Sharp counterexamples in unique continuation for second order elliptic equations

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Abstract

We construct nontrivial solutions with compact support for the elliptic equation $\Delta u = Vu$ with $V \in L^p$, $p < n/2$ or $V \in L^{n/2}_w$ for $n \geq 3$ and with $V \in L^1$ for $n = 2$. The same method also yields nontrivial solutions with compact support for the elliptic equation $\Delta u = W\nabla u$ with $W \in L^q$, $p < n$ or $W \in L^n_w$ for $n \geq 2$.

For the second order elliptic equations

$$\Delta u = Vu \quad \text{in} \;\; \mathbb{R}^n \tag{1}$$

respectively

$$\Delta u = W\nabla u \quad \text{in} \;\; \mathbb{R}^n \tag{2}$$

we define the (weak) unique continuation property **(UCP)** , respectively the strong unique continuation property **(SUCP)** as follows:

Let u be a solution to (1) which vanishes in an open set. Then $u = 0$. **(UCP)**

Let u be a solution to (1) which vanishes of infinite order at some point $x_0 \in \mathbb{R}^n$. Then $u = 0$. **(SUCP)**

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The same definitions apply to solutions to (2).

If $V = W = 0$ then **(SUCP)** holds trivially because of the analyticity of u. The natural question to ask is for which class of potentials does the unique continuation property hold. Scale invariant classes of potentials are $V \in L^{\frac{n}{2}}$, respectively $W \in L^n$. It has been believed for some time that this should be the threshold: **(SUCP)** was expected to hold above it and there is a simple counterexample below it, namely

$$u(x) = e^{-|\ln x|^{1+\epsilon}}.$$

On the positive side, **(SUCP)** was proved in Jerison-Kenig [1] for $L^{\frac{n}{2}}_{loc}$ potentials, $n \geq 3$ and for L^p potentials, $p > 1$, $n = 2$. Stein [1] shows that **(SUCP)** holds for potentials which are small in $L^{\frac{n}{2}}_w$, $n \geq 3$. See also [3] for the most recent positive results. The counterexamples with $V \in L^{n/2}_w$ of Wolff [6] for **(SUCP)** show that these results are optimal.

On the other hand, the counterexamples to **(UCP)** are rather scarce. Recently Kenig-Nadirashvili [2] have obtained a counterexample to unique continuation for (1) with $V \in L^1$ for $n \geq 2$, while Mandache [4] found a counterexample to unique continuation for (2) with $W \in L^q$, $q < 2$, for $n \geq 2$. Our aim here is to close the gap for $n \geq 3$ and obtain counterexamples for (1) with $V \in L^{n/2}_w$ and for (2) with $W \in L^n_w$ and also smooth counterexamples with $V \in L^p$, $p < n/2$ or $W \in L^q$, $q < n$. For $n = 2$ we only improve the regularity of the counterexamples compared to [2], [4]. We are grateful to both Kenig-Nadirashvili and Mandache for making their articles [2], respectively [4], available prior to publication.

Let $\mathcal{H}^1$ be the Hardy space, H^1 the space of functions with square integrable derivatives and H^{-1} its dual. Let L^p_w is the weak L^p space and let $L^{p,1} \subset L^p$ be the Lorentz space.

Theorem 1. *a) Let $n \geq 2$, $p < \frac{n}{2}$, $q < n$. Then there exists a nontrivial smooth compactly supported function u so that*

$$\frac{\Delta u}{u} \in L^p(\mathbb{R}^n), \quad \frac{\Delta u}{|\nabla u|} \in L^q(\mathbb{R}^n),$$

b) Let $n \geq 3$. There exists a nontrivial compactly supported function $u \in L^{\frac{n}{n-2},1}$ so that $\Delta u \in \mathcal{H}^1$ and

$$\frac{\Delta u}{u} \in L^{\frac{n}{2}}_w(\mathbb{R}^n)$$

c) Let $n \geq 2$. There exists a nontrivial compactly supported vector valued function $u \in L^{\frac{n}{n-2},1}$ (C_0 for $n = 2$) so that $\nabla u \in L^{\frac{n}{n-1},1}$, $\Delta u \in \mathcal{H}^1$ and

$$\frac{|\Delta u|}{|\nabla u|} \in L^n_w(\mathbb{R}^n)$$

d) Let $n = 2$. Then there exists a nontrivial continuous compactly supported function $u \in H^1$ so that $\Delta u \in \mathcal{H}^1 \subset H^{-1}$ and

$$\frac{\Delta u}{u} \in L^1(\mathbb{R}^2)$$

The above quotients are set to 0 whenever the denominator vanishes. This is acceptable provided that the set where the numerator is zero but the denominator is not has measure zero. Such a condition is always satisfied for smooth functions u, and our counterexamples are smooth except at most for a sphere, which has measure zero.

