estimates are much less obvious. We start again with a bound on the kernels $K_\lambda$. This time, however, we need to combine the convexity of $h$ with the hypothesis (20) to get

$$|K_\lambda(t,s)| \lesssim \lambda^{-1} e^{-c(1+|\lambda-h'(s)|)|t-s|}, \qquad \frac{\tau}{2} \leq \lambda \leq 4\tau. \tag{30}$$

We start with the $L^{p'} \to L^p$ bound:

$$\begin{aligned} \|K^\tau(t,s)\|_{L^{p'}\to L^p} &\lesssim \sum_{\lambda=[\tau/2]+1}^{[4\tau]} \lambda^{-1}\lambda^{\frac{n-2}{n}} e^{-c(1+|\lambda-h'(s)|)|t-s|} \\ &\lesssim \tau^{-\frac{2}{n}} \int_0^\tau e^{-c\lambda|t-s|} d\lambda \\ &\lesssim \tau^{-\frac{2}{n}} \tau (1+\tau|t-s|)^{-1} \\ &\lesssim |t-s|^{-\frac{n-2}{n}}. \end{aligned}$$

Then the global $L^{p'} \to L^p$ estimate follows using the Hardy-Littlewood-Sobolev inequality.

For the $L^{p'} \to L^2$ bound we proceed in a similar fashion, using (30) and (25):

$$\begin{aligned} \|K^\tau(t,s)\|_{L^{p'}\to L^2} &\lesssim \Big( \sum_{\lambda=[\tau/2]+1}^{[4\tau]} \lambda^{\frac{n-2}{n}} \lambda^{-2} e^{-c(1+|\lambda-h'(s)|)|t-s|} \Big)^{\frac{1}{2}} \\ &\lesssim e^{-c|t-s|} \tau^{-\frac{1}{n}} (1+\tau|t-s|)^{-\frac{1}{2}}. \end{aligned}$$

This can be bound by $\tau^{-\frac{1}{2}-\frac{1}{n}}|t-s|^{-\frac{n-1}{n}}$ to obtain the $L^{p'} \to L^2$ estimate. The $L^2 \to L^p$ bound follows in an identical manner.

The $L^2 \to L^2$ bound is even simpler, for by (30)

$$\|K^\tau(t,s)\|_{L^2\to L^2} \lesssim \tau^{-1} e^{-c|t-s|}.$$

Finally, the estimates involving the gradients in $f$ and/or the gradient of $K^\tau f$ bring nothing new since both differentiation in $s,t$ and multiplication by $\lambda$ simply increase the size of $K_\lambda(t,s)$ by a factor of $\tau$ for $\lambda = O(\tau)$.

**Proof of Proposition 4.2:** In the $(s,\theta)$ coordinates (22) becomes

$$\sum_{j=0,1} \|e^{h(s)} \tilde{\nabla}^j v\|_{\tilde{X}^j_{\tau,\varepsilon}(\mathcal{C})} \lesssim \|e^{h(s)} \tilde{\Delta} v\|_{\tilde{Y}_{\tau,\varepsilon}(\mathcal{C})} \tag{31}$$

We work with the same $K$, $K_\lambda$ as before. The estimates we proved for $K^l$, $K^h$ are already stronger than we need, so we should concentrate on $K^\tau$. If $\lambda = O(\tau)$ then the $\varepsilon$ convexity for $\varphi$ implies

$$h(s) - h(t) \leq h'(s)(s-t) - \epsilon|t-s|^2.$$

In the previous section we only used the first term on the right hand side. The second term implies that $K_\lambda$ satisfies the stronger Gaussian bound

$$|K_\lambda(t,s)| \lesssim \lambda^{-1} e^{-c\varepsilon\tau(t-s)^2} e^{-c(1+|\lambda-h'(s)|)|t-s|} \qquad \frac{\tau}{2} \leq \lambda \leq 4\tau \tag{32}$$

