Homogeneously Suslin sets

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I. What is (pure) descriptive set theory about?

Answer #1 Definable sets of reals.

- but "definable" is vague
- DST is about projective, $L(\mathbb{R})$-definable etc., sets, but it goes further.
- So far, DST is not about arbitrary $OD(\mathbb{R})$ sets of reals.

Answer #2 Homogeneously Suslin sets.

- So far, yes.
- Maybe not forever.

Plan

1. Definition of "homogeneously Suslin", and related concepts.

2. • Closure properties of Hom
   • Generic absoluteness and Hom

3. • Iteration strategies in Hom
   • Generic absoluteness from iterable mice

4. Does Hom go "all the way"?

Rmks Will assume large cardinal hypotheses without regret.

This is a survey, but not a history.

<u>Def</u> For T a tree on $\omega \times Z$ and $x \in \omega^\omega$

$$x \in p[T] \text{ iff } \exists f \in Z^\omega \, \forall n \, (x \restriction n, f \restriction n) \in T$$

<u>Def</u> A <u>homogeneity system</u> for T is a map $s \mapsto \mu_s$ $(s \in \omega^{<\omega})$ s.t.

- μ_s is a countably complete u.f. s.t. $\mu_s(T_s) = 1$
- $s \subseteq t \Rightarrow \mu_s$ compatible with μ_t
- $x \in p[T]$ iff $\langle \mu_{x \restriction n} \mid n < \omega \rangle$ is a countably complete tower

(iff dir lim $\mathrm{Ult}(V, \mu_{x \restriction n})$ is wellfounded)

<u>Def</u> T is <u>κ-homogeneous</u> iff T admits a homogeneity system consisting of κ-complete measures.

Also call $p[T]$ κ-homogeneous

Def $Hom_\kappa = \{A \subseteq {}^\omega\omega \mid A \text{ is } \kappa\text{-homogeneous}$

$$Hom_{<\lambda} = \bigcap_{\kappa<\lambda} Hom_\kappa$$

$$Hom_\infty = Hom_{<ORD}$$

Rmks

- Hom_κ is closed downward under $\leq_W$.
- $\alpha \leq \beta \Rightarrow Hom_\beta \subseteq Hom_\alpha$. Since $\leq_W$ is wellfounded, there are only finitely many different pointclasses Hom_α.
- The concept was explicitly isolated by Martin and Kechris.

Thm (Martin '68) Every Hom_κ set is determined.

Thm (Martin '68) If κ is measurable, then every $\underset{\sim}{\Pi}^1_1$ set is Hom_κ.

Prop. (K. Windsens) Equivalent are:

1. $A \in \mathrm{Hom}_\kappa$

2. There are $i_{st} : M_s \to M_t$, for $s \subseteq t$, such that $i_{su} = i_{tu} \circ i_{st}$ for $s \subseteq t \subseteq u$, and each M_s is transitive and closed under $2^{\aleph_0}$ sequences, and

$$x \in A \iff \operatorname{dir\,lim}_n M_{x \restriction n} \text{ is wellfounded}$$

and $\operatorname{crit} i_{st} \geq \kappa$ for all $s \subseteq t$, and $M_0 = V$.

Rmk So A is homogeneously Suslin iff A is continuously reducible to wellfoundedness of towers of measures.

Def A is <u>κ-weakly homogeneous</u> iff for some κ-homogeneous $B \subseteq {}^\omega\omega \times {}^\omega\omega$,

$$A(x) \iff \exists y\, B(x,y)$$

If T on $(\omega \times \omega) \times Z$ witnesses B is κ-homogeneous, then regarding T as a tree on $\omega \times (\omega \times Z)$, we call T κ-weakly homogeneous.

Prop T is κ-weakly homogeneous iff there is a countable set σ of κ-comp measures s.t.

$x \in p[T]$ iff $\exists$ tower $\langle \mu_i \mid i<\omega \rangle$ s.t.

- each $\mu_i \in \sigma$ and $\mu_i(T_{x \restriction i}) = 1$

and

- $\langle \mu_i \mid i<\omega \rangle$ is countably complete

The Martin-Solovay tree

Let $\mu_{s,t}$ for $s,t \in \omega^{<\omega}$ witness that T is weakly homogeneous. Let $\langle t_i \mid i<\omega \rangle$ enumerate $\omega^{<\omega}$ naturally. Fix

$$(s, \langle \alpha_0, \dots, \alpha_{lh s - 1} \rangle) \in U$$

$$\text{iff} \quad \alpha_0 < \gamma \text{ and } \forall i < j \leq lh s - 1$$

$$t_i \subsetneq t_j \Rightarrow \pi(\alpha_i) > \alpha_j,$$

for

$$\pi : Ult(V, \mu_{s \restriction lh t_i, t_i}) \to Ult(V, \mu_{s \restriction$$