The reason for using Lorentz spaces in (b) above is that we want to set $V = \Delta u/u$ and then verify that (1) is satisfied. This works due to the multiplicative property

$$L^n_w \cdot L^{\frac{n-2}{n},1} \subset L^1$$

A similar comment applies to (c).

Besides being counterexamples for unique continuation, our counterexamples serve also as examples for embedded eigenvalues and compactly supported eigenfunctions. The construction can easily be modified for nonzero eigenvalues.

The proof is inspired from the work of Kenig-Nadirashvili [2]. The novelty here is that we obtain more precise quantitative estimates which allow us to bridge the gap between L^1 and $L^{\frac{n}{2}}_w$ potentials for $n \geq 3$.

We start with a sequence of disjoint increasing annuli centered at the origin

$$A_k = \{r_k - a_k \leq |x| \leq r_k\}$$

so that the thickness a_k of A_k is equal to its the distance to A_{k+1},

$$r_{k+1} - r_k = a_k + a_{k+1}, \qquad r_0 = a_0 \geq \frac{1}{2}.$$

Here a_k is a decreasing slowly varying sequence. We want all the A_k to be contained in a fixed ball $B(0, 2)$. This is equivalent to a summability condition for the sequence a_k

$$\sum_{k=0}^{\infty} a_k = 1.$$

Corresponding to the sequence A_k we define a sequence of compactly supported cutoff functions χ_k so that $\chi(0) = 1$ and $\nabla\chi_k$ is supported in A_k. More precisely we require that with a constant δ which will be chosen later we have

$$\text{dist (supp } \nabla\chi_k, \partial A_k) \geq \delta a_k/(2\sqrt{n}).$$

Then we define inductively a sequence u_n as follows. Set $u_1 = \chi_1$. For the inductive step we start with some f_k of the form

$$f_k = \sum f_k^j \phi_{\epsilon_k^j}(x - x_n^j)$$

which is "close" to Δu_k in A_k. Here $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$ where ϕ is smooth, nonnegative, supported in the unit ball and having integral 1. Let v_k be the solution to the elliptic problem

$$\begin{aligned} \Delta v_k &= \begin{cases} \Delta u_k & \text{outside } A_k \\ f_k & \text{in } A_k \end{cases} \\ v_k = 0 & \quad \text{on } \partial B(0, r_{k+1}). \end{aligned} \tag{3}$$

Then set $u_{k+1} = \chi_{k+1} v_k$. We will produce the counterexample u as the limit of u_k.

1 The choice of f_k

Here we want to construct f_k using as few x_k^j's as possible so that for a large range of ϵ_k^j we can still make sure that the sequence

$$|u_{k+1} - u_k|$$

decays exponentially (with respect to k) away from A_k. This suffices in order to insure the convergence of u_k. For now we are not concerned at all about the size of $\frac{\Delta u_k}{u_k}$.

Given $\delta > 0$ we define the set $X_k = \{x_k^j\}$ by

$$X_k = A_k \cap \delta a_k \mathbb{Z}^n.$$

Then, with $N_k = |X_k|$ denoting the number of points in X_k,

$$N_k \lesssim \delta^{-n} a_k^{1-n}, \qquad |x_k^j - x_k^l| \geq \delta a_k \quad \text{if } j \neq l.$$

Let ψ be a smooth function supported in $[-3/4, 3/4]^n$ so that

$$\sum_{m \in \mathbb{Z}^n} \psi(x - m) = 1.$$

Correspondingly we get a partition of unit on A_k

$$1 = \sum_j \psi_k^j, \quad \psi_k^j = \psi\Big(\frac{x - x_k^j}{\delta a_k}\Big).$$

The following result is the basis of our iterative construction of the functions u_k.

