More mice

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Plan:

I. General background.

II. Constructing inner models via a core model induction.

III. Recent progress.

The Interpretability Hierarchy

For T a theory, let Arithmetic_T be the set of consequences of T in the language of first order arithmetic.

Phenomenon: If T is a natural extension of ZFC, then there is an extension S axiomatized by large cardinal hypotheses such that

$$\text{Arithmetic}_T = \text{Arithmetic}_S.$$

Moreover, if T and U are natural extensions of ZFC, then

$$\text{Arithmetic}_T \subseteq \text{Arithmetic}_U,$$

or

$$\text{Arithmetic}_U \subseteq \text{Arithmetic}_T.$$

In practice, $\text{Arithmetic}_T \subseteq \text{Arithmetic}_U$ iff PA proves $\text{Con}(U) \Rightarrow \text{Con}(T)$.

Core model program: Develop a general method for constructing inner models with large cardinals, and thereby obtaining optimal consistency-strength lower bounds.

Definition 1 A *premouse* is a structure of the form $\mathcal{M} = (J_\gamma^{\vec{E}}, \in, \vec{E})$, where $\vec{E}$ is a *coherent sequence of extenders*.

Coherence: for all $\alpha \leq \gamma$, $E_\alpha = \emptyset$, or E_α is an extender (system of ultrafilters) with support α over $\mathcal{M}|\alpha = (J_\alpha^{\vec{E}\restriction\alpha}, \in, \vec{E} \restriction \alpha)$ coding

$$i\colon \mathcal{M}|\alpha \to \mathcal{N} = \mathrm{Ult}(\mathcal{M}|\alpha, E_\alpha)$$

such that

$$i(\vec{E} \restriction \alpha) \restriction \alpha = \vec{E} \restriction \alpha \text{ and } i(\vec{E} \restriction \alpha)_\alpha = \emptyset.$$

Remark. The extenders in a coherent sequence appear in order of their strength, without leaving gaps.

Proper class premice are sometimes called extender models.

A *mouse* is an iterable premouse.

The iteration game

Let $\mathcal{M}$ be a premouse. In $\mathcal{G}(\mathcal{M}, \theta)$, players I and II play for θ rounds, producing a tree $\mathcal{T}$ of models, with embeddings along its branches, and $\mathcal{M} = \mathcal{M}_0^{\mathcal{T}}$ at the base.

Round $\beta + 1$*:* I picks an extender E_β from the sequence of $\mathcal{M}_\beta$ with support $\geq$ the supports of all earlier extenders chosen . Let ξ be least such that $\mathrm{crit}(E_\beta) <$ support (E_ξ). We set

$$\mathcal{M}_{\beta+1} = \mathrm{Ult}_k(\mathcal{M}_\xi | \eta, E_\beta),$$

where $\langle \eta, k \rangle$ is as large as possible.

Round λ*, for* λ *limit:* II picks a branch b of $\mathcal{T}$ which is cofinal in λ, and we set

$$\mathcal{M}_\lambda = \text{ dirlim }_{\alpha \in b} \mathcal{M}_\alpha.$$

As soon as an illfounded model $\mathcal{M}_\alpha$ arises, player I wins. If this has not happened after θ rounds, then II wins.

Definition 2 A *θ-iteration strategy for $\mathcal{M}$* is a winning strategy for II in $\mathcal{G}(\mathcal{M}, \theta)$. We say $\mathcal{M}$ is *θ-iterable* just in case there is such a strategy. If Σ is a strategy for II in $\mathcal{G}(\mathcal{M}, \theta)$, and $\mathcal{P} = \mathcal{M}^{\mathcal{T}}_{\alpha}$ for some $\mathcal{T}$ played by Σ, then we call $\mathcal{P}$ a *Σ-iterate* of $\mathcal{M}$.

Theorem 3 (Comparison Lemma) *Let Σ and Γ be $\omega_1 + 1$ iteration strategies for countable premice $\mathcal{M}$ and $\mathcal{N}$ respectively. Then either*

(a) there is a Γ-iterate $\mathcal{P}$ of $\mathcal{N}$, and a map $j\colon \mathcal{M} \to \mathcal{P}|\eta$ produced by Σ-iteration, or

(b) there is a Σ-iterate $\mathcal{P}$ of $\mathcal{M}$, and a map $j\colon \mathcal{N} \to \mathcal{P}|\eta$ produced by Γ-iteration $\mathcal{M}$.