We write

$$U = ms(T, \vec{\mu}, \gamma)$$

For γ sufficiently large

$$x \notin p[T] \text{ iff } \forall y \; Ult(V, \langle \mu_{x \restriction n, y \restriction n} \mid n<\omega \rangle) \text{ is illfounded}$$

$$\text{iff } x \in p[ms(T, \vec{\mu}, \gamma)]$$

Thm (Martin, Solovay) Let $\vec{\mu}$ witness that T is κ-weakly homogeneous, and G be V-generic / $\mathbb{P}$, where $|\mathbb{P}| < \kappa$. Then for $\gamma > |T|$

$$V[G] \models p[ms(T, \vec{\mu}, \gamma)] = {}^{\omega}\omega - p[T]$$

Rmk We say that T and $ms(T, \vec{\mu}, \gamma)$ are <u>κ-absolute complements</u>, and call $p[T]$ <u>κ-universally Baire</u>. (Feng, Magidor, Woodin)

Cor (Martin, Solovay) If κ is measurable, G is V-generic / $\mathbb{P}$ for $|\mathbb{P}| < \kappa$, and φ is Σ^1_3, then for all $x \in {}^{\omega}\omega \cap V$

$$V \models \varphi[x] \text{ iff } V[G] \models \varphi[x]$$

Prf homogeneous T for Π^1_1 yields weakly homogeneous T for Σ^1_2 yields $ms(T, \vec{\mu}, \gamma)$ for Π^1_2.

Thm (Martin, Steel '85) ~~Let~~ Let T be δ^+-weakly homogeneous via $\vec{\mu}$, where δ is Woodin. Then for all sufficiently large γ

$$ms(T, \vec{\mu}, \gamma) \text{ is } \alpha\text{-homogeneous}$$

for all $\alpha < \delta$.

Cor If λ is a limit of Woodin cardinals then $Hom_{<\lambda}$ is closed under $\exists^{\mathbb{R}}$, $\forall^{\mathbb{R}}$, $\neg$

Cor If λ is a limit of Woodins, then for any Σ^1_n φ, any $x \in {}^{\omega}\omega \cap V$:

$$V \models \varphi[x] \text{ iff } V[G] \models \varphi[x]$$

for G V-gen/$\mathbb{P}$, with $|\mathbb{P}| < \lambda$.

Rmk The corollary was first proved in '84 by Woodin, using generic embedding techniques of Foreman, Magidor, and Shelah.

Thm (Woodin, late 80's) Let δ be Woodin, and T, T^* be δ^+-absolute complements. Then T is α-weakly homogeneous, for all $\alpha < \delta$.

Rmk First proved using stationary tower forcing. There is a proof using iteration trees and iterations to make reals generic. Shows $\mathrm{Hom}_\infty = \infty$-univ. Baire.

Woodin proved a "tree production lemma", which yields absolutely complementing trees projecting to $\{x \mid \varphi(x)\}$ and $\{x \mid \neg\varphi(x)\}$ from a certain generic absoluteness property of φ.

Stationary tower forcing implies that kind of absoluteness for the φ defining, e.g., $\mathbb{R}^\#$.

Thm (Woodin) $\mathbb{R}^{\#} \in \mathrm{Hom}_\infty$; in fact, $A \in \mathrm{Hom}_\infty \Rightarrow A^{\#} \in \mathrm{Hom}_\infty$. (Provided there are arbitrarily large Woodin cardinals.

Thm (S.) Every Hom_∞ set admits a Hom_∞ scale. (Provided there are arbitrarily large Woodin cardinals.)

Rmk In sum

generic absoluteness of $\varphi(\cdot) \Rightarrow$

$\{x \mid \varphi(x)\}$ univ. Baire $\Rightarrow$

$\{x \mid \varphi(x)\}$ weakly homog. $\Rightarrow$

$\{x \mid \varphi(x)\}$ homogeneous

The Derived model theorem

Let λ be a limit of Woodins, and G generic for $Col(\omega, <\lambda)$ and

$$\mathbb{R}^* = \bigcup_{\alpha<\lambda} \mathbb{R} \cap V[G \cap Col(\omega, <\alpha)]$$

Put

$A \in Hom^* \iff A = p[T] \cap \mathbb{R}^*$, for some T s.t. $\exists \alpha < \lambda$ $V[G \cap Col(\omega, <\alpha)] \models T$ is $<\lambda$ weakly homogeneous

<u>Thm</u> (Woodin)

(a) $L(\mathbb{R}^*, Hom^*) \models AD^+$

(b) $Hom^* =$ Suslin-co-Suslin sets of $L(\mathbb{R}^*, Hom^*)$

(c) The Suslin-co-Suslin sets of any model of AD^+ (eg any initial seg. of Hom_∞) can be realized this way.