Proposition 2. *There exist a small constant $\delta > 0$ and a large constant $C > 0$ independent of k so that the following statement is true:*

For each v_{k-1} harmonic in A_k and satisfying

$$|\partial^\alpha v_{k-1}| \leq 2^{-2k} a_k^{1-|\alpha|} \quad \text{in } A_k \quad , |\alpha| \leq 1$$

there exist $(f_k^j)_{1 \leq j \leq N_k}$ so that

$$\delta^{n+1} a_k^{n-1} 2^{-2k} \leq |f_k^j| \leq C \delta^n a_k^{n-1} 2^{-2k} \tag{4}$$

and for all $\epsilon_k^j \leq \delta a_k$ we have

$$|\partial^\alpha v_k| \leq 2^{-2(k+1)} a_{k+1}^{1-|\alpha|} \quad \text{in } A_{k+1} \qquad |\alpha| \leq 1 \tag{5}$$

and

$$|\partial^\alpha (v_k - v_{k-1})| \leq 2^{-2(k+1)} a_k^{1-|\alpha|} \quad \text{in } B(0, r_k - 2a_k) \qquad |\alpha| \leq 1. \tag{6}$$

Proof: In A_k we have

$$g_k = \Delta u_k = v_{k-1} \Delta \chi_k + 2 \nabla v_{k-1} \nabla \chi_k$$

therefore

$$|g_k| \lesssim 2^{-2k} a_k^{-1}. \tag{7}$$

The function

$$w_k = v_k - \chi_k v_{k-1}$$

satisfies

$$\Delta w_k = f_k - g_k, \qquad \text{supp } w_k \subset B(0, r_{k+1}). \tag{8}$$

Then we define

$$f_k^j = \begin{cases} \delta^{n+1} a_k^{n-1} 2^{-2k} & \text{if } |\int \psi_k^j g_k dx| \leq \delta^{n+1} a_k^{n-1} 2^{-2k} \\ \int \psi_k^j g_k dx & \text{otherwise} \end{cases}$$

Given (7), this implies (4). It remains to establish the bounds (5) and (6), i.e. show that w_k is small away from the annulus A_k. Without any restriction in generality we can rescale and assume that $r_{k+1} = 1$. In order to obtain bounds for w_k we need the fundamental solution for the Laplacian in the unit ball,

$$K(x, y) = c_n \Big(|x - y|^{2-n} - |x|^{n-2} |\bar{x} - y|^{2-n} \Big), \quad \bar{x} = \frac{x}{|x|^2}, \quad n \geq 3$$

respectively

$$K(x, y) = c_2 \Big(- \ln |x - y| + \ln |\bar{x} - y| - \ln |x| \Big), \quad \bar{x} = \frac{x}{|x|^2}, \quad n = 2.$$

Note that K is symmetric and satisfies the bounds

$$\begin{aligned}
|K(x,y)| &\lesssim \frac{(1-|x|)(1-|y|)}{|x-y|^n}, \\
|\partial_y K(x,y)| &\lesssim \frac{1-|x|}{|x-y|^n}, \\
|\partial^2 K(x,y)| &\lesssim \frac{1}{|x-y|^n}.
\end{aligned} \tag{9}$$

We can compute

$$\begin{aligned}
w_k(x) &= \int (f_k - g_k)(y) K(x,y) dy \\
&= \sum_j \int (\psi_k^j(y) g_k(y) - f_k^j \phi_{\epsilon_k^j}(y - x_k^j)) K(x,y) dy \\
&= \sum_j \int (\psi_k^j(y) g_k(y) - f_k^j \phi_{\epsilon_k^j}(y - x_k^j))(K(x,y) - K(x, x_k^j)) dy \\
&+ \sum_j \Big(\int \psi_k^j(y) g_k(y) dy - f_k^j \Big) K(x, x_k^j)
\end{aligned}$$

For x at distance at least a_k from A_k and $|y - x_k^j| \lesssim \delta a_k$ we have

$$|K(x,y) - K(x, x_k^j)| \lesssim \delta a_k (1 - |x|)|x - x_k^j|^{-n},$$

$$|\nabla_x (K(x,y) - K(x, x_k^j))| \lesssim \delta a_k |x - x_k^j|^{-n}.$$

Then

$$\begin{aligned}
|w_k(x)| &\lesssim \delta a_k (1-|x|) \sum_j |x_k^j - x|^{-n} \int \psi_k^j(y) |g_k(y)| + |f_k^j| \phi_{\epsilon_k^j} dy \\
&\quad + \delta^{n+1} a_k^{n-1} 2^{-2k} \sum_j K(x, x_k^j) \\
&\lesssim \delta a_k (1-|x|) \sum_j |x_k^j - x|^{-n} \int \psi_k^j(y) |g_k(y)| dy \\
&\quad + \delta^{n+1} a_k^{n-1} 2^{-2k} \sum_j K(x, x_k^j) \\
&\lesssim \delta a_k (1-|x|) \int |g_k(y)| |x-y|^{-n} dy + \delta a_k^{-1} 2^{-2k} \int_{A_k} K(x,y) dy \\
&\lesssim \delta 2^{-2k} (1-|x|) \int_{A_k} |x-y|^{-n} dy \\
&\lesssim \delta a_k (1-|x|) d(x, A_k)^{-1} 2^{-2k}.
\end{aligned}$$