Corollary 4 *If $\mathcal{M}$ is an $\omega_1 + 1$-iterable premouse, and $x \in \mathbb{R} \cap \mathcal{M}$, then x is ordinal definable.*

Constructing $\omega_1 + 1$-iterable countable mice is the central problem of core model theory. The way to do it is to construct an absolutely definable ω_1-strategy.

Woodin cardinals in $\mathcal{M}$ make its ω_1-strategy harder to define, at the same time making $\mathcal{M}$ more correct, if it is $\omega_1 + 1$-iterable.

One more Woodin cardinal in $\mathcal{M}$ adds one real quantifier to the complexity of its ω_1-strategy, and one real quantifier to its degree of correctness.

Definition 5 Let $\kappa < \delta$, and $A \subseteq \delta$; then

- κ is A-reflecting in δ iff for all $\eta < \delta$ there is $j\colon V \to M$ such that $\mathrm{crit}(j) = \kappa$, and $j(A) \cap \eta = A \cap \eta$.
- δ is Woodin with respect to A iff some $\kappa < \delta$ is A-reflecting in δ.
- δ is Woodin iff $\forall A \subseteq \delta(\delta$ is Woodin with respect to A.)

Woodin cardinals mark a critical transition point for core model theory. Work of Dodd, Jensen, Mitchell, Schimmerling, Steel, Zeman, and others led to

Theorem 6 (Jensen, Steel 2007) *Suppose there is no proper class extender model with a Woodin cardinal. Then there is a definable extender model K such that*

(1) (Weak covering) $(\alpha^+)^K = \alpha^+$, for any singular cardinal α.

(2) (Generic absoluteness) $K^{V[G]} = K$, for any G set-generic over V.

(3) (Fine structural hierarchy with condensation) K satisfies Jensen's $\diamond$, $\Box$ at all κ, etc.

If some T is of consistency strength at least one Woodin cardinal, it can usually be shown using this theorem.

One goes beyond one Woodin cardinal by relativising the theorem above, to extenders models built over a set, and closed explicitly under some function with condensation.

Definition 7 A *model operator with parameter* a is a function F defined on transitive J-structures $\mathcal{P}$ such that $a \in P$ such that

(1) $F(\mathcal{P}) = (Q, \in, \mathcal{P}, ...)$, where $Q = \text{Hull}_1^{F(\mathcal{P})}(P \cup \{\mathcal{P}\})$,

(2) (Condensation) If $\pi\colon \mathcal{R} \to F(\mathcal{P})$ is Σ_1, and $a \cup \{a\} \subset \text{ran}(\pi)$, then $\mathcal{R} = F(\pi^{-1}(\mathcal{P}))$,

(3) $F(\mathcal{P})$ is "amenable over $\mathcal{P}$", and

(4) (Generic interpretability) F determines a unique extension of itself to any $V[G]$.

An example of such an F is just $F(\mathcal{P}) = J_1(\mathcal{P}) =$ rudimentary closure of $P \cup \{\mathcal{P}\}$. An F-extender model is just like an ordinary one, except the next level after $\mathcal{P}$ is $F(\mathcal{P})$. We get

Theorem 8 *Let F be a model operator with parameter a, and let $a \in X$, where X is transitive. Suppose there is no proper class F-extender model over X with a Woodin cardinal. Then there is a definable F-extender model $K^F(X)$ such that*

(1) (Weak covering) $(\alpha^+)^{K^F(X)} = \alpha^+$, for any singular cardinal $\alpha > |X|$.

(2) (Generic absoluteness) $(K^F(X))^{V[G]} = K^F(X)$, for any G set-generic over V.

(3) (Fine structural hierarchy with condensation) K satisfies Jensen's $\diamond$, $\square$ at all $\kappa > |X|$, etc.

Definition 9 An *ideal* on ω_1 is a family $I \subsetneq P(\omega_1)$ such that

(1) $(A \subseteq B \in I) \Rightarrow A \in I$,

(2) I is closed under countable unions.

E.g., I could be the family of nonstationary sets.

Definition 10 An ideal I on ω_1 is ω_2-*saturated* if there is no family of ω_2 many I-positive sets such that the intersection of any pair of them is in I.

Definition 11 An ideal I on ω_1 is ω_1-*dense* iff there are I-positive sets $\{B_\alpha \mid \alpha < \omega_1\}$ such that every I-positive set almost contains some B_α.

Clearly, ω_1-dense implies ω_2-saturated.

