Determinacy, large cardinals, and inner models

John R. Steel
University of California, Berkeley

Joint Mathematics Meeting
San Francisco, January 2024

Plan:

I. Large cardinal hypotheses and the consistency strength hierarchy.
II. Infinite games.
III. Large cardinals imply determinacy.
IV. Inner model theory.

Basic set theory

The *ordinals* are:

$0, 1, 2, \ldots\omega, \omega{+}1, \omega{+}2, \ldots\omega{+}\omega, \ldots, \ldots, \ldots\omega^2, \ldots\ldots\omega_1, \ldots\ldots\omega_2, \ldots.$

They are the stages in a definition or proof by induction.

Basic set theory

The *ordinals* are:

$0, 1, 2, ...\omega, \omega+1, \omega+2, ...\omega+\omega, ..., ..., ...\omega^2, ...\omega_1, ...\omega_2, ...$

They are the stages in a definition or proof by induction.

Definition

$$V_0 = \emptyset,$$

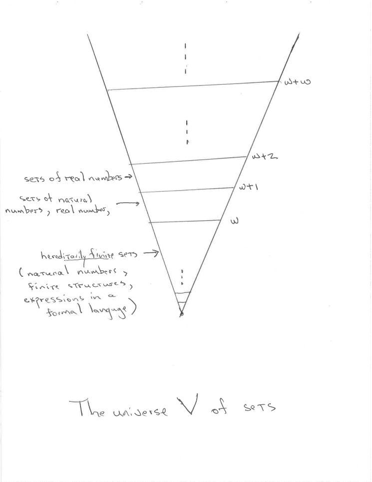
$$V_{\alpha+1} = \{x \mid x \subseteq V_\alpha\},$$

and for λ a limit ordinal,

$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha.$$

The universe V of all sets is the union of the V_α.
One can show that $V_\alpha \subsetneq V_\beta$ whenever $\alpha < \beta$. In fact, $V_{\alpha+1}$ has strictly greater cardinality than V_α, for all α.
(Cantor 1873.)

Cantor’s paradise



Definition

The *language of set theory* has binary relation symbols $=$ and $\in$, as well the logical symbols $\forall, \exists, \wedge, \vee, \neg, \rightarrow, \leftrightarrow, (,)$, and variables.

ZFC is the theory in this language whose axioms are Extensionality, Regularity, Nullset, Pairing, Union, Infinity, Powerset, Separation, Collection, and Choice.

Except for Extensionality and Regularity, these axioms are *set-existence* axioms.

Definition

The *language of set theory* has binary relation symbols $=$ and $\in$, as well the logical symbols $\forall, \exists, \wedge, \vee, \neg, \rightarrow, \leftrightarrow, (,)$, and variables.

ZFC is the theory in this language whose axioms are Extensionality, Regularity, Nullset, Pairing, Union, Infinity, Powerset, Separation, Collection, and Choice.

Except for Extensionality and Regularity, these axioms are *set-existence* axioms.
There is no ambiguity as to what constitutes a sentence of LST, an axiom of ZFC, or a proof from ZFC. One can program a computer to determine whether a given string of symbols has these properties. This makes *metamathematics* possible.

Incompleteness

What is the status of ZFC as a foundation for mathematics?

(a) (Late 1800s–1927) The theorems of "ordinary mathematics" can be stated in the language of set theory, and proved from the axioms of ZFC.

Incompleteness

What is the status of ZFC as a foundation for mathematics?

(a) (Late 1800s–1927) The theorems of “ordinary mathematics” can be stated in the language of set theory, and proved from the axioms of ZFC.

(b) (Cantor) There is no set of all sets.

(c) (Tarski 1936) Truth in LST is not definable in LST.

Incompleteness

What is the status of ZFC as a foundation for mathematics?

(a) (Late 1800s–1927) The theorems of "ordinary mathematics" can be stated in the language of set theory, and proved from the axioms of ZFC.

(b) (Cantor) There is no set of all sets.

(c) (Tarski 1936) Truth in LST is not definable in LST.

(d) (Gödel 1931.) No consistent finite extension of ZFC proves its own consistency.

Incompleteness

What is the status of ZFC as a foundation for mathematics?

(a) (Late 1800s–1927) The theorems of “ordinary mathematics” can be stated in the language of set theory, and proved from the axioms of ZFC.

(b) (Cantor) There is no set of all sets.

(c) (Tarski 1936) Truth in LST is not definable in LST.

(d) (Gödel 1931.) No consistent finite extension of ZFC proves its own consistency.

(e) (Gödel 1936, Cohen 1963) ZFC neither proves nor refutes the Continuum Hypothesis.

Descriptive Set Theory

The Continuum Hypothesis concerns *arbtirary* sets of reals, sets having no perhaps no definition or construction. Such sets can be pathological from the point of view of Analysis.

Thesis of Descriptive Set Theory (Borel, Baire, Lebesgue ca. 1904) Definable sets of reals are free of the pathologies that follow from a wellorder of the reals.

Definition

A set of reals is *Borel* iff it can be obtained from open sets via countable unions, intersections, and complements. A set is *projective* iff it can be obtained from a Borel subset of some $\mathbb{R}^n$ via projections and complements.

Remark A subset of $\mathbb{R}^n$ is projective iff it is definable from parameters over $(V_{\omega+1}, \in)$.

Classical descriptive set theorists (1904-mid 1930s) verified their thesis for low-level projective sets. For example, if $A \subset \mathbb{R}$ is the projection of a Borel $B \subset \mathbb{R}^2$, then A is Lebesgue measurable, has the property of Baire, and has many other "regularity properties".

(f) (Gödel 1936, Solovay 1970) ZFC neither proves nor refutes the assertion that there is a (projectively) definable wellorder of the reals. ZFC neither proves nor refutes the assertion that all (projectively) definable sets of reals are Lebesgue measurable.

Strong axioms of infinity

How can we strengthen ZFC so as to remove some of its incompleteness?

Strong axioms of infinity

How can we strengthen ZFC so as to remove some of its incompleteness? Strengthening the Axiom of Infinity is the most fruitful approach we know.

Informal Reflection Principle: Suitable properties of V are shared by some V_α.
The idea is that the V_α's go on as long as possible. If something happened for the first time at V, we should have declared that level a V_α, and kept going.

Some examples:

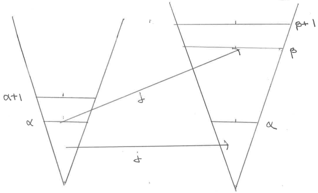
(1) There is an α such that V_α is infinite.

(2) There is an α such that ZFC is true in V_α.

(3) There is an α such that "second-order ZFC" is true in $V_{\alpha+1}$ (that is, α is *strongly inaccessible*).

(4) There is a *1-extendible* α.