If $x \in A_{k+1}$ then $1 - |x| \le a_{k+1}$ and $d(x, A_k) \approx a_k$ therefore we get

$$|w_k(x)| \lesssim \delta a_{k+1} 2^{-2k}$$

On the other hand if $|x| < r_k - 2a_k$ then $1 - |x| \approx d(x, A_k)$ therefore

$$|w_k(x)| \lesssim \delta a_k 2^{-2k}.$$

The computation for the derivative of w_k is similar, the only difference is that the factor $1 - |x|$ no longer appears. Thus we obtain

$$|\nabla w_k(x)| \lesssim \delta a_k d(x, A_k)^{-1} 2^{-2k}.$$

Then the conclusion follows if we choose δ sufficiently small.

2 Bounds for $u_{k+1} - u_k$

The estimate (5) is the only one needed in order to carry out the iterative process. However, at each step we can also obtain additional information about higher order derivatives of the functions u_k. Since we are interested in the convergence of the sequence u_k, we collect below the information we can obtain about the difference $u_{k+1} - u_k$. The easy part is to get estimates away from A_k:

Proposition 3. *Assume that u_k, v_k, x_k^j, ϵ_k^j and f_k^j are inductively chosen as in Proposition 2. Then*

$$|\partial^\alpha (u_{k+1} - u_k)(x)| \le c_\alpha a_k d(x, A_k)^{-\alpha} 2^{-2k} \quad x \in B(0, r_k - 2a_k) \tag{10}$$

and

$$|\partial^\alpha (u_{k+1} - u_k)(x)| \le c_\alpha a_{k+1}^{1-\alpha} 2^{-2k} \quad x \in A_{k+1} \tag{11}$$

($u_k = 0$ in A_{k+1}, so this is really a bound on u_{k+1}).

For $|\alpha| \le 1$ this is an immediate consequence of Proposition 2. The higher order derivatives are obtained in a similar fashion since each derivative on the kernel K beyond the second simply produces an additional $|x - y|^{-1}$ factor.

It remains to bound $u_{k+1} - u_k$ in a neighborhood of A_k, more precisely in the larger annulus $\{r_k - 2a_k \le |x| \le r_k + a_k\}$. This estimate is not as simple since the difference should have spikes concentrated near the points in X_k.

Proposition 4. *Assume that u_k, v_k, x_k^j, ϵ_k^j and f_k^j are inductively chosen as in Proposition 2. Then for $r_k - 2a_k \le |x| \le r_k + a_k$ we have*

$$|\partial^\alpha (u_{k+1} - u_k)(x)| \le c_\alpha a_k^{n-1} 2^{-2k} \max_j (\epsilon_k^j + |x - x_k^j|)^{2-n-|\alpha|}, \quad n \ge 3. \tag{12}$$

For $n = 2$ (12) still holds for $\alpha \ne 0$, while for $\alpha = 0$ we get

$$|(u_{k+1} - u_k)(x)| \le c_\alpha a_k^{n-1} 2^{-2k} \max_j - \ln(\epsilon_k^j + |x - x_k^j|), \quad n = 2. \tag{13}$$

Proof: We prove the result for $n \geq 3$. The argument also works for $n = 2$ with obvious changes. In the desired range we have $u_{k+1} - u_k = w_k$, so we need to get bounds for w_k, which solves (8). The function $g_k = \Delta u_{k-1}|_{A_k}$ can be estimated using (11),

$$|\partial^\alpha g_k(x)| \leq c_\alpha a_k^{-1-\alpha} 2^{-2k} \quad x \in A_k. \tag{14}$$