We only need to prove the bounds involving the convexity parameter $\varepsilon$. The analysis is fairly simple since we can use the earlier computation with the added Gaussian. For the $L^{p'} \to L^2$ bound, for instance, we get

$$\begin{aligned} \|K^\tau(t,s)\|_{L^{p'}\to L^2} &\lesssim e^{-c\varepsilon\tau(t-s)^2} \tau^{-\frac{1}{n}} (1+\tau|t-s|)^{-\frac{1}{2}} \\ &\lesssim |t-s|^{-\frac{n-1}{n}} \tau^{-\frac{1}{2}-\frac{1}{n}} (\varepsilon\tau)^{-\frac{1}{4}+\frac{1}{2n}} \end{aligned}$$

The $L^2 \to L^p$ estimate is similar. The corresponding $L^2 \to L^2$ bound is now

$$\|K(t,s)\|_{L^2\to L^2} \lesssim \tau^{-1} e^{-c\varepsilon\tau(t-s)^2}$$

Now the $L^1$ norm of the kernel on the right is $\tau^{-1}(\varepsilon\tau)^{-\frac{1}{2}}$, which gives exactly the desired $L^2 \to L^2$ bound.

The estimates involving the gradients in $f$ and/or the gradient of $K^\tau f$ follow in the same way as in the proof of Theorem 4.1.

# 5 A perturbation argument

Here we explain to what extent the results proved in the previous section for the flat case transfer to operators with variable coefficients. At the same time, we want our estimates to be stable with respect to "reasonable" perturbations of the weight $h$,

$$\varphi(x) = h(-ln|x|) + k(-\ln|x|, \theta)$$

**Proposition 5.1.** *Let $\tau \gg 1$. Consider a convex function $h$ satisfying (20) for which $|h'| \in [\tau, 2\tau]$. Assume that*

$$|g - I_n| + |x||\nabla g| \ll \tau^{-1}, \quad |k| + |x||\nabla k| \ll 1. \tag{33}$$

*Then*

$$\|e^\varphi u\|_{X_{\tau,0}} \lesssim \|e^\varphi P u\|_{X'_{\tau,0}} \tag{34}$$

*for all functions $u$ vanishing of infinite order at $0$ and $\infty$.*

This indicates that perturbations of the order of $\tau^{-1}$ are always acceptable. Since $\tau$ can be arbitrarily large, this is clearly not sufficient. To handle larger perturbations, we need to use additional convexity on $h$.

**Proposition 5.2.** *Let $\tau \geq 1$ and $0 < \varepsilon < 1$, $\varepsilon\tau > 1$. Consider a convex function $h$ satisfying*

$$|h'| \in [\tau, 2\tau], \quad h'' \in [\varepsilon\tau, \tau], \quad |h'''| \leq \tau$$

*in some interval $I$ of length $|I| \lesssim 1$. Assume that*

$$|g - I_n| + |x||\nabla g| \ll \varepsilon, \qquad k + |x|^2|\nabla^2 k| \ll \varepsilon\tau.$$

*Then*

$$\|e^{\varphi}u\|_{X_{\tau,\varepsilon}} \lesssim \|e^{\varphi}Pu\|_{X'_{\tau,\varepsilon}} \tag{35}$$

*for all functions $u$ supported in $\{x : |x| \in e^{-I}\}$.*

**Proof of Proposition 5.2:** We first prove the estimate in the case when $u$ is supported in a ball $B$ of size $C|x_0|(\varepsilon\tau)^{-\frac{1}{2}}$ centered at some $x_0 \in \mathbb{R}^n$. Here $C$ is a large parameter which we fix later on. After rescaling and rotation in $\theta$ we can assume that $x_0 = e_1$. It is important that our estimates and assumptions are scale invariant.