A 1-extendibility embedding



$$j : V_{\alpha+1} \longrightarrow V_{\beta+1}$$

$V_{\alpha+1} \models \varphi[x]$ if and only if $V_{\beta+1} \models \varphi[j(x)]$

Between inacessible and extendible:

Definition

(a) α is *measurable* iff there is an elementary embedding $j\colon V_{\alpha+1} \to M$, where M is transitive, such that $\alpha = \mathrm{crit}(j)$.

(b) If in addition $V_\beta \subseteq M$, then j witnesses that α is β-*strong*.

(c) If in addition $j(A) \cap \beta = A \cap \beta$, then j witnesses that κ is (β, A)-*strong*.

(c) If $V_{j(\alpha)} \subseteq M$, then j witnesses that α is *superstrong*.

Definition

δ is *Woodin* iff δ is inaccessible, and
$\forall A \subseteq \delta \exists \kappa < \delta \forall \beta < \delta(\kappa$ is (β, A)-strong.)

These properties can be reformulated as requiring the existence of two-valued measures (ultrafilters). For example

Lemma

The following are equivalent

(a) κ *is measurable.*

(b) *There is a* κ*-complete, nonprincipal ultrafilter on* κ*, and* $\kappa > \omega$

(c) *There is an elementary* $j \colon V \to M$*, where* M *is a transitive class, such that* $\kappa = \mathrm{crit}(j)$*.*

Proof.

For $(a) \Rightarrow (b)$: for $A \subseteq \kappa$, let

$$A \in U \text{ iff } \kappa \in j(A).$$

For $(b) \Rightarrow (c)$: let $M = \mathrm{Ult}(V, U)$ be the ultrapower of V by U, and let j be the canonical embedding of V into $\mathrm{Ult}(V, U)$. □

Stronger properties reflect the weaker ones. The reflection occurs at the critical point of the stronger embedding. For example:

Lemma

(a) Let κ be measurable; then κ is an inaccessible limit of inaccessible cardinals.

(b) Let κ be $\kappa + 2$-strong; then κ is a measurable limit of measurable cardinals.

(c) Let δ be Woodin; then δ is a limit of κ such that κ is β-strong for all $\beta < \delta$.

(c) Let κ be superstrong; then κ is a Woodin limit of Woodin cardinals.

Proof.

For (b): Let $j\colon V \to M$ with $\mathrm{crit}(j) = \kappa$ and $V_{\kappa+2} \subseteq M$. Since κ is measurable, we have a nonprincipal, κ-complete ultrafilter U on $P(\kappa)$. Since $U \in V_{\kappa+2}$, $U \in M$, and

$$M \models \kappa \text{ is measurable.}$$

For any $\beta < \kappa$, $j(\beta) = \beta$, so

$$M \models \exists\alpha(j(\beta) < \alpha < j(\kappa) \wedge \alpha \text{ is measurable}),$$

so by the elementarity of j

$$V \models \exists\alpha(\beta < \alpha < \kappa \wedge \alpha \text{ is measurable}).$$

□

In general, the closer M is to V, the stronger is the assertion that there is a nontrivial elementary $j\colon V \to M$. Kunen (1971) proved that $M = V$ is impossible.

The consistency strength hierarchy

There are many mutually incompatible extensions of ZFC, but there seem to be no incompatibilities among their most concrete consequences.

The consistency strength hierarchy

There are many mutually incompatible extensions of ZFC, but there seem to be no incompatibilities among their most concrete consequences.

For T a theory, let $(\Pi^0_\omega)_T$ be the set of consequences of T of the form "$(V_\omega, \in) \models \varphi$" (equivalently, in the language of first order arithmetic).

The consistency strength hierarchy

There are many mutually incompatible extensions of ZFC, but there seem to be no incompatibilities among their most concrete consequences.

For T a theory, let $(\Pi^0_\omega)_T$ be the set of consequences of T of the form "$(V_\omega, \in) \models \varphi$" (equivalently, in the language of first order arithmetic).

Phenomenon: If T is a natural extension of ZFC, then there is an extension S axiomatized by large cardinal hypotheses such that

$$(\Pi^0_\omega)_T = (\Pi^0_\omega)_S.$$

The consistency strength hierarchy

There are many mutually incompatible extensions of ZFC, but there seem to be no incompatibilities among their most concrete consequences.

For T a theory, let $(\Pi^0_\omega)_T$ be the set of consequences of T of the form "$(V_\omega, \in) \models \varphi$" (equivalently, in the language of first order arithmetic).

Phenomenon: If T is a natural extension of ZFC, then there is an extension S axiomatized by large cardinal hypotheses such that

$$(\Pi^0_\omega)_T = (\Pi^0_\omega)_S.$$

Moreover, if T and U are natural extensions of ZFC, then

$$(\Pi^0_\omega)_T \subseteq (\Pi^0_\omega)_U \text{ or } (\Pi^0_\omega)_U \subseteq (\Pi^0_\omega)_T.$$

The consistency strength hierarchy

There are many mutually incompatible extensions of ZFC, but there seem to be no incompatibilities among their most concrete consequences.

For T a theory, let $(\Pi^0_\omega)_T$ be the set of consequences of T of the form "$(V_\omega, \in) \models \varphi$" (equivalently, in the language of first order arithmetic).

Phenomenon: If T is a natural extension of ZFC, then there is an extension S axiomatized by large cardinal hypotheses such that

$$(\Pi^0_\omega)_T = (\Pi^0_\omega)_S.$$

Moreover, if T and U are natural extensions of ZFC, then

$$(\Pi^0_\omega)_T \subseteq (\Pi^0_\omega)_U \text{ or } (\Pi^0_\omega)_U \subseteq (\Pi^0_\omega)_T.$$

In practice, $(\Pi^0_\omega)_T \subseteq (\Pi^0_\omega)_U$ iff ZFC proves $\mathrm{Con}(U) \Rightarrow \mathrm{Con}(T)$.

Definition

For any theory T in LST, $(\Pi^1_\omega)_T$ is the set of consequences of T of the form "$V_{\omega+1} \models \varphi$" (equivalently, in the language of 2nd order arithmetic).

In practice, $(\Pi^0_\omega)_T \subseteq (\Pi^0_\omega)_U$ iff ZFC proves $\mathrm{Con}(U) \Rightarrow \mathrm{Con}(T)$.

Definition

For any theory T in LST, $(\Pi^1_\omega)_T$ is the set of consequences of T of the form "$V_{\omega+1} \models \varphi$" (equivalently, in the language of 2nd order arithmetic).