We consider only the more difficult case case when x is close to some x_k^j, $|x - x_k^j| \leq \delta a_k$. We decompose w_k into three components,

$$w_k = u_k^j + v_k^j + w_k^j$$

so that u_k^j solves

$$\Delta u_k^j = f_k^j \phi_{\epsilon_k^j}(x - x_n^j), \quad u_k^j = 0 \ \text{ on } \partial B(0, r_{k+1})$$

and v_k^j satisfies

$$\Delta v_k^j = -\chi_k^j(x) g(x), \quad v_k^j = 0 \ \text{ on } \partial B(0, r_{k+1}).$$

Observe that Δw_k^j is supported at least $O(a_k)$ away from $B(x_k^j, \epsilon_k^j)$. Then the analysis in the proof of Proposition 2 applies and we obtain

$$|\partial^\alpha w_k^j| \lesssim c_\alpha a_k^{1-\alpha} 2^{-2k} \quad \text{ in } B(x_k^j, \epsilon_k^j).$$

On the other hand for v_k^j we can use the fundamental solution for Δ. It is convenient to treat separately the two terms in the kernel K,

$$K_0(x, y) = c_n |x - y|^{2-n}, \quad K_1(x, y) = |x|^{n-2} |\bar{x} - y|^{2-n},$$

Correspondingly we decompose

$$u_k^j = u_{k,0}^j + u_{k,1}^j, \quad v_k^j = v_{k,0}^j + v_{k,1}^j.$$

We are in the range where $|x| \approx 1$, $|\bar{x} - y| \approx a_k$, therefore

$$|\partial_x^\alpha K_1(x, y)| \leq c_\alpha a_k^{2-n-\alpha}.$$

This implies that the corresponding components $u_{k,1}^j$, $v_{k,1}^j$ of u_k^j, v_k^j satisfy a bound similar to the bound for w_k^j. Indeed,

$$|\partial_x^\alpha u_{k,1}^j| = \left| \partial_x^\alpha \int K_1(x, y) f_k^j \phi_{\epsilon_k^j}(y - x_n^j) dy \right| \leq c_\alpha a_k^{2-n-\alpha} |f_k^j| \leq c_\alpha a_k^{1-\alpha} 2^{-2k}.$$

The same argument applies to $v_{k,1}^j$ as well.

For $v_{k,0}^j$ we get the same bound,

$$\begin{aligned} |\partial_x^\alpha v_{k,0}^j| &= |\partial_x^\alpha(\chi_k^j g_k * |x|^{2-n})| \\ &= |\partial_x^\alpha \chi_k^j g_k * |x|^{2-n}| \\ &\le c_\alpha a_k^{1-\alpha} 2^{-2k}. \end{aligned}$$

The only contribution which can be worse comes from

$$u_{k,0}^j = f_k^j \phi_{\epsilon_k^j}(x - x_k^j) * |x|^{2-n}.$$

But this can be bounded by

$$|\partial^\alpha u_{k,0}^j| \le c_\alpha a_k^{n-1}(\epsilon_k^j + |x - x_k^j|)^{2-n-|\alpha|} 2^{-2k}.$$

Putting together all the pieces we get (12) and (13).

3 Convergence

Proposition 5. *Assume that u_k, v_k, x_k^j, ϵ_k^j and f_k^j are inductively chosen as in Proposition 2. Then there exists a function $u \in L_w^{\frac{n}{n-2}}$ if $n \ge 3$ and $u \in VMO$ if $n = 2$ supported in $B_2(0)$ with $\Delta u \in L^1$ so that*

- *The sequence Δu_k converges in L^1 to Δu.*
- *u_k converges to u uniformly on compact sets.*
- *$|u - u_{k+1}| \le a_k 2^{-2k}$ in A_k.*

Proof: Observe first that Δu_k is supported in $\cup_{j=0}^k A_j$ and $\Delta u_{k+1} - \Delta u_k$ is supported in $A_k \cup A_{k+1}$. In A_k we have

$$\Delta u_{k+1} - \Delta u_k = f_k - g_k$$

while in A_{k+1} we get

$$\Delta u_{k+1} - \Delta u_k = g_{k+1}.$$

Using the bounds in Proposition 2 we obtain

$$\|\Delta u_{k+1} - \Delta u_k\|_{L^1(\mathbb{R}^n)} \lesssim 2^{-2k}$$

This guarantees the L^1 convergence of Δu_k, which implies the convergence of u_k in $L_w^{\frac{n}{n-2}}$ if $n \ge 3$: The space L_w^p, $p > 1$, is a Banach space when equipped with the norm

$$\|f\|_{L_w^p} = \sup_{A \subset \mathbb{R}^n} |A|^{-\frac{1}{p'}} \int_A |f|\, dx$$

where $|A|$ denotes measure of the set A and the supremum is taking over all measurable sets. Then

$$\|f\|_{L^p_w} \sim \sup_{t>0} t|\{x : |f(x)| > t\}|^{1/p}$$

and $|x|^{2-n} \in L_w^{\frac{n}{n-2}}$, hence

$$\| |x|^{2-n} * f \|_{L_w^{\frac{n}{n-2}}} \leq \| |x|^{2-n} \|_{L_w^{\frac{n}{n-2}}} \|f\|_{L^1}.$$