Reduction to $(\Sigma_1^2)^{Hom_\infty}$

Def. A $(\Sigma_1^2)^{Hom_\infty}$ sentence (formula) is one of the form

$$\exists A \in Hom_\infty \quad (HC, \in, A) \models \varphi$$

A $(\Sigma_1^2)^{Hom_\infty}$ set is one defined by such a formula.

Rmk. Let $A \in \mathrm{Hom}_{<\lambda}$, and G be V-generic for $\mathbb{P}$ s.t. $|\mathbb{P}| < \lambda$. We set

$$A^{V[G]} = p[T]^{V[G]},$$ where T witnesses that A is $|\mathbb{P}|^+$-homogeneous

(Independent of T!)

If λ is a limit of Woodins, we get

$$(HC^V, \in, A) \prec (HC^{V[G]}, \in, A^{V[G]})$$

Theorem (Woodin) $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ sentences are absolute for set forcing, granted that there are arbitrarily large Woodin cardinals.

Proof Upward absoluteness was just done.

Now let

$$V[g] \models \exists A \in \mathrm{Hom}_\infty\ \varphi$$

Pick δ Woodin, $\delta > |\mathbb{P}|$ where g is $\mathbb{P}$-gen

Let

$$j : V \to M \subseteq V[G]$$

be the generic embedding associated to a stationary tower generic G s.t. $g \in V[G]$. Then

$$V[G] \models \exists A \in \mathrm{Hom}_\infty\ \varphi$$

and

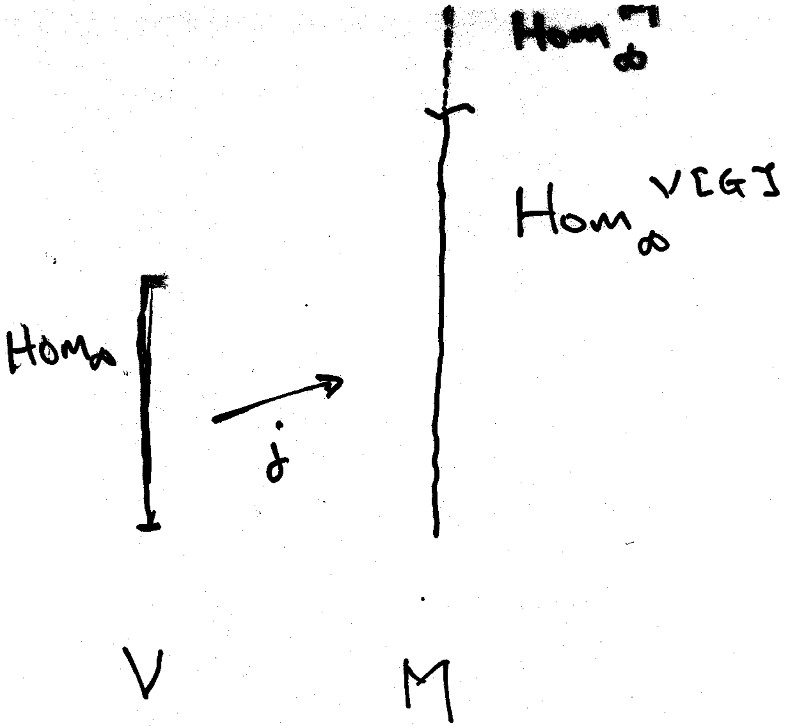
$$\mathrm{Hom}_\infty^{V[G]} \subseteq j(\mathrm{Hom}_\infty)$$

so

$$M \models \exists A \in \mathrm{Hom}_\infty\ \varphi$$

so

$$V \models \exists A \in \mathrm{Hom}_\infty\ \varphi$$



- $\mathbb{R}^M = \mathbb{R}^{V[G]}$
- $Hom_\infty^{V[G]}$ is a Wadge initial segment of Hom_∞^M

<u>Rmk</u> *All generic-absoluteness-of-φ proofs so far can be cast: large card. hypothesis H is s.t.*

$$H \vdash \varphi \leftrightarrow \varphi^*$$

where φ is $(\Sigma_1^2)^{Hom_\infty}$.