The questions of descriptive set theory about the projective sets can usually be stated in the language of 2nd order arithmetic. So if we are liberal about "natural", there are many mutually incompatible possibilities for $(\Pi^1_\omega)_T$ among natural T extending ZFC. But

Phenomenon: Let T, U be natural theories strong enough to prove the consistency of ZFC + "There are infinitely many Woodin cardinals"; then

$$(\Pi^1_\omega)_T \subseteq (\Pi^1_\omega)_U \text{ or } (\Pi^1_\omega)_U \subseteq (\Pi^1_\omega)_T.$$

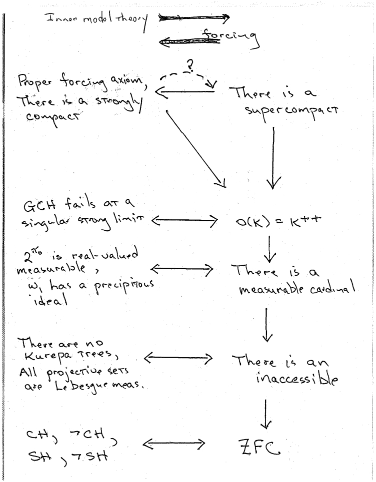
This corresponds to the fact that "there are infinitely many Woodin cardinals" decides all those classical questions about projective sets.

Phenomenon: Let T, U be natural theories strong enough to prove the consistency of ZFC + "There are infinitely many Woodin cardinals"; then

$$(\Pi^1_\omega)_T \subseteq (\Pi^1_\omega)_U \text{ or } (\Pi^1_\omega)_U \subseteq (\Pi^1_\omega)_T.$$

This corresponds to the fact that "there are infinitely many Woodin cardinals" decides all those classical questions about projective sets.

CH is a sentence about $V_{\omega+2}$. Nothing like our current large cardinal hypotheses can decide CH. (Levy, Solovay 1967).


Inner model theory
forcing
?
Proper forcing axiom, There is a strongly compact
There is a supercompact
GCH fails at a singular strong limit
$o(\kappa) = \kappa^{++}$
$2^{\aleph_0}$ is real-valued measurable, ω_1 has a precipitous ideal
There is a measurable cardinal
There are no Kurepa trees, All projective sets are Lebesgue meas.
There is an inaccessible
CH, $\neg$CH, SH, $\neg$SH
ZFC

38

Strong compact, forcing axioms

PFA, MM $\longleftarrow$ supercompact

$\downarrow$

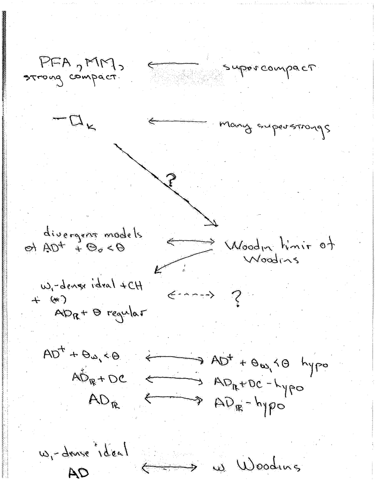
$\neg\Box_\kappa$, for κ singular strong limit $\longleftarrow$ many superstrongs

ω_1-dense ideal

ZF+AD $\longleftrightarrow$ ω Woodins

ω_2 saturated ideal on ω_1

Δ^1_2-det $\longleftrightarrow$ one Woodin cardinal


PFA, MM, strong compact
supercompact
$\neg\square_\kappa$
many superstrongs
?
divergent models of $AD^+ + \Theta_0 < \Theta$
Woodin limit of Woodins
ω_1-dense ideal + CH + (*)
$AD_{\mathbb{R}} + \Theta$ regular
?
$AD^+ + \Theta_{\omega_1} < \Theta$
$AD^+ + \Theta_{\omega_1} < \Theta$ hypo
$AD_{\mathbb{R}} + DC$
$AD_{\mathbb{R}} + DC$ - hypo
$AD_{\mathbb{R}}$
$AD_{\mathbb{R}}$ - hypo
ω_1-dense ideal
AD
ω Woodins

Infinite games

Large cardinals decide the questions of classical descriptive set theory left undecided by ZFC.

Infinite games

Large cardinals decide the questions of classical descriptive set theory left undecided by ZFC.

Let A be a set of infinite sequences of elements of some set Z. A is the *payoff set* for a game G_A:

$$\begin{array}{c|cccccccccc} I & n_0 & & n_2 & & \dots & & n_{2i} & & \dots & \\ \hline II & & n_1 & & n_3 & & \dots & & n_{2i+1} & & \dots \end{array}$$

where player I wins iff $\langle n_0, n_1, n_2, ...\rangle \in A$. We say G_A (or A) is *determined* iff one of the two players has a winning strategy.

$Z = \omega$ is a very interesting special case. Then the payoff set A is essentially a set of reals. (The *Baire space* ${}^\omega\omega$ is homeomorphic to the irrationals, and to $({}^\omega\omega)^k$ for all $k \leq \omega$.))

Determinacy as a law of Logic

Two of Aristotle's laws:

$$\neg\exists n P \Leftrightarrow \forall n \neg P,$$

$$\neg\forall n P \Leftrightarrow \exists n \neg P.$$

Determinacy as a law of Logic

Two of Aristotle's laws:

$\neg\exists nP \Leftrightarrow \forall n\neg P$,

$\neg\forall nP \Leftrightarrow \exists n\neg P$.

The determinacy of G_A can be expressed:

$$\neg\exists n_0\forall n_1\exists n_2...A(n_0, n_1, n_2, ...)$$

if and only if

$$\forall n_0\exists n_1\forall n_2...\neg A(n_0, n_1, n_2, ...).$$

For games of finite length, this follows at once from Aristotle's laws.

Determinacy as a law of Logic

Two of Aristotle's laws:

$\neg\exists n P \Leftrightarrow \forall n \neg P$,

$\neg\forall n P \Leftrightarrow \exists n \neg P$.

The determinacy of G_A can be expressed:

$$\neg\exists n_0 \forall n_1 \exists n_2 ... A(n_0, n_1, n_2, ...)$$

if and only if

$$\forall n_0 \exists n_1 \forall n_2 ... \neg A(n_0, n_1, n_2, ...).$$

For games of finite length, this follows at once from Aristotle's laws.The Law of the Excluded Middle states that games with no moves are determined.

Borel determinacy

It is easy to construct a non-determined $A \subseteq {}^{\omega}\omega$ using the Axiom of Choice. But

Theorem (Open determinacy, Gale-Stewart 1951)

Let A be given by:

$$A(n_0, n_1, n_2, ...) \Leftrightarrow \exists k B(n_0, ..., n_k);$$

then G_A is determined.

Borel determinacy

It is easy to construct a non-determined $A \subseteq {}^{\omega}\omega$ using the Axiom of Choice. But

Theorem (Open determinacy, Gale-Stewart 1951)

Let A be given by:

$$A(n_0, n_1, n_2, \ldots) \Leftrightarrow \exists k B(n_0, \ldots, n_k);$$

then G_A is determined.

Proof.

Suppose that I has no winning strategy. Then II has the strategy: avoid positions from which I has a winning strategy. Since $\neg A$ is closed in ${}^{\omega}Z$, II wins if he never reaches a losing position. Thus this strategy is winning for II. □

Borel determinacy

It is easy to construct a non-determined $A \subseteq {}^\omega\omega$ using the Axiom of Choice. But

Theorem (Open determinacy, Gale-Stewart 1951)

Let A be given by:

$$A(n_0, n_1, n_2, ...) \Leftrightarrow \exists k B(n_0, ..., n_k);$$

then G_A is determined.