Similarly u_k converge in VMO if $n = 2$: the functions u_k are smooth, hence in VMO, and they converge in BMO since $\ln|x| \in BMO$, thus $u_k \to u$ in VMO. Note that

$$\Delta u = \sum_k \sum_j f_k^j \phi_{\epsilon_k^j}(x - x_n^j). \tag{15}$$

Moreover

$$\Delta(u - u_k)(x) = 0$$

for x in $B(0, \rho)$ with $\rho < 2$ and k sufficiently large. This, together with convergence in $L_w^{\frac{n}{n-2}}$ implies uniform convergence of all derivatives on $B(0, \rho)$. Finally, the last assertion in the theorem is a straightforward consequence of (10).

We can also sum up the estimates in Propositions 3 and 4 to obtain pointwise bounds on u and its derivatives:

Proposition 6. *Let $n \geq 3$. For $r_k - 2a_k \leq |x| \leq r_k + a_k$ we have*

$$|\partial^\alpha u(x)| \leq 2^{-2k} a_k^{n-1} \sup_j |\epsilon_k^j + dist(x, X_k)|^{2-n-|\alpha|}.$$

The modification for $n = 2$ is obvious.

4 Bounds on $\Delta u/u$, $\Delta u/|\nabla u|$

By construction Δu is supported in $\cup_{k,j} B(x_k^j, \epsilon_k^j)$. The following proposition provides a bound from above on $\Delta u/u$ on each such ball.

Proposition 7. *Assume that u_k, v_k, x_k^j, ϵ_k^j and f_k^j are inductively chosen as in Proposition 2.*

a) Then

$$\left|\frac{\Delta u}{u}\right| \lesssim \begin{cases} (\epsilon_k^j)^{-2} & \text{if } n \geq 3 \\ (\epsilon_k^j)^{-2} |\ln(\epsilon_k^j a_k^{-1})|^{-1} & \text{if } n = 2 \end{cases} \quad \text{in } B(x_k^j, \epsilon_k^j) \tag{16}$$

b) Assume in addition that the mollifier ϕ is spherically symmetric and non-increasing with respect to the radial variable. Then there is a small constant c and points $y_k^j \in B(x_k^j, c\epsilon_k^j)$

$$\frac{|\Delta u|}{|\nabla u|} \lesssim |x - y_k^j|^{-1} \quad \textit{in } B(x_k^j, \epsilon_k^j) \tag{17}$$

Proof: We restrict ourselves to the proof for $n \geq 3$. The modifications for $n = 2$ are obvious.

a) In $B(x_k^j, \epsilon_k^j)$ we have

$$\Delta u = f_k^j \phi_{\epsilon_k^j}(x - x_n^j),$$

therefore

$$|\Delta u| \lesssim |f_k^j|(\epsilon_k^j)^{-n} \lesssim 2^{-2k} a_k^{n-1} (\epsilon_k^j)^{-n}.$$

Then it remains to show that in the same region we have

$$|u| \gtrsim 2^{-2k} a_k^{n-1} (\epsilon_k^j)^{2-n}.$$

By the last part of Proposition 5 and by (11) it suffices to show that

$$|u_{k+1} - u_k| \gtrsim 2^{-2k} a_k^{n-1} |\epsilon_k^j|^{2-n} \quad \text{in } B(x_k^j, \epsilon_k^j).$$

In A_k we have $u_{k+1} - u_k = w_k$. For w_k we use the decomposition in the proof of Proposition 4. All terms except for $u_{k,0}^j$ are bounded by $2^{-2k} a_k$, which is negligible. Then it remains to show that

$$|u_{k,0}^j| \gtrsim 2^{-2k} a_k^{n-1} |\epsilon_k^j|^{2-n} \quad \text{in } B(x_k^j, \epsilon_k^j). \tag{18}$$