Theorem 12 (Jensen, Shelah, S., Woodin) *The following are equiconsistent*

(1) ZFC + *"there is a Woodin cardinal",*

(2) ZFC + *"there is an ω_2-saturated ideal on ω_1,*

(3) ZFC + Δ^1_2*-determinacy.*

Our theorem on K is used for $\mathrm{Con}(2) \to \mathrm{Con}(1)$. It can also be used for $\mathrm{Con}(3) \to \mathrm{Con}(1)$, though Woodin's first proof did not go that way.

Theorem 13 (Woodin 1988-91) *The following are equiconsistent*

(1) ZFC + *"there are infinitely many Woodin cardinals",*

(2) ZFC + *"there is an ω_1-dense ideal on ω_1,*

(3) ZF + AD.

The theorem on K^F plays a central role in $\mathrm{Con}(2) \to \mathrm{Con}(1)$. It can also be used for $\mathrm{Con}(3) \to \mathrm{Con}(1)$, though Woodin's first proof did not go that way.

Definition 14 $\square_\kappa$ asserts that there is a sequence C_α, for $\alpha < \kappa^+$ a limit ordinal, such that for all α

(1) C_α is a club in α, with order type $\leq \kappa$, and

(2) if β is a limit point of C_α, then $C_\beta = C_\alpha \cap \beta$.

Schimmerling and Zeman showed (iterable) extender models satisfy $\forall\kappa\square_\kappa$. So if K computes κ^+ correctly, then $\square_\kappa$ holds in V. Thus our first theorem on K gives one Woodin cardinal from the failure of $\square_\kappa$ at some singular cardinal κ.

Our theorem on K^F gives

Theorem 15 (S., 2004) *Suppose $\neg\square_\kappa$ for some singular strong limit κ; then* AD *holds in $L(\mathbb{R})$, and in fact for any n, there is an extender model with $\omega \cdot n$ Woodin cardinals.*

The core model induction method

In using the theorem on K^F to construct mice with more than one Woodin cardinal, two issues arise:

(1) Where do the F's come from?

(2) How do we translate F-mice with one Woodin into ordinary mice with many Woodins?

Vaguely speaking, the answer to (1) is: The F's are constructed by an induction. At reasonably closed stages, the earlier model operators on H_ν give us a model M^ν of AD containing the reals. The next F comes from analyzing HOD^{M^ν} as a *hod mouse*.

In the $\neg\Box_\kappa$ application, $\nu = \kappa^+$. In getting strength from am ω_1-dense ideal + CH, we use $\nu = \omega_2$.

The maximal model M^ν

For $A \subseteq \mathbb{R}$, let

$$A \in C^\nu \quad \Leftrightarrow \quad \exists F(F \text{ is a model operator on } H_\nu \text{ with parameter in } \mathbb{R}, \text{ and } A \text{ is definable over } (H_{\omega_1}, \in, F \restriction H_{\omega_1}).$$

With ν adjusted to suit our hypothesis, we have that C^ν is a projectively closed model of AD + "All sets are Suslin", using our K^F theorem. $C^\nu$is the class of Suslin, co-Suslin sets of our M^ν.

Definition 16 Let $A \subseteq \mathbb{R}$. Let M be a countable transitive model of ZFC, and Σ an iteration strategy for M. Let δ be Woodin in M, and τ be a $\mathrm{Col}(\omega, \delta)$-term in M. Then (M, Σ, τ) *captures* A iff whenever $i \colon M \to N$ is a Σ-iteration map, and g is $\mathrm{Col}(\omega, i(\delta))$ generic over N, then

$$i(\tau)_g = A \cap N[g].$$

Remark. It follows by Woodin's genericity iterations that

$$A = \bigcup \{ i(\tau)_g \mid i \colon M \to N \text{ is by } \Sigma \}.$$

Remark. Suppose (M, Σ, τ) captures A at δ. Let $\mu < \delta$ be Woodin in M; then there is a term ρ such that (M, Σ, ρ) captures $\exists^{\mathbb{R}} A$ at μ.

Using the theorem on K^F and one of our strong hypotheses (strong enough to refute the existence of K^F, with the appropriate H_ν), we have that the following are equivalent:

(1) $A \in C^\nu$,

(2) A is captured by some (M, Σ, τ), where Σ is a ν-iteration strategy that condenses to itself and determines itself on generic extensions,

(3) A is definable over $(H_{\omega_1}, \in, \Sigma \upharpoonright H_{\omega_1})$, for some Σ as in (2).