We claim that we can find a change of coordinates $\chi$ around $e_1$ satisfying

$$\chi(e_1) = e_1, \qquad D\chi = I_n + o(\varepsilon)$$

so that in the new coordinates we have

$$g^{ij}(e_1) = I_n, \qquad k = 0.$$

We know that

$$g(e_1) = I_n + o(\varepsilon), \quad \nabla\varphi(e_1) = \nabla h(e_1) + o(\varepsilon\tau).$$

Then we can make a linear transformation of the form $I_n + o(\varepsilon)$ around $x_0$ to insure that $g(e_1) = I_n$ and $\nabla\varphi(e_1) = h'(0)e_1$, i.e. $\nabla k(e_1) = 0$. Without any restriction in generality we can also assume that $k(e_1) = 0$.

To set $k$ identically 0 we make another change of coordinates,

$$\frac{y}{|y|} = \frac{x}{|x|}, \qquad h(-\ln(|y|)) = \varphi(-\ln(|x|)).$$

Since $\nabla k(e_1) = 0$, the Jacobian of the change of coordinates equals $I_n$ at $e_1$, therefore the condition $g(e_1) = I_n$ is preserved. Furthermore, the bounds on the derivatives of $h, k$ imply that

$$\frac{\partial y}{\partial x} = I_n + o(\varepsilon), \quad \frac{\partial^2 y}{\partial x^2} = o(\varepsilon),$$

therefore this change of coordinates does not affect our hypothesis on $g$. This concludes the reduction.

Since $g(e_1) = I_n$, within $B$ we get

$$\sup_B |g - I_n| \ll \varepsilon^{\frac{1}{2}} \tau^{-\frac{1}{2}}. \tag{36}$$

This will allow us to use a perturbation argument to transfer the estimate from $\Delta$ to $P$.

We know that (35) is true with $P$ replaced by $\Delta$:

$$\|e^{\varphi} u\|_{X_{\tau,\varepsilon}} \lesssim \|e^{\varphi} \Delta u\|_{X'_{\tau,\varepsilon}}. \tag{37}$$

Then we need to estimate the difference in the right hand side. We claim that the following $L^2$ estimate holds for all $w$ supported in $B$:

$$\|e^{\varphi}(P - \Delta)u\|_{L^2 + \tau^{-1} H^{-1}} \ll \tau(1 + \varepsilon\tau)^{\frac{1}{2}} \|e^{\varphi} u\|_{L^2 + \tau H^1}. \tag{38}$$

Given the definition of our spaces, this suffices in order to get (35) from (37). Note that the powers of $x$ in our norms are all irrelevant since here we are doing estimates in $B$ which is away from 0. To prove (38) it suffices to expand the difference

$$P - \Delta \;=\; \partial_i (g^{ij} - \delta^{ij}) \partial_j$$

and use (36) together with $|x||\nabla\varphi| \lesssim \tau$.

It remains to remove the support assumption on $u$, which we do using a partition of unit. We choose a smooth partition of unity with functions $\psi_k$ supported in balls of radius $C(\epsilon\tau)^{-1/2}$. We apply the estimate (35) to $\psi_k v$:

$$\|e^{\varphi} \psi_k u\|_{X_{\tau,\varepsilon}} \lesssim \|e^{\varphi} \psi_k P u\|_{X'_{\tau,\varepsilon}} + \|e^{\varphi} [\psi_k, P] v\|_{X'_{\tau,\varepsilon}}. \tag{39}$$

This implies (35) after summation without the additional restriction on the support, provided we can handle the commutators. This we do again using $L^2$ estimates. We claim that

$$\sum_k \|e^{\varphi} [\psi_k, P] u\|_{L^2} \leq C^{-1} \tau (\varepsilon\tau)^{\frac{1}{2}} \|e^{\varphi} u\|_{L^2 + \tau H^1} \tag{40}$$

which is exactly what we need provided that $C$ is chosen sufficiently large. For this we produce a pointwise bound on the commutators:

$$\begin{aligned} |e^{\varphi}[\psi_k, P]u| &\lesssim e^{\varphi}(|\nabla\psi_k|(|\nabla u| + |u|) + |\nabla^2 \psi_k||u|) \\ &\lesssim e^{\varphi}(|\nabla\psi_k|(|\nabla(e^{\varphi} u)| + \tau|u|) + |\nabla^2\psi_k||u|). \end{aligned}$$