But

$$u_{k,0}^j = f_k^j \phi_{\epsilon_k^j}(x - x_k^j) * |x|^{-\frac{n}{2}},$$

therefore for $x \in B(x_k^j, \epsilon_k^j)$ we get

$$|u_{k,0}^j| \geq |f_k^j| (\epsilon_k^j)^{2-n}.$$

Then (18) follows from the bound from below for f_k^j in (4).

b) This time we need to show that

$$|\nabla u| \gtrsim 2^{-2k} a_k^{n-1} (e_k^j)^{-n} |x - y_k^j| \quad \text{in } B(x_k^j, \epsilon_k^j)$$

We decompose u as before into

$$u = u_{k,0}^j + (u - u_{k,0}^j)$$

For the second component we use the bounds in the proof of Proposition 4 to get

$$|\partial^\alpha (u - u_{k,0}^j)| \leq c_\alpha 2^{-2k} a_k^{1-\alpha}$$

On the other hand, u_k^0 is the spherically symmetric (around x_k^j) solution to

$$\Delta u_{k,0}^j = f_k^j \phi_{\epsilon_k^j}(x - x_k^j)$$

We translate and rescale to the case when $x_k^j = 0$, $\epsilon_k^j = 1$. We also use the bound from below for f_k^j in (4) to eliminate the factor $2^{-2k} a_k^{n-1}$ and reduce the problem to the case when $f_k^j = 1$. Then we need to prove the estimate

$$|\nabla(u_0 + u_1)(x)| \gtrsim |x - y| \quad \text{in } B(0,1) \tag{19}$$

for some small y, where u_0 is the bounded spherically symmetric solution to

$$\Delta u_0 = \phi$$

and u_1 is a small C^2 perturbation. It is easy to see that u_0 has an unique critical point, which is a non-degenerate minimum at 0. Then the perturbed function $u_0 + u_1$ will still have an unique non-degenerate critical point y close to 0. Hence (19) follows.

5 Conclusion

Here we finish the proof of our main result. The question is how to choose the parameters a_k, ϵ_k^j. There are two competing factors. In order to get good bounds on $\Delta u/u$ and $\Delta u/|\nabla u|$ we would like to have ϵ_k^j as small as possible, while in order to get better regularity for u we need ϵ_k^j as large as possible. Balancing these two factors yields Theorem 1.

Proof of Theorem 1: a) Here we can choose all ϵ_k^j equal to some ϵ_k. Then by Proposition 7 we get

$$\left\| \frac{\Delta u}{u} \right\|_{L^p}^p \lesssim \sum_k a_k^{1-n} \epsilon_k^{n-2p}, \quad \left\| \frac{\Delta u}{|\nabla u|} \right\|_{L^q}^q \lesssim \sum_k a_k^{1-n} \epsilon_k^{n-q}$$

while

$$\|u\|_{C^m(\mathbb{R}^n)} \lesssim \sup_k 2^{-2k} a_k^{n-2} \epsilon_k^{2-n-m}.$$

Since $p < n/2$ and $q < n$, in order to have all three norms finite it suffices to choose a_k and ϵ_k with polynomial decay in k. This construction is rough and one should even be able to get Gevrey type estimates for u.

b) The difficulty here is that the ratio $\frac{\Delta u}{u}$ is a sum of "bumps" which have comparable $L^{n/2}$ norm. Hence the only way of getting a bounded $L^{n/2}_w$ norm is if the size of these "bumps" decreases exponentially,

$$\{\epsilon^{j_m}_{k_m}\} = \{2^{-m};\ m \in \mathbb{N}\}$$

where $(k_m, j_m)_m$ is an enumeration of the set $\{(k,j) : 1 \le j \le N_j\}$ such that $k_m \le k_{\tilde{m}}$ if $m \le \tilde{m}$. For such a choice of ϵ^j_k we compute the regularity of Δu. It suffices to show that $\Delta u \in \mathcal{H}^1$. Then all second derivatives are in $\mathcal{H}^1 \subset L^1$ and the regularity of u by a variant of the Hardy-Littlewood-Sobolev inequality:

$$\|\Delta^{-1} f\|_{L^{\frac{n}{n-2},1}} \le c\|f\|_{\mathcal{H}^1}$$

whenever the right hand side is bounded. The left hand side is defined by convolution with the fundamental solution. We define the norm on the left hand side by duality with $L^{n/2}_w$. It suffices to check this inequality for atoms, see Stein [5]. After scaling it suffices to prove the estimate for bounded f supported in the unit ball with mean zero, in which case the proof is simple.