Each of the above implies A is $< \nu$ universally Baire, and if ν is a limit of Woodin cardinals, they are equivalent to being $< \nu$ universally Baire.

An iteration strategy beyond M^ν

C^ν comprises the Suslin-co-Suslin sets of our maximal model M^ν. We need to do more to get the full M^ν. Once this is done, we try to show M^ν is not maximal after all, by using it to construct model operator F not in C^ν.

This next F comes from an iteration strategy Σ for some countable *hod mouse* H such that $(\text{HOD}|\theta)^{M^\nu}$ is the direct limit of all countable Σ-iterates of H.

The mouse set conjecture

The passage from F-mice to ordinary mice involves proving instances of:

Mouse Set Conjecture: Assume AD^+, and that there is no countable mouse with a superstrong. Then for x, y reals, the following are equivalent:

(1) x is ordinal definable from y,

(2) x belongs to some mouse over y.

In this approach, the analysis of HOD^{M^ν} needed to get the next model operator beyond M^ν, and the proof of the instance of MSC needed to translate F-mice into ordinary ones, are interwoven.

Here are some applications of earlier work in this direction.

Definition 17 (1) $\mathsf{AD}_{\mathbb{R}}$ is the axiom of determinacy for games of length ω in which the individual moves are real numbers.

(2) $\Theta = \sup\{\alpha \mid \exists f(f\colon \mathbb{R} \to \alpha \text{ and } f \text{ is surjective})\}$.

Definition 18 A premouse $\mathcal{M}$ is *tame* iff whenever $\mathrm{crit}(E_\alpha^{\mathcal{M}}) \leq \delta < \alpha$, then $\mathcal{M}|\alpha \models \delta$ is not Woodin.

Definition 19 The $\mathsf{AD}_{\mathbb{R}}$-*hypothesis* is the assertion that there is a λ such that λ is a limit of Woodin cardinals, and of κ which are V_λ-reflecting in λ.

Definition 20 The $\mathsf{AD}_{\mathbb{R}}$ + DC - *hypothesis* is the assertion that there is a λ such that λ is a limit of Woodin cardinals, and the set of κ which are V_λ-reflecting in λ has order type λ.

Making heavy use of work by Woodin which analyzed HOD^M, and proved the Mouse Set Conjecture, in the case M is the minimal model of $\mathsf{AD}_{\mathbb{R}}$ containing all reals and ordinals, one gets:

Theorem 21 (Ketchersid 2000) *Suppose there is an ω_1-dense ideal on ω_1, CH holds, and $(*)$. Then there is a non-tame mouse.*

Theorem 22 (Woodin late 90's, S. 2007) *(1)* $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ *is equiconsistent with* ZFC + *the* $\mathsf{AD}_{\mathbb{R}}$*-hypothesis.*

(2) ZF + $\mathsf{AD}_{\mathbb{R}}$ + DC *is equiconsistent with* ZFC + *the* $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$*- hypothesis.*

Grigor Sargsyan has recently made significant progress on the twin problems of analyzing HOD in models of AD^+, and proving the Mouse Set Conjecture.

Theorem 23 (Sargsyan 2008) *The Mouse Set Conjecture holds in the minimal model of* $\mathsf{AD}_{\mathbb{R}}$ + *"*Θ *is regular" containing all reals and ordinals.*

Theorem 24 (Sargsyan 2008) *Assume* AD^+ *and there is no model of* $\mathsf{AD}_{\mathbb{R}}$ + *"*Θ *is regular" containing all reals and ordinals. The HOD is a hod-mouse.*

Applications include

Theorem 25 (Woodin 90's, Sargsyan 2008) *The following are equiconsistent*

(1) ZFC + *"there is an ω_1-dense ideal on ω_1* + CH + $(*)$,

(2) $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ + *"Θ is regular"*.

Theorem 26 (Sargsyan 2008) *Con*(ZFC$+$ *"there is a Woodin limit of Woodin cardinals") implies Con*($\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}+$ *"Θ is regular")*.

Theorem 27 (Sargsyan, S. 2008) *The following are equiconsistent:*

(1) $\mathsf{ZF} + \mathsf{AD}^{+} + \Theta_{\omega_1} < \Theta$,

(2) ZFC$+$ *"there is λ which is a limit of Woodin cardinals, and $\kappa < \lambda$ which is V_λ-reflecting in λ of rank λ"*.