Then (40) follows if we use the bounds on the derivatives of $\psi_k$,

$$|\partial^{\alpha} \psi_k| \leq c_{\alpha} \big( C(\varepsilon\tau)^{-\frac{1}{2}} \big)^{-|\alpha|}.$$

**Proof of Proposition 5.1:** By the same arguments as above we may assume $k \equiv 0$. We assume first that $u$ is supported in a $e^I$ where $I$ is an interval of length $|I| \lesssim 1$ to simplify the notation. We may again assume that $0 \in I$. The result follows by the same perturbation argument as above, but now applied to $u$ without additional restrictions on the support. It suffices to observe that

$$\|e^{\varphi}(P-\Delta)v\|_{L^2+\tau^{-1}H^{-1}} \quad \ll \quad \tau\|e^{\varphi}v\|_{L^2+\tau H^1}.$$

Finally we observe that the restriction on the size of $I$ is inessential since all estimates are scale invariant.

# 6 The construction of $h$ and global estimates

In this section we detail the construction of $h$ and we prove a stronger form of (14) in the flat case. In what follows $C$ is a fixed large constant to be chosen later.

**Lemma 6.1.** *Let $\{a_j\}_{j\in\mathbb{Z}}$ be a nonnegative sequence so that*

$$\|\{a_j\}\|_{l^1_w} \leq \frac{1}{4} \tag{41}$$

*and which is slowly varying,*

$$\frac{1}{2} \leq \frac{a_{j+1}}{a_j} \leq 2 \qquad \forall j \in \mathbb{Z} \tag{42}$$

*Then for each large $\tau$ there exists a function $h : \mathbb{R} \to \mathbb{R}$ with the following properties:*

*(i) $\tau \leq \partial_s h(s) \leq \tau^2$*
*(ii) $h'(s)a_j \leq h''(s) \leq 2h'(s)a_j$ if $j \leq s \leq j+1$ and $a_j \geq C\tau^{-1}$.*
*(iii) $|2h'(s) - \mathbb{Z}| \geq \frac{1}{4}$ if $j \leq s \leq j+1$ and $a_j \leq C\tau^{-1}$.*
*(iv) $|h'''(s)| \leq 4h'(s)a_j$ if $j \leq s \leq j+1$.*

The condition (42) is only a convenient technical assumption which can always be achieved at the expense of a slight increase of the $a_j$'s. More precisely, if $\|\{a_j\}\|_{l^1_w} \leq \frac{1}{12}$ then the sequence $|a_j| * 2^{-|j|}$ satisfies both (42) and (41).

**Proof:** Define the function $h$ by $\partial_s h(-\infty) = [\tau] + 5/4$ and

$$\partial_s^2 h = \sum b_j \chi_{[j,j+1]}$$

where $b_j$'s are integers which satisfy the inductive relation

$$\begin{array}{ll} b_j = 0 & \text{if } a_j \leq Ch'(j)^{-1} \\ b_j \in [h'(j)a_j, 2h'(j)a_j] & \text{if } a_j \geq Ch'(j)^{-1} \end{array}$$

Note that in effect the inductive definition of $b_j$ does not start at $-\infty$ but at some finite integer since there are only finitely may $j$ for which $a_j \geq C\tau^{-1}$.