Proof.

Suppose that I has no winning strategy. Then II has the strategy: avoid positions from which I has a winning strategy. Since $\neg A$ is closed in ${}^\omega Z$, II wins if he never reaches a losing position. Thus this strategy is winning for II. □

Remark. The proof is highly non-constructive.

Theorem

(1) [Martin 1975] All Borel games are determined.

(2) [Friedman 1974] You need the V_α's, for $\alpha < \omega_1$, to prove this.

Theorem

(1) [Martin 1975] All Borel games are determined.

(2) [Friedman 1974] You need the V_α's, for $\alpha < \omega_1$, to prove this.

Remark. Borel determinacy can be stated in the language of $(V_{\omega+1}, \in)$. It is not provable in Zermelo set theory, which does not prove that $V_{\omega+\omega}$ exists.

The determinacy proof uses:

Martin's Auxiliary Game Method. Given a game G on X, associate an open game G^* on some $X \times Y$, and show that winning strategies in G^* yield winning strategies for the same player in G.

Suslin representations

Definition

A *tree on* X is a set $T \subseteq X^{<\omega}$ of finite sequences from X that is closed under initial segment. For $f \in {}^{\omega}X$,

$$f \in [T] \text{ iff } \forall k(f \restriction k \in T).$$

If T is a tree on $X \times Y$, then for $x \in {}^{\omega}X$,

$$x \in p[T] \text{ iff } \exists g \in {}^{\omega}Y \forall k(\langle x \restriction k, g \restriction k \rangle \in T).$$

A set $A \subseteq {}^{\omega}X$ is κ-*Suslin* iff $A = p[T]$ for some tree T on $X \times \kappa$.

Suslin representations

Definition

A *tree on* X is a set $T \subseteq X^{<\omega}$ of finite sequences from X that is closed under initial segment. For $f \in {}^{\omega}X$,

$$f \in [T] \text{ iff } \forall k (f \restriction k \in T).$$

If T is a tree on $X \times Y$, then for $x \in {}^{\omega}X$,

$$x \in p[T] \text{ iff } \exists g \in {}^{\omega}Y \forall k (\langle x \restriction k, g \restriction k \rangle \in T).$$

A set $A \subseteq {}^{\omega}X$ is κ*-Suslin* iff $A = p[T]$ for some tree T on $X \times \kappa$.

Every subset of ${}^{\omega}\omega$ is 2^{ω}-Suslin in a trivial way. A Suslin representation T can be nontrivial if T is definable in some way, or we have a nontrivial bound on its size, or structural information like homogeneity. Nontrivial Suslin representations are a crucial tool in Descriptive Set Theory.

Definition

A set $A \subseteq ({}^\omega\omega)^k$ is

(1) $\mathbf{\Sigma}^1_1$ iff A is the projection of a Borel subset of some $({}^\omega\omega)^l$,

(2) $\mathbf{\Pi}^1_1$ iff $({}^\omega\omega)^k \setminus A$ is $\mathbf{\Sigma}^1_1$, and

(3) $\mathbf{\Delta}^1_1$ iff it is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

Theorem

(Suslin 1917)

(a) $\Sigma^1_1 = \omega$*-Suslin.*

(b) $\Delta^1_1 =$ *Borel.*

We can we use Suslin representations to prove determinacy.

Given $A = p[T]$, associate to G_A the auxiliary game G^*_T:

$$\begin{array}{c|ccccccccc} I & n_0, \alpha_0 & & n_2, \alpha_1, \alpha_2 & & \ldots & n_{2i}, \alpha_{2i-1}, \alpha_{2i} & & \ldots \\ \hline II & & n_1 & & n_3 & & \ldots & n_{2i+1} & \ldots \end{array}$$

where player I wins iff $\forall k(\langle (n_0, \alpha_0), ..., (n_k, \alpha_k)\rangle \in T)$.
In general G^*_T is not equivalent to G_A. For example, you can't use ω-Suslin representations to prove $\mathbf{\Sigma}^1_1$-determinacy. Player I's additional moves may force him to give too much information.

Homogeneously Suslin sets

Definition

A *tower of measures on Y* is a sequence $\langle \mu_n \mid n < \omega \rangle$ of countably complete ultrafilters such that

(i) $\mu_n(Y^n) = 1$ for all n, and

(ii) if $m \leq n$, then μ_m is compatible with μ_n; i.e. whenever $\mu_m(A) = 1$, then $\mu_n(\{u \mid u \upharpoonright m \in A\}) = 1$.

We say the tower $\vec{\mu}$ is *countably complete* iff whenever $\mu_n(A_n) = 1$ for all n, then $\exists f \forall n (f \upharpoonright n \in A_n)$.

If $\vec{\mu}$ is a tower of measures, then $i_{m,n}([f]_{\mu_m}) = [f^*]_{\mu_n}$, where $f^*(u) = f(u \upharpoonright m)$. Then

$$\vec{\mu} \text{ is countably complete iff } \lim_n \mathrm{Ult}(V, \mu_n) \text{ is wellfounded.}$$

Definition
For T a tree on $X \times Y$ and $s \in X^{<\omega}$, $T_s = \{u \mid (s, u) \in T\}$.

Definition
Let T be a tree on $X \times Y$. A *homogeneity system for T* is a family $\langle \mu_s \mid s \in X^{<\omega} \rangle$ of countably complete ultrafilters such that

(a) for all s, $\mu_s(T_s) = 1$,
(b) if $s \subseteq t$ then μ_s is compatible with μ_t, and
(c) $x \in p[T]$ iff $\langle \mu_{x \restriction n} \mid n < \omega \rangle$ is countably complete.

We say A is *κ-homogeneous* iff $A = p[T]$ for some T that admits a homogeneity system consisting of κ-complete ultrafilters.

Theorem

(Martin 1968) Let $A \subseteq {}^{\omega}X$ be $|X|^+$-homogeneously Suslin; then G_A is determined.

Proof.

Let $A = p[T]$ where $\vec{\mu}$ is a $|X|^+$-complete homogeneity system for T. So G^*_T is determined. Clearly, if I has a ws in G^*_T, he has a ws in G_A.

So suppose σ^* is a ws for II in G^*_T. We define σ for II in G_A by: if s is a position with II to move

$$\sigma(s) = x \text{ iff for } \mu_s\text{-a.e. } u\ (\sigma^*(\langle s, u\rangle) = x).$$

σ is well defined by $|X|^+$-completeness. Suppose toward contradiction that x is an infinite play according to σ and $x \in p[T]$. By the countable completeness of the tower $\langle \mu_{x\restriction k} \mid k < \omega\rangle$ there is f such that $(x, f) \in [T]$ and (x, f) is a play by σ^*. So (x, f) is a loss for σ^*, contradiction. □

Theorem

(Martin 1968) Let κ be a measurable cardinal; then every $\mathbf{\Pi}^1_1$ set is κ-homogeneous.