Let

$$b_k = C\sum_{j=1}^{k} a_k^{1-n}.$$

Then the sharp bound from below for ϵ^j_k is

$$\epsilon^j_k \ge 2^{-b_k}.$$

We fix such a sequence ϵ^j_k and compute the $\mathcal{H}^1$ norm of Δu. We recall that

$$\Delta u = \sum_{j,k} f^j_k \phi_{\epsilon^j_k}(x - x^j_n).$$

The function u is supported in $B(0,2)$ and $\Delta u \in L^1$. Let $\phi \in C^\infty_0(\mathbb{R}^n)$ with $\int \phi\, dx = 1$ and $\phi_R(x) = R^{-n}\phi(x/R)$. Then

$$\|\Delta u\|_{\mathcal{H}^1} \sim \left\| \sup_R |\phi_R * \Delta u| \right\|_{L^1}.$$

Since Δu has compact support and $\int \Delta u\, dx = 0$ we have

$$\left\| \sup_{R\ge 1} |\phi_R * \Delta u| \right\|_{L^1} \le c\|\Delta u\|_{L^1}.$$

Abusing the notation slightly we consider only radii $R \le 1$ in the $\mathcal{H}^1$ norm below:

$$\|\Delta u\|_{\mathcal{H}^1} \lesssim \sum_{k,j} \|f_k^j \phi_{\epsilon_k^j}(x - x_n^j)\|_{\mathcal{H}^1} \lesssim \sum_{k,j} 2^{-2k} |\ln \epsilon_k^j| \lesssim \sum_k 2^{-2k} a_k^{1-n} b_k.$$

If we choose a_k with polynomial decay then b_k has polynomial growth and the sum converges.

c) We would like to repeat the argument in the previous case. This does not work in the scalar case because the bound in Proposition 7(b) is not as good as the bound in Proposition 7(a). What we would like to have instead of (17) is

$$\frac{|\Delta u|}{|\nabla u|} \lesssim |x - y_k^j|^{-1} \quad \text{in } B(x_k^j, \epsilon_k^j) \tag{20}$$

Then the previous argument can be applied.

To achieve this we use the same procedure to construct two functions, u_1 and u_2, using the same x_k^j, ϵ_k^j but different choices ϕ_1, ϕ_2 for ϕ. Here ϕ_2 is chosen so that

$$\phi_2(x) = \phi_1(x + e_1)$$

where e_1 is the first unit vector in the canonical basis. The bounds from above for Δu_1, Δu_2 and the bounds from below for $|\nabla u_1|$, $|\nabla u_2|$ remain the same as in Proposition 7. Thus for $u = (u_1, u_2)$ we get

$$\frac{|\Delta u|}{|\nabla u|} \lesssim (|x - y_{k,1}^j|^{-1} + |x - y_{k,2}^j|)^{-1} \quad \text{in } B(x_k^j, 2\epsilon_k^j) \tag{21}$$

At this point the shift in ϕ plays an essential role. By the argument in Proposition 7, $y_{k,1}^j$ is close to x_k^j, $|y_{k,1}^j - x_k^j| \le c\epsilon_k^j$ with a small constant c. On the other hand, because of the shift we get $|y_{k,2}^j - (x_k^j + \epsilon_k^j e_1)| \le c\epsilon_k^j$. This implies that $|y_{k,1}^j - y_{k,2}^j| \gtrsim \epsilon_k^j$. Using this in (21) we get some form of (20).

d) By Proposition 7 we know that $\Delta u/u \in L^1$ if

$$\sum_{j,k} |\ln \epsilon_k^j|^{-1} < \infty$$

On the other hand we can bound the $\mathcal{H}^1$ norm of u as in the previous case,

$$\|\Delta u\|_{\mathcal{H}^1} \lesssim \sum_{j,k} 2^{-2k} |\ln(\epsilon_k^j)|.$$

Both sums are finite if we choose a_k with polynomial decay at ∞ and

$$\epsilon_{k_m}^{j_m} \lesssim 2^{-m^\kappa}, \quad \kappa > 1.$$

The space BMO is the dual space of $\mathcal{H}^1$. Since $\ln|x| \in BMO$, $\Delta u \in \mathcal{H}^1$ implies boundedness of u, as well as uniform convergence of u_k. Hence we obtain continuity of u. As above one sees that $\nabla u \in L^2$.

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