Then $h$ clearly satisfies the condition (ii). Furthermore, on each interval where $a_j \leq Ch'(j)^{-1}$ the function $\partial_s h$ is constant and takes values in $\mathbb{Z} + \frac{1}{2}$. Thus (iii) also holds. Now we verify (i). If we denote

$$J = \{j \in \mathbb{Z};\ a_j \geq Ch'(j)^{-1}\}$$

then

$$h'(k) \leq \tau \prod_{j \in J} (1 + 2a_j). \tag{43}$$

The right hand side becomes larger under rearrangement of the numbers $a_j$ such that $a_{j+1} \leq a_j$. The worst case is when

$$a_j = \|\{a_k\}\|_{l^1_w}/j.$$

Then we get

$$h'(k) \leq \tau \prod_{k=1}^{\frac{1}{4C}h'(k)} (1 + \frac{k}{2}) \leq \tau e^{\frac{1}{2}\ln(\frac{1}{4C}h'(k))} \leq \tau h'(k)^{\frac{1}{2}}$$

which implies (i).

To fulfill also (iv) take the above defined function and regularize it on the scale of 1,

$$h := h * \eta$$

where $\eta$ is smooth, nonnegative, supported in $[-1, 1]$ and with integral 1. This is where the slowly varying assumption is used.

Now we can state our main global estimate:

**Theorem 5.** *Let $\{a_j\}_{j\in\mathbb{Z}}$ and $h$ be as in Lemma 6.1 and $k(x)$ be a function satisfying*

$$|x||\nabla k| + |x|^2|\nabla^2 k| \ll a_j h'(j), \qquad s \in [j, j+1] \tag{44}$$

*Consider a metric $g$ satisfying*

$$|g - I_n| + |x||\nabla g| \ll a_j \qquad -\ln(|x|) \in [j, j+1] \tag{45}$$

*Then the estimate*

$$\|e^{\varphi} u\|_{l^q(X_{h',a})} \leq \|e^{\varphi} P u\|_{l^q(X'_{h',a})} \tag{46}$$

*holds for all $v$ vanishing of infinite order at 0 and $\infty$.*

Here the spaces $X_{h',a}$ are defined in each dyadic annulus $2^{-k-1} \leq |x| \leq 2^{-k}$ as $X_{h'(k),a_k}$ (Recall that $s = -\ln x$ so this corresponds to a partition into intervals of size 1 with respect to $s$). On this we superimpose the $l^q$ summation. This implies (15) uniformly with respect to the class of weights allowed above.

**Proof:** The proof does not require new ideas. Using a partition of unit in $s$ we localize to $s$-intervals of fixed size. Then on each such interval we can apply either Theorem 5.2 (if $a_j > C\tau^{-1}$) or Theorem 5.1 (if $a_j < C\tau^{-1}$).

# 7 Wolff's lemma and the gradient term

In the previous section we have obtained uniform Carleman estimates for family of weights $\varphi$ given as sum of a function $h$ satisfying the conditions in Lemma 6.1 and a perturbation $k$ subject to (44). Wolff's idea to overcome the missing Carleman estimate for controlling the gradient term is the following: We should use the flexibility in choosing the weights $\varphi$, to pick one of them so that $|e^{\varphi} W_1 \nabla u|^{p'}$ and $|e^{\varphi} W_2 u|^{p'}$ are concentrated on a "small" set - it should be so small so that the $L^p$ estimate suffices to bound the worst contributions of the gradient term.

We claim that we can choose some function $\varphi$ satisfying the conditions in Lemma 6.1 so that (17) holds. Verifying that claim completes the proof of Theorem 1.

Without any restriction in generality we assume that

$$\|W_i\|_{l^1_w(L^n)} = 1$$

Then we can choose the convexity parameters $a_j$ satisfying the assumptions of Lemma 6.1 so that (45) holds and, in addition,

$$\|\tilde{W}_i\|_{L^n([j,j+1]\times\mathbb{S}^{n-1})} \lesssim a_j \tag{47}$$

The estimate (46) implies that (15) holds uniformly for all weights $\varphi$ we consider, therefore it remains to show that we can choose the weight $\varphi$ so that

$$\|W_1 e^{\varphi} \nabla u\|_{L^{p'}} + \||\nabla\Phi| W_2 e^{\varphi} u\|_{L^{p'}} \lesssim \|e^{\varphi} u\|_{l^{p'}(X_{\varphi})}$$