Proof.

(Sketch.) Let $\neg A = p[U]$, where U is a tree on $\omega \times \omega$. So

$$x \in A \text{ iff } [U_x] = \emptyset,$$

where $U_x = \bigcup_k U_{x \restriction k}$. Our T on $\omega \times \kappa$ is such that $(x, f) \in [T]$ iff f determines a rank function for U_x.
So long as $\kappa \geq \omega_1$, we get $A = p[T]$. (Novikov-Kondo 1930s). If κ is measurable, then T admits κ-complete homogeneity measures. □

Π^1_1 determinacy

Corollary

(Martin, 1968) If there is a measurable cardinal, then all $\boldsymbol{\Pi}^1_1$ *games are determined.*

The concept of homogeneity was abstracted independently by Kechris and Martin from Martin's proof of Π^1_1 determinacy. Being homogeneously Suslin is the fundamental regularity property for sets of reals.

Π^1_1 determinacy

Corollary

(Martin, 1968) If there is a measurable cardinal, then all $\boldsymbol{\Pi}^1_1$ *games are determined.*

The concept of homogeneity was abstracted independently by Kechris and Martin from Martin's proof of Π^1_1 determinacy. Being homogeneously Suslin is the fundamental regularity property for sets of reals.

Π^1_1-determinacy requires something approximating the measurable cardinal:

Theorem

(Martin, Harrington 1975) Π^1_1 *determinacy is equivalent to the existence of a "standard" model of a certain fragment of* ZFC + *"there is a measurable cardinal".*

The model one gets from Π^1_1 determinacy is the *canonical inner model* for the associated large cardinal hypothesis.

Projective determinacy

Theorem

(Martin-Solovay 1969, Martin-S. 1985) Let $A \subseteq ({}^\omega\omega)^{k+1}$ *be* δ^+*-homogeneously Suslin, where* δ *is a Woodin cardinal. Let*

$$B(\vec{x}) \text{ iff } \forall y \neg A(\vec{x}, y);$$

then for all $\gamma < \delta$*, B is* γ*-homogeneously Suslin.*

Projective determinacy

Theorem

(Martin-Solovay 1969, Martin-S. 1985) Let $A \subseteq ({}^\omega\omega)^{k+1}$ be δ^+-homogeneously Suslin, where δ is a Woodin cardinal. Let

$$B(\vec{x}) \text{ iff } \forall y \neg A(\vec{x}, y);$$

then for all $\gamma < \delta$, B is γ-homogeneously Suslin.

Martin-Solovay constructed the Suslin representations. The idea: let $\langle \mu_s \mid s \in \omega^{<\omega} \rangle$ be a homogenity system for T. We get a Suslin representation for $\neg p[T]$ by using

$$x \notin p[T] \text{ iff } \lim_n \mathrm{Ult}(V, \mu_{x \restriction n}) \text{ is illfounded.}$$

Paths through the *Martin-Solovay tree* $\mathrm{ms}(T, \vec{\mu})$ projecting to $\neg p[T]$ are pairs (x, f) such that f witnesses the illfoundedness of $\lim_n \mathrm{Ult}(V, \mu_{x \restriction n})$.

Projective determinacy

Theorem

(Martin-Solovay 1969, Martin-S. 1985) Let $A \subseteq ({}^{\omega}\omega)^{k+1}$ be δ^+-homogeneously Suslin, where δ is a Woodin cardinal. Let

$$B(\vec{x}) \text{ iff } \forall y \neg A(\vec{x}, y);$$

then for all $\gamma < \delta$, B is γ-homogeneously Suslin.

Martin-Solovay constructed the Suslin representations. The idea: let $\langle \mu_s \mid s \in \omega^{<\omega} \rangle$ be a homogenity system for T. We get a Suslin representation for $\neg p[T]$ by using

$$x \notin p[T] \text{ iff } \lim_n \mathrm{Ult}(V, \mu_{x\restriction n}) \text{ is illfounded.}$$

Paths through the *Martin-Solovay tree* $\mathrm{ms}(T, \vec{\mu})$ projecting to $\neg p[T]$ are pairs (x, f) such that f witnesses the illfoundedness of $\lim_n \mathrm{Ult}(V, \mu_{x\restriction n})$.

Martin-S. constructed the homogeneity measures on $\mathrm{ms}(T, \vec{\mu})$. The idea for that comes out of inner model theory.

Corollary

(1) *If there are n Woodin cardinals with a measurable cardinal above them all, then all $\mathbf{\Pi}^1_{n+1}$ sets are homogeneously Suslin, and hence determined.*

(2) *If there are infinitely many Woodin cardinals, then all projective sets are homogeneously Suslin, and hence determined. .*

Corollary

(1) If there are n Woodin cardinals with a measurable cardinal above them all, then all $\mathbf{\Pi}^1_{n+1}$ sets are homogeneously Suslin, and hence determined.

(2) If there are infinitely many Woodin cardinals, then all projective sets are homogeneously Suslin, and hence determined. .

Let PD be the assertion that all projective games on ω are determined. Kechris, Martin, Mycielski, Moschovakis, Solovay, and others had by 1985 shown that PD decides the main questions about projective sets left open by classical descriptive set theory, in a way that extrapolates naturally the classical theory of low-level projective sets.

Corollary

(1) If there are n Woodin cardinals with a measurable cardinal above them all, then all $\mathbf{\Pi}^1_{n+1}$ sets are homogeneously Suslin, and hence determined.

(2) If there are infinitely many Woodin cardinals, then all projective sets are homogeneously Suslin, and hence determined. .

Let PD be the assertion that all projective games on ω are determined. Kechris, Martin, Mycielski, Moschovakis, Solovay, and others had by 1985 shown that PD decides the main questions about projective sets left open by classical descriptive set theory, in a way that extrapolates naturally the classical theory of low-level projective sets.

A large-cardinal equivalent of PD

Theorem (Martin, S., Woodin, late 80's)

The following are equivalent

(1) PD,

(2) For all n there is a Σ^1_n*-correct inner model of* ZFC + *"there are n Woodin cardinals",*

(3) For all n, every consequence of ZFC + *"There are n Woodin cardinals" of the form "*$V_{\omega+1} \models \varphi$*" is true.*

A large-cardinal equivalent of PD

Theorem (Martin, S., Woodin, late 80's)

The following are equivalent

(1) PD*,*

(2) For all n there is a Σ^1_n*-correct inner model of* ZFC + *"there are n Woodin cardinals",*

(3) For all n, every consequence of ZFC + *"There are n Woodin cardinals" of the form "*$V_{\omega+1} \models \varphi$*" is true.*

That (2) $\Rightarrow$ (3) is general logic, and that (3) $\Rightarrow$ (1) is immediate from the corollary above.