(Almost, anyway.) E.g. $L(\mathbb{R})$ truth, or truth in the minimal model of $AD_{\mathbb{R}}$, or ... is provably (from some large cardinal) $(\Sigma_1^2)^{Hom_\infty}$.

Question Does some large cardinal hypo. (e.g. proper class of supercompacts) imply statements of the form

$$\forall x \in {}^{\omega}\omega \; \exists A \in \mathrm{Hom}_{\infty} \; \varphi$$

(where φ has only real quantifiers) are absolute for set forcing?

Question Does some large cardinal hypothesis (e.g. proper class of supercompacts) imply "There is a real which is not $(\Sigma^2_1)^{\mathrm{Hom}_{\infty}}$ in a countable ordinal".

The current limits of generic absoluteness:

<u>Def</u> Let

$$x \in A \Longleftrightarrow \exists n_0 \, \forall n_1 \, \exists n_2 \cdots\cdots \bigwedge_{\alpha<\omega_1} R(x, \langle n_\beta \mid \beta<\alpha \rangle)$$

where R is Π^1_1. Then we say

A is $\Game^{\omega_1}(\text{closed-}\Pi^1_1)$

<u>Prop</u> $CH \Rightarrow \Game^{\omega_1}(\text{closed-}\Pi^1_1) = \Sigma^2_1$

<u>Theorem</u> (Woodin '88?) Assume there are arbitrarily large measurable Woodin cardinals. Then

(a) $\Game^{\omega_1}(\text{closed-}\Pi^1_1) \subseteq \mathrm{Hom}_\infty$

(b) $\Game^{\omega_1}(\text{closed-}\Pi^1_1)$ statements are absolute for set forcings.

Rmks • We don't yet have a reduction of $\supset^{\omega_1}$(closed-Π^1_1) statements to $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ statements. A proof of the determinacy of the long games would probably give this. Alternatively, it would follow from the existence of a Hom_∞ iteration strategy for a countable $M \models \exists$ measurable Woodin.

• The analog of the theorem for $\supset^{\omega_1}$(open-Π^1_1) is an open problem

<u>Corollary</u> (Woodin '88) Assume there are arbitrarily large measurable Woodin cardinals. Let $V[g_1]$ and $V[g_2]$ be set generic extensions satisfying CH. Then for any Σ_1^2 φ:

$$V[g_1] \models \varphi \text{ iff } V[g_2] \models \varphi.$$

<u>Questions</u>

- Is $\Diamond$ a "complete invariant" at the Σ_2^2 level?
- Are there "conditional generic absoluteness" theorems at the Σ_n^2 level, for all n?

<u>Theorem</u> (Woodin, late 90's)
The universal $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ set is Σ^2_3.

<u>Cor.</u> Π^2_3 truth cannot be reduced to $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ truth.

<u>Rmk</u> So the existence of conditional generic absoluteness theorems at the Σ^2_3 level contradicts Woodin's Ω conjecture.

III Hom_∞ iteration strategies

The Iteration game

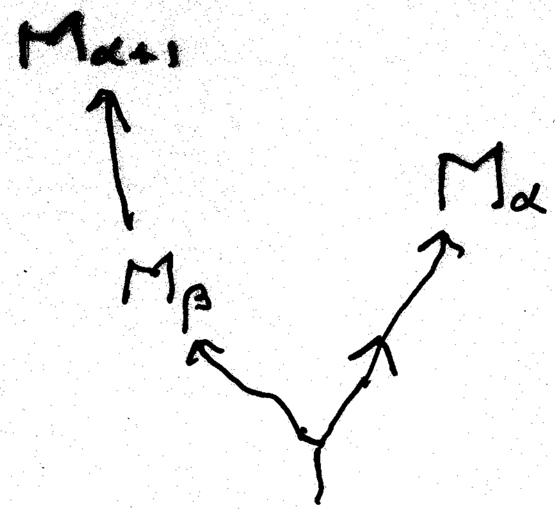
Given a transitive M and ordinal θ, we say M is θ-iterable iff II has a winning strategy in the following game of θ moves:

Set $M_0 = M$.

Move $\alpha+1$: Player I picks an extender F_α from M_α, and some $\beta < \alpha$ s.t.

$$M_{\alpha+1} = \mathrm{Ult}(M_\beta, F_\alpha)$$

makes sense. I wins if $M_{\alpha+1}$ is illfounded. Otherwise, the game continues.



<u>Move λ</u> II picks a branch b of the tree built so far s.t.

$$M_\lambda = \operatorname{dir\,lim}_{\alpha \in b} M_\alpha$$

is wellfounded, and s.t. b is cofinal in λ. If he doesn't, he loses.

If II hasn't lost after θ moves then he wins.