Expanding the $X_{\varphi}$ norm, this amounts to

$$\|W_1 e^{\varphi} \nabla u\|_{L^{p'}} + \||\nabla\Phi| W_2 e^{\varphi} u\|_{L^{p'}} \lesssim \|e^{\varphi} \nabla u\|_{l^{p'}(L^2+|\nabla\varphi| L^p)} + \|e^{\varphi} u\|_{l^{p'}(L^p)}. \tag{48}$$

Now we think of $u$ and $\nabla u$ as independent functions. We decompose $e^{\varphi}\nabla u$ into an $l^{p'}(L^2)$ and a $l^{p'}(L^p)$ part. The estimate of the $l^{p'}(L^2)$ part follows by Hölder's inequality since $W_1 \in l^1_w(L^n) \subset l^{\infty}(L^n)$. Therefore, by a slight

imprecision of notation, we drop the $L^2$ norm on the right hand side of (48). Then we can combine the remaining estimate to

$$\|W|\nabla\Phi|e^{\varphi}w\|_{L^{p'}} \lesssim \|e^{\varphi}w\|_{l^{p'}(L^p)} \tag{49}$$

where

$$W := |W_1| + |W_2|, \qquad w := |u| + |\nabla\varphi|^{-1}|\nabla u|.$$

It remains to show that the weight $\varphi$ can be chosen so that (49) is true.

The analysis is simpler in the $s, \theta$ coordinates. Then the above estimate is equivalent to

$$\|\tilde{W}|\tilde{\nabla}\varphi|e^{\varphi}v\|_{L^{p'}} \lesssim \|e^{\varphi}v\|_{l^{p'}(L^p)} \tag{50}$$

where $\tilde{W} = |x|W$ and $v = |x|^{\frac{n-2}{2}}w$. Clearly

$$\|\tilde{W}\|_{l^1_w(L^n)(\mathcal{C})} = \|W\|_{l^1_w(L^n)(\mathbb{R}^n)}.$$

Now we use the freedom in the choice of $k$ to osculate $\varphi$ independently within each unit $s$-interval. We consider weight functions of the form

$$\varphi(s,\theta) = h(s) + k(s,\theta)$$

where $k$ satisfies condition (44) of Theorem 5.

STEP 1. Within each $s$-interval $[j, j+1] \times \mathbb{S}^{n-1}$ choose a cube $F_j$ of size $1/10$ so that

$$\|e^{h(s)}|\tilde{\nabla}\varphi|\tilde{W}v\|_{L^{p'}([j,j+1]\times\mathbb{S}^{n-1})} \leq c\|e^{h(s)}|\tilde{\nabla}\varphi|\tilde{W}v\|_{L^{p'}(F_j)} \tag{51}$$

If any two such cubes are closer than $1/4$ then we eliminate the one in which the norm is smaller. Then we have still

$$\|e^{h(s)}|\tilde{\nabla}\varphi|\tilde{W}v\|^{p'}_{L^{p'}} \lesssim \sum_j \|e^{h(s)}|\tilde{\nabla}\varphi|\tilde{W}v\|^{p'}_{L^{p'}(F_j)}$$

where the sum is only over those $j$'s we keep. Next we choose $k$ of the form

$$k(s,\theta) := ch'(j)(-k_0(s,\theta) + \frac{1}{10}l(s,\theta)), \qquad s \in [j, j+1]$$

where $c$ is a small parameter, $k_0 \geq 0$ satisfies

$$\begin{array}{ll} k_0 = 0 & \text{in } F_j \\ k_0 \geq a_j & \text{in } [j,j+1]\times\mathbb{S}^{n-1} \setminus 2F_j \\ |\tilde{\nabla}k_0| \lesssim a_j & \text{in } [j,j+1]\times\mathbb{S}^{n-1} \end{array}$$

and $l$ satisfies

$$|l|, |\tilde{\nabla}l| \leq a_j$$

Such a modification is small enough so that it satisfies (44). The effect of this change is that we concentrate the function $e^{h(s,\theta)}\tilde{W}v$ to the set $\cup 2F_j$ while still retaining the ability to modify $h$ further by osculating $l$ within the allowed margins. The point of this argument is to reduce the problem to a fixed unit $s$-interval.