A large-cardinal equivalent of PD

Theorem (Martin, S., Woodin, late 80's)

The following are equivalent

(1) PD*,*

(2) For all n there is a Σ^1_n-correct inner model of ZFC *+ "there are n Woodin cardinals",*

(3) For all n, every consequence of ZFC *+ "There are n Woodin cardinals" of the form "$V_{\omega+1} \models \varphi$" is true.*

That (2) $\Rightarrow$ (3) is general logic, and that (3) $\Rightarrow$ (1) is immediate from the corollary above.That (1) $\Rightarrow$ (2) comes out inner model theory, and we hope to say something about it later

A large-cardinal equivalent of PD

Theorem (Martin, S., Woodin, late 80's)

The following are equivalent

(1) PD*,*

(2) For all n there is a Σ^1_n*-correct inner model of* ZFC *+ "there are n Woodin cardinals",*

(3) For all n, every consequence of ZFC *+ "There are n Woodin cardinals" of the form "*$V_{\omega+1} \models \varphi$*" is true.*

That (2) $\Rightarrow$ (3) is general logic, and that (3) $\Rightarrow$ (1) is immediate from the corollary above.That (1) $\Rightarrow$ (2) comes out inner model theory, and we hope to say something about it later

The moral is that determinacy axiomatizes the consequences of large cardinal hypotheses in the theory of the reals.

Beyond projective

Work of Foreman, Magidor, and Shelah (1984) had motivated the results above. Using their machinery more heavily, Woodin showed

Theorem

(Woodin 1986) If there are ω Woodin cardinals with a measurable above them all, then all sets of reals in $L(\mathbb{R})$ are homogeneously Suslin, and hence determined. Thus $L(\mathbb{R}) \models \mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$.

Let us set

$$\mathrm{Hom}_\kappa = \{A \subseteq {}^\omega\omega \mid A \text{ is } \kappa\text{-homogeneous}\}$$
$$\mathrm{Hom}_\infty = \bigcap_\kappa \mathrm{Hom}_\kappa\,.$$

Theorem

(Woodin 1986) Suppose there are arbitrarily large Woodin cardinals; then for all $A \in \mathrm{Hom}_\infty$, $P(\mathbb{R}) \cap L(A, \mathbb{R}) \subseteq \mathrm{Hom}_\infty$, and thus $L(A, \mathbb{R}) \models \mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$.

Inner model theory

Goal: Associate to each large cardinal hypothesis H a canonical minimal universe M_H satisfying H whose structure can be analyzed in detail.
The paradigm is Gödel's universe of constructible sets:

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= \{A \subseteq L_\alpha \mid A \text{ is definable over } (L_\alpha, \in)\} \\ L_\lambda &= \bigcup_{\alpha<\lambda} L_\alpha, \text{ for } \lambda \text{ limit.} \end{aligned}$$

Inner model theory

Goal: Associate to each large cardinal hypothesis H a canonical minimal universe M_H satisfying H whose structure can be analyzed in detail.
The paradigm is Gödel's universe of constructible sets:

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= \{A \subseteq L_\alpha \mid A \text{ is definable over } (L_\alpha, \in)\} \\ L_\lambda &= \bigcup_{\alpha<\lambda} L_\alpha, \text{ for } \lambda \text{ limit.} \end{aligned}$$

We enlarge Gödel's L by adding at certain stages an *extender* over the model we have so far.

Any $j \colon M \to N$ can be captured by system of measures on M. To capture j and N up to λ, for $a \subseteq \lambda$ finite put

$$X \in E_a \text{ iff } a \in j(X).$$

One has the diagram

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[dr, "i_E"'] & & N \\ & \lim_a \mathrm{Ult}(M, E_a) \arrow[ur, "k"'] & \end{tikzcd}$$

with $k \restriction \lambda = \text{ id}$.

Any $j\colon M \to N$ can be captured by system of measures on M. To capture j and N up to λ, for $a \subseteq \lambda$ finite put

$$X \in E_a \text{ iff } a \in j(X).$$

One has the diagram

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[dr, "i_E"'] & & N \\ & \lim_a \mathrm{Ult}(M, E_a) \arrow[ur, "k"'] & \end{tikzcd}$$

with $k \restriction \lambda = \text{ id}$.

Definition

An *extender E over M with support* λ is a system of compatible measures $\langle E_a \mid a \in [\lambda]^{<\omega} \rangle$ on M coding an elementary $i_E\colon M \to \mathrm{Ult}(M, E) = \lim_a \mathrm{Ult}(M, E_a)$.

Any $j\colon M \to N$ can be captured by system of measures on M. To capture j and N up to λ, for $a \subseteq \lambda$ finite put

$$X \in E_a \text{ iff } a \in j(X).$$

One has the diagram

$$\begin{tikzcd}
M \arrow[r, "j"] \arrow[dr, "i_E"'] & N \\
& \lim_a \mathrm{Ult}(M, E_a) \arrow[u, "k"']
\end{tikzcd}$$

with $k \restriction \lambda = \text{ id}$.

Definition

An *extender E over M with support* λ is a system of compatible measures $\langle E_a \mid a \in [\lambda]^{<\omega} \rangle$ on M coding an elementary $i_E\colon M \to \mathrm{Ult}(M, E) = \lim_a \mathrm{Ult}(M, E_a)$.$E$ is *short* if $\lambda = i_E(\mathrm{crit}(E))$.

Short extenders can represent superstrong embeddings, but not extendibility embeddings.

Incompleteness
Large cardinal hypotheses
Determinacy
Large cardinals and determinacy
Inner model theory

Our expansion of L has a hierarchy whose levels are *premice*.

Definition

A *premouse* is a structure of the form $M = (J_\gamma^{\vec{E}}, \in, \vec{E})$, where $\vec{E}$ is a *coherent sequence of short extenders*.

Coherence: for all $\alpha \leq \gamma$, $E_\alpha = \emptyset$, or E_α is an extender with support α over $M|\alpha = (J_\alpha^{\vec{E}\restriction\alpha}, \in, \vec{E} \restriction \alpha)$ coding

$$i\colon M|\alpha \to N = \mathrm{Ult}(\mathcal{M}|\alpha, E_\alpha)$$

such that

$$i(\vec{E} \restriction \alpha) \restriction \alpha = \vec{E} \restriction \alpha \text{ and } i(\vec{E} \restriction \alpha)_\alpha = \emptyset.$$

The extenders in a coherent sequence appear in order of their strength, without leaving gaps.

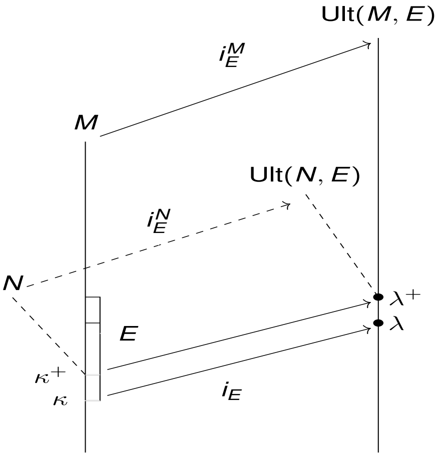
Definition

$\rho_n(\mathcal{M})$ is the Σ_n-projectum of $\mathcal{M}$. We say $\mathcal{M}$ is *n-sound* iff

$$\mathcal{M} = \mathrm{Hull}_n^{\mathcal{M}}(\rho_n(\mathcal{M}) \cup r),$$

for r the n-th standard parameter of $\mathcal{M}$.