A winning strategy for II is a <u>θ-iteration strategy for M.</u>

Question. Is every countable $M \prec V$ ω_1+1 - iterable ?

Question. Let $L[\vec{E}]$ be a canonical inner model constructed using full background extenders. Is every countable $M \prec L_\alpha[\vec{E}]$ ω_1+1 iterable?

Remark In each case, can lift plays of the iteration game (iteration trees) on M to iteration trees on V. Suppose we try to iterate M by making sure that the corresponding branch of the lifted tree "is wellfounded". (I.e. yields a wellfounded direct limit model".

Thm (Martin, S. 1986) This works so long as there is exactly one cofinal wellfounded branch in the lifted tree on V.

<u>Theorem</u> (Woodin) Assume $\exists j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$

Then there is a countable iteration tree on V having distinct cofinal wellfounded branches.

For $L[\vec{E}]$ models which are not too complicated, we have had some success. E.g.

<u>Theorem</u> (Mitchell, S. 1990) Assume there are arbitrarily large Woodin cardinals. Then there is an canonical minimal $L[\vec{E}]$ model

$$M_\omega^\# \models \exists\ \omega \text{ Woodin cardinals}$$

which has a unique θ^{+1} iteration strategy for all regular θ.

__Remark__ An ω_1-iteration strategy for a countable M is essentially a set of reals. The known iterability proofs give Hom_∞ iteration strategies. Hom_∞ strategies extend to uncountable trees.

__Cor to prf__ Granted arbitrarily large Woodin cardinals, $M_\omega^\#$ has a Hom_∞ iteration strategy.

__Correctness of iterable mice; generic absoluteness from iterable mice:__

__Thm__ (Woodin, S. early 90's) There is a recursive $\varphi \mapsto \varphi^*$ s.t.

$$L(\mathbb{R}) \models \varphi \quad \text{iff} \quad M_\omega^\# \models \varphi^*$$

(Granted that $M_\omega^\#$ is ω_1+1 iterable – and exist

<u>Rmk</u> Thus $L(\mathbb{R})$ truth is $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$, and absolute for set forcing, granted arbitrarily large Woodin cardinals.

<u>Rmk</u> A similar result holds for $\Game^{\omega_1}$(closed-Π^1_1) truth vis-a-vis truth in the minimal $L[\vec{E}]$ model satisfying "there is a measurable Woodin" which is ω_1+1 iterable. (Provided there is such a model.) (Woodin, early 90's)

This would reduce $\Game^{\omega_1}$(closed-Π^1_1) truth to $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ truth, provided we got Hom_∞ iteration strategies.

<u>Question</u> How far to Hom_∞ iteration strategies go?

Some warning signs at the level of supercompacts:

1) There are iteration trees on V of length ω having no cofinal branch if there is an ω-extendible card. and "long extenders" are allowed in our trees. (Neeman)

2) In trees with long extenders, it is impossible to completely avoid "moving generators", so the proof of comparison ~~at the~~ seems to break down.

In sum, we don't really ~~know~~ have even a good iterability conjecture at the level of supercompacts.

Can some other $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ condition identify the canonical mice at the supercompact level?

<u>Thm</u> (Woodin, late 90's) Let κ be extendible, and G be V-generic for $\mathrm{Col}(\omega,\kappa)$. Then

$$L(\mathrm{Hom}_\infty, \mathbb{R})^{V[G]} \equiv L(\mathrm{Hom}_\infty, \mathbb{R})^{V[G][H]}$$

for all H generic over $V[G]$, moreover

$$L(\mathrm{Hom}_\infty, \mathbb{R})^{V[G]} \models AD_{\mathbb{R}}.$$

<u>Cor</u> No wellorder of $P(\kappa)\cap V$ can be $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ in $V[G]$.

<u>Seven conjectures</u>

Assume there are arb. large supercompacts.

1. There is a Hom_∞ iteration strategy for an $L[\vec{E}]$ model with a superstrong
2. There is a Hom_∞ set of reals coding an $L[\vec{E}, \mathbb{R}]$ model with a superstrong
3. No countable $M \prec V$ has a Hom_∞ iteration strategy
4. The Ω conjecture is false.
5. There are Σ^2_n conditional generic absoluteness theorems, for all n.
6. There is no ~~model~~ canonical inner model with an extendible containing all the reals and coded by a Hom_∞ set of reals.
7. There is a canonical minimal model with an extendible, but its truth set is not $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$.

In this picture, Hom does ~~do~~ not go "all the way".

There are generic absoluteness theorems, and notions of canonicity for mice, which are not enforced by Hom_∞ sets.