STEP 2. Introduce flat local coordinates $y$ in $4F_j$ so that $0 \in F_j$. Let $\chi(y)$ be a cutoff function supported in $4F_j$ which is 1 in $2F_j$. Then we seek $l$ of the form

$$l(y) = \chi(y) a_j b \cdot y,$$

where $|b| \leq 1$. To continue we need the $n$-dimensional version of Wolff's lemma from [29], Lemma 1:

**Lemma 7.1.** *(Wolff) Let $\mu$ be a positive compactly supported measure in $\mathbb{R}^n$. Define $\mu_k$ by $d\mu_k(x) = e^{k\cdot x} d\mu(x)$. Suppose $B$ is a convex body in $\mathbb{R}^n$. Then there is a sequence $\{k_i\} \subset B$ and, for each $i$, a convex body $E_{k_i}$ with*

$$\mu_{k_i}(\mathbb{R}^n \setminus E_{k_i}) \leq \frac{1}{2}\|\mu_{k_i}\| \tag{52}$$

*such that $\{E_{k_i}\}$ are pairwise disjoint and*

$$\sum_j |E_{k_i}|^{-1} \geq C|B| \tag{53}$$

*where $C$ is a positive constant depending only on $n$, and where $|B|$, $|E_{k_i}|$ denote the Lebesgue measures of $B$ and $E_{k_i}$.*

We apply the lemma for the measure

$$d\mu = \left| |\tilde{\nabla}\varphi| e^{\varphi(s,\theta)} \tilde{W} v \right|^{p'} ds d\theta$$

in $2F_j$, with $B = B_{ch'(j)a_j}(0)$. Then

$$|B| \approx (a_j h'(j))^n$$

therefore

$$\sum_i |E_{k_i}|^{-1} \gtrsim (a_j h'(j))^n.$$

On the other hand we have

$$\sum_i \|\tilde{W}\|^n_{L^n(E_{k_i})} \leq \|\tilde{W}\|^n_{L^n(2F_j)}.$$

Then for some $i$ we obtain

$$\|\tilde{W}\|^n_{L^n(E_{k_i})} (a_j h'(j))^n \lesssim \|\tilde{W}\|^n_{L^n(2F_j)} |E_{k_i}|^{-1}$$

which by (47) yields

$$\|\tilde{W}\|_{L^n(E_{k_i})} \lesssim |h'(j)|^{-1}|E_{k_i}|^{-\frac{1}{n}}.$$

We set $b = k_i(ca_j)^{-1}$, choose $l$ as described above and denote the set $E_{k_i}$ by $E_j \subset 2F_j$. Then on one hand $e^{\varphi}|\tilde{\nabla}\varphi|\tilde{W}v$ is concentrated on $\cup E_j$,

$$\|e^{\varphi}|\tilde{\nabla}\varphi|\tilde{W}v\|_{L^{p'}} \lesssim \|e^{\varphi}|\tilde{\nabla}\varphi|\tilde{W}v\|_{L^{p'}(\cup E_j)} \tag{54}$$

while, on the other hand, within the sets $E_j$ we can estimate

$$\begin{aligned} \|e^{\varphi}|\tilde{\nabla}\varphi|\tilde{W}v\|_{L^{p'}(E_j)} &\lesssim h'(j)\|e^{\varphi}v\|_{L^2(E_j)}\|\tilde{W}\|_{L^n(E_j)} \\ &\lesssim |E_j|^{-\frac{1}{n}}\|e^{\varphi}v\|_{L^2(E_j)} \\ &\lesssim \|e^{\varphi}v\|_{L^p(E_j)}. \end{aligned}$$

Now (50) follows. □

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