We require of premice that proper initial segments be n-sound for all n. It follows that for all κ, $P(\kappa) \cap \mathcal{M} \subseteq \mathcal{M}|(\kappa^+)^{\mathcal{M}}$. This is a strong, local form of GCH.



M agrees with Ult(M, E) and Ult(N, E) to $(\lambda^+)^{\mathrm{Ult}(M,E)}$.

The iteration game

Let M be a premouse. In $\mathcal{G}(M, \theta)$, players I and II play for θ rounds, producing a tree $\mathcal{T}$ of models, with embeddings along its branches, and $M = \mathcal{M}_0^{\mathcal{T}}$ at the base.

Round $\alpha + 1$*:* I picks an extender $E_\alpha^{\mathcal{T}}$ from the sequence of $\mathcal{M}_\alpha^{\mathcal{T}}$ with support $\geq$ the supports of all earlier extenders chosen . Let β be least such that $\mathrm{crit}(E_\alpha^{\mathcal{T}}) <$ support $(E_\beta^{\mathcal{T}})$. We set

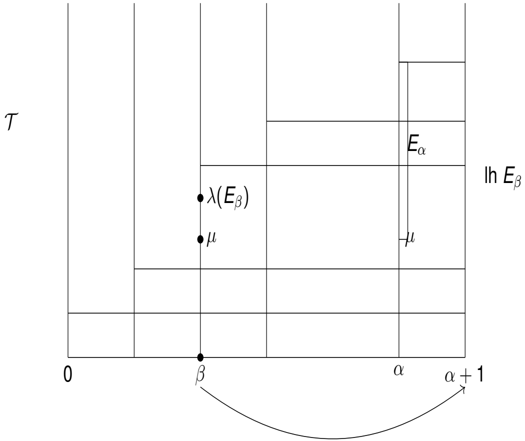
$$\mathcal{M}_{\alpha+1}^{\mathcal{T}} = \mathrm{Ult}_k(\mathcal{M}_\beta^{\mathcal{T}} | \eta, E_\alpha^{\mathcal{T}}),$$

where $\langle \eta, k \rangle$ is as large as possible.

Round λ*, for* λ *limit:* II picks a branch b of $\mathcal{T}$ which is cofinal in λ, and we set

$$\mathcal{M}_\lambda^{\mathcal{T}} = \text{ dirlim }_{\alpha \in b} \mathcal{M}_\alpha^{\mathcal{T}}.$$

As soon as an illfounded model $\mathcal{M}_\alpha^{\mathcal{T}}$ arises, player I wins. If this has not happened after θ rounds, then II wins.



The vertical lines represent the models of $\mathcal{T}$, and the horizontal ones their agreement with one another. β is the $\mathcal{T}$-predecessor of $\alpha + 1$.

Definition

A *θ-iteration strategy for M* is a winning strategy for II in $\mathcal{G}(M, \theta)$. We say *M* is *θ-iterable* just in case there is such a strategy. If Σ is a strategy for II in $\mathcal{G}(M, \theta)$, and $P = \mathcal{M}_\alpha^\mathcal{T}$ for some $\mathcal{T}$ played by Σ, then we call *P* a *Σ-iterate* of *M*.

Theorem (Comparison Lemma)

Let Σ and Γ be $\theta + 1$ iteration strategies for M and N respectively, where $\theta =$ max $(|\mathcal{M}|, |\mathcal{N}|)^+$. Then either

(a) there is a Γ-iterate P of N, and a map $j\colon M \to P|\eta$ produced by Σ-iteration, or

(b) there is a Σ-iterate P of M, and a map $j\colon N \to P|\eta$ produced by Γ-iteration M.

Corollary

If M and N are $\omega_1 + 1$-iterable, then $M|\alpha = N|\alpha$, where $\alpha = \inf(\omega_1^M, \omega_1^N)$. That is, their canonical wellorders of the reals by stage-of-construction are compatible.

Proof. If $j \colon M \to P$ is produced by iteration, then $P|\omega_1^P = M|\omega_1^M$. So we can apply the comparison lemma. □

Corollary

If M is an $\omega_1 + 1$-iterable premouse, and $x \in \mathbb{R} \cap M$, then x is ordinal definable.

Proof. Let x be the α-th real in M. Then $y = x$ iff y is the α-th real in some $\omega_1 + 1$- iterable premouse. □

Definition

Let $n \leq \omega$; then $M_n^\sharp$ is the minimal $\omega_1 + 1$- iterable, sound, active premouse satisfying "there are n Woodin cardinals". M_n is the result of iterating the last extender of M_n through the ordinals.

Thus M_n is the canonical minimal proper class extender model with n Woodins, and $M_n^\sharp$ is its sharp. $M_0 = L$. The basic theory of these and somewhat larger extender models with many Woodin cardinals was developed by Martin, Mitchell, and Steel. The optimal correctness results for these models were established by Woodin.

This led to

Theorem (Martin, Mitchell, S., Woodin 1985-1990)

Suppose there are ω Woodin cardinal, plus a measurable cardinal above them all; then

(1) for any $n < \omega$,

(a) $\mathbb{R} \cap M_n = \{x \mid x \text{ is } \Delta^1_{n+2} \text{ in a countable ordinal}\}$, and

(b) $M_n \models$ "$\mathbb{R}$ has a Δ^1_{n+2} wellorder.

This led to

Theorem (Martin, Mitchell, S., Woodin 1985-1990)

Suppose there are ω Woodin cardinal, plus a measurable cardinal above them all; then

(1) for any $n < \omega$,

   (a) $\mathbb{R} \cap M_n = \{x \mid x \text{ is } \Delta^1_{n+2} \text{ in a countable ordinal}\}$, and

   (b) $M_n \models$ "$\mathbb{R}$ has a Δ^1_{n+2} wellorder.

(2) (a) $\mathbb{R} \cap M_\omega = \{x \in \mathbb{R} \mid x \text{ is } OD^{L(\mathbb{R})}\}$, and

   (b) $M_\omega \models$ "$\mathbb{R}$ has an $OD^{L(\mathbb{R})}$ wellorder".

This led to

Theorem (Martin, Mitchell, S., Woodin 1985-1990)

Suppose there are ω Woodin cardinal, plus a measurable cardinal above them all; then

(1) for any $n < \omega$,

(a) $\mathbb{R} \cap M_n = \{x \mid x \text{ is } \Delta^1_{n+2} \text{ in a countable ordinal}\}$, and

(b) $M_n \models$ "$\mathbb{R}$ has a Δ^1_{n+2} wellorder.

(2) (a) $\mathbb{R} \cap M_\omega = \{x \in \mathbb{R} \mid x \text{ is } OD^{L(\mathbb{R})}\}$, and

(b) $M_\omega \models$ "$\mathbb{R}$ has an $OD^{L(\mathbb{R})}$ wellorder".

The proof involves showing that the mice in question have ω_1-iteration strategies that are homogeneously Suslin. This is a very general phenomenon: $\omega_1 + 1$ strategies come from extending absolutely definable ω_1-strategies.

A version of part (1), on the existence of the $M_n^\sharp$'s for $n < \omega$, can be proved assuming only PD, rather than the existence of infinitely many Woodin cardinals. This leads to a proof of

Theorem (Martin, S., Woodin, late 80's)

The following are equivalent

(1) PD,

(2) *For all n there is a* Σ^1_n*-correct inner model of* ZFC + *"there are n Woodin cardinals",*

That $L(\mathbb{R}) \models$ AD is also equivalent to a mouse-existence assertion, but the following is neater:

Theorem

(Woodin, late 1980s) The following theories are equiconsistent

(a) ZF + AD,

(b) ZFC+ *"There are infinitely many Woodin cardinals".*

Iteration trees yield homogeneity measures

Lemma

(Windszus.) Equivalent are

(1) $A \in \mathrm{Hom}_\kappa$,

(2) *There are* M_s *and* $i_{s,t}$ *for* $s, t \in \omega^{<\omega}$ *such that*

  (a) $M_\emptyset = V$, *and each* M_s *is transitive and closed under* κ*-sequences,*

  (b) *for* $s \subseteq t$, $i_{s,t} \colon M_s \to M_t$ *is elementary and* $\mathrm{crit}(i_{s,t} \geq \kappa$,

  (c) $s \subseteq t \subseteq u$ *implies* $i_{s,u} = i_{t,u} \circ i_{s,t}$, *and*

  (d) $x \in A$ *iff* $\lim_n M_{x\restriction n}$ *is wellfounded.*

So homogeneity is equivalent to being continuously reducible to wellfoundedness of direct limit systems on V.

Theorem

(Martin, S. 1985) Let $\mathcal{T}$ be a nice iteration tree on V of length ω; then $\mathcal{T}$ has at least one cofinal welfounded branch.

Theorem

(Martin, S. 1985) Let $\mathcal{T}$ be a nice iteration tree on V of length ω; then $\mathcal{T}$ has at least one cofinal welfounded branch.

Definition

An *alternating chain* is an iteration tree of length ω with exactly two branches, $b = \{2n \mid n < \omega\}$ and $c = \{0\} \cup \{2n+1 \mid n < \omega\}$.

So if $\mathcal{T}$ is an alternating chain on V, then at least one of $\mathcal{M}_b^{\mathcal{T}}$ and $\mathcal{M}_c^{\mathcal{T}}$ is wellfounded. (It is open whether both can be wellfounded.)

Now suppose δ is Woodin, A is δ^+-homogeneous, and $\kappa < \delta$. One can construct a map $s \mapsto \mathcal{T}_s$, for $s \in \omega^{<\omega}$, such that for all $x \in {}^\omega\omega$

(1) $\mathcal{T}_x = \bigcup_n \mathcal{T}_{x \restriction n}$ is an alternating chain on V, with all critical points $\geq \kappa$,

(2) $x \in A \Rightarrow M_b^{\mathcal{T}_x}$ is illfounded, and

(3) $x \notin A \to M_c^{\mathcal{T}_x}$ is illfounded.

By (3), $x \notin A \Rightarrow M_b^{\mathcal{T}_x}$ is wellfounded. So

$$x \notin A \text{ iff } M_b^{\mathcal{T}_x} \text{ is wellfounded},$$

and $\neg A \in \mathrm{Hom}_\kappa$ by Windszus' theorem.

Higher up

The main open problems have to do with the existence of iteration strategies.

Conjecture. Suppose there is a superstrong cardinal with a measurable above it; then there is a countable $\omega_1 + 1$-iterable premouse M such that $M \models \text{ZFC}+$ "There is a superstrong cardinal".

Itay Neeman has proved the conjecture with "there is a superstrong cardinal" weakened to "there is a Woodin limit of Woodin cardinals".

Higher up

The main open problems have to do with the existence of iteration strategies.

Conjecture. Suppose there is a superstrong cardinal with a measurable above it; then there is a countable $\omega_1 + 1$-iterable premouse M such that $M \models \text{ZFC}+$ "There is a superstrong cardinal".

Itay Neeman has proved the conjecture with "there is a superstrong cardinal" weakened to "there is a Woodin limit of Woodin cardinals".

The conjecture follows from

Conjecture.(Unique Branches Hypothesis) Let $\mathcal{T}$ be a nice iteration tree on V of limit length; then $\mathcal{T}$ has at most one cofinal, wellfounded branch.

Conjecture. Suppose there is a singular strong limit cardinal κ such that $\Box_\kappa$ fails; then there is a countable $\omega_1 + 1$-iterable premouse *M* such that $M \models$ ZFC$+$ “There is a superstrong cardinal”.

Conjecture. Suppose there is a singular strong limit cardinal κ such that $\square_\kappa$ fails; then there is a countable $\omega_1 + 1$-iterable premouse M such that $M \models$ ZFC$+$ "There is a superstrong cardinal".

Here the hypothesis is consistent with there being no measurable cardinals. (It is known to be consistent relative to many superstrongs, a result of Martin Zeman.)

Conjecture. Suppose there is a singular strong limit cardinal κ such that $\Box_\kappa$ fails; then there is a countable $\omega_1 + 1$-iterable premouse *M* such that *M* $\models$ ZFC$+$ "There is a superstrong cardinal".
Here the hypothesis is consistent with there being no measurable cardinals. (It is known to be consistent relative to many superstrongs, a result of Martin Zeman.)

Our most all-purpose method for constructing inner models with large cardinals under such hypotheses is known as the *core model induction* method. The current best result on the conjecture is due to Grigor Sargsyan and Nam Trang (ca. 2018). It is somwhat below a Woodin limit of Woodins.

PFA, MM, strong compact $\longleftarrow$ supercompact

$\neg\square_\kappa$ $\longleftarrow$ many superstrongs

?

divergent models of $AD^+ + \Theta_0 < \Theta$ $\longleftrightarrow$ Woodin limit of Woodins

ω_1-dense ideal + CH + (*) $\dashleftarrow\dashrightarrow$?
$AD_{\mathbb{R}} + \Theta$ regular

$AD^+ + \Theta_{\omega_1} < \Theta$ $\longleftrightarrow$ $AD^+ + \Theta_{\omega_1} < \Theta$ hypo
$AD_{\mathbb{R}}$ + DC $\longleftrightarrow$ $AD_{\mathbb{R}}$ + DC-hypo
$AD_{\mathbb{R}}$ $\longleftrightarrow$ $AD_{\mathbb{R}}$-hypo

ω_1-dense ideal
AD $\longleftrightarrow$ ω Woodins

Thank you!