Letter 7

Construction of a model of **NFU**$*$from an inaccessible

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§1 Introduction

We are currently working in the theory **ZFC** + **V**=**L** + "There is an inaccessible cardinal". We let θ be an inaccessible cardinal. Our goal is to prove the consistency of **NFU**$*$

Our goal in this letter is to carry out the construction of a term model which meets the sufficient conditions of letter 6 and so yields a model of **NFU**$*$.

The construction will proceed in ω stages. At stage i we will define a lbfp $\alpha_i < \theta$ together with an equivalence relation $\equiv_i$ on the closed terms of rank at most (α_i, i). We will have

$$\alpha_0 < \alpha_1 < \alpha_2 \dots$$

We will let α be the supremum of the α_i's.

1. 1

Since we will not know the value of α until the end of our construction the following trivial point is important to us. Let $\alpha < \theta$ be a lbfp. So as discussed in letter 4, α determines a term language $\mathcal{L}$.

Lemma 1.1 Let $\gamma < \alpha$. Let $n \in \omega$. The collection of closed terms of rank at most (γ, n) depends only on γ and n and not on α.

Proof: In fact this collection is just the set of closed terms of the following term language: (Cf. letter 5 section 2.1.)

1. It has a constant ξ_i for each i with $|i| \leq n$.
2. It has an m-ary function symbol $f_{m,i}$ for each $m, i \in \omega$ with $m > 0$.
3. It has a unary function symbol h_i for each $i < n$.
4. It has a constant symbol $\bar{\beta}$ for each $\beta < \gamma$.

□Lemma

1. 2

In order to carry the induction along, we shall need to assume that the equivalence relation $\equiv_n$ is *very well instantiated* in the sense of the following definition:

Definition 1.2 Let $n \in \omega$. Let $\alpha_n < \theta$ be a lbfp. Let $\equiv_n$ be an equivalence relation on the closed terms of rank at most (α_n, n). Then $\equiv_n$ is *very well instantiated* if for every cardinal $\eta < \theta$ there is a (α_n, n) pre-instantiation model $\langle M, Y \rangle$ which well-instantiates $\equiv_n$ and is such that the cardinality of Y is at least η.

1. 3

We let α_0 be the least lbfp. To start things off we need the following lemma;

Lemma 1.3 There is an equivalence relation $\equiv_0$ on the closed terms of rank at most $(\alpha_0, 0)$ which is very well instantiated.

Proof: An easy application of the Erdös-Rado theorem yields the following claim:

For every cardinal $\eta < \theta$, there is an equivalence relation $\equiv_\eta$ [on the closed terms of rank at most $(\alpha_0, 0)$] which is well-instantiated by an $(\alpha_0, 0)$ pre-instantiation model $\langle M_\eta, Y_\eta \rangle$ with Y_η of cardinality at least η.

Now the number of possible choices of $\equiv_\eta$ clearly is less than θ. Hence there is a fixed equivalence relation $\equiv_0$ which is equal to $\equiv_\eta$ for cofinally many $\eta < \theta$. $\square_{\text{Lemma}}$

§2 Continuing the construction

We consider the following situation. We are given the following data:

1. A number $n \in \omega$.
2. An ordinal $\alpha_n < \theta$ which is a lbfp.
3. An equivalence relation $\equiv_n$ on the closed terms of rank at most (α_n, n) which is very well instantiated.

We shall, in this situation, define the following:

1. An lbfp α_{n+1} with $\alpha_n < \alpha_{n+1} < \theta$.
2. An equivalence relation $\equiv_{n+1}$ on the closed terms of rank at most $(\alpha_{n+1}, n+1)$ which is very well instantiated and which, in an evident sense, prolongs $\equiv_n$.

Of course, once we do this, we will have inductively defined the sequences

$$\langle \alpha_n \mid n \in \omega \rangle \quad \text{and} \quad \langle \equiv_n \mid n \in \omega \rangle \ .$$

2. 1

We say that an *instantiation function* is a function F with domain θ such that:

1. For each $\eta < \theta$, $F(\eta)$ will be an ordered pair $\langle M, Y \rangle$.
2. M will be some flavor of model of set-theory. In particular, the "underlying set" of M will be an L_λ where λ is a lbfp less than θ.
3. Y will be a subset of the underlying set of M consisting of lbfps. The order type of Y will be at least η.

2. 2 Stage 1

We start with an instantiation function F_0 that witness that $\equiv_n$ is very well-instantiated.

If $n = 0$, we will bypass stage 1.

Suppose that τ is a closed term of rank at most $(\alpha_{n-1}, n-1)$. Then $j(\tau)$ has rank at most (α_n, n). We say that τ is Cantorian if $\equiv_n$ decrees that $\tau = j(\tau)$.

Let W_n be the set of closed terms τ of rank at most $(\alpha_{n-1}, n-1)$ such that:

1. τ is Cantorian.
2. $\equiv_n$ "decrees" that τ is an ordinal.

We let $W_n^\star$ be the set of equivalence clases of W_n under the equivalence relation on W_n induced from $\equiv_n$.

We put a well-ordering on $W_n^\star$ as follows. Let τ_1 and τ_2 be elements of W_n. Then $[\tau_1] < [\tau_2]$ provided that $\equiv_n$ "decrees" that $\tau_1 < \tau_2$.

2. 3

Let F and G be instantiation functions. We say that G is a *refinement* of F if the following obtains:

For every $\eta < \theta$, there is an $\eta' < \theta$ such that (letting $\langle M, Y \rangle = G(\eta)$ and $\langle M', Y' \rangle = F(\eta')$):

1. The underlying sets of M and M' coincide.
2. $Y \subseteq Y'$.
3. $H_i^M = H_i^{M'}$.

2. 4

The following lemma will be left "to the reader".

Lemma 2.1 There is a refinement F_1 of F_0 with the following property:

Let $\tau \in W_n$. Let $\eta < \theta$. Then there is an ordinal $\gamma < \theta$ such that if τ has the value γ in any instantiation derived from $F_1(\eta)$.

From now on we work with F_1 [and further refinements of it].

We shall refer to the ordinal γ provided by the lemma as the value of τ in $F_1(\eta)$

2. 5

Let $\tau \in W_n$. τ is *divergent* if the following obtains:

For every $\eta < \theta$, there is an $\eta' < \theta$ such that the value of τ in $F_1(\eta') > \eta$.

2. 6

We first consider the trivial case that that there are no divergent τ's in W_n. In that case, we define an instantiation function F_2 as follows.

Let $\eta < \theta$. Suppose that $F_1(\eta) = \langle M, Y, \rangle$. Then we set $F_2(\eta) = \langle M', Y' \rangle$ where:

1. $Y' = Y$.
2. The structure M is a reduct of the structure M'. The only additional structure for M' is that a meaning is given to the function symbol h_n. It denotes the identially 0 function with domain the underlying set of M.

2. 7

Let us now take up the more interesting case where there are divergent τ's.

In the first place, it is clear that whether or not τ is divergent depends only on its equivalence class with respect to $\equiv_n$.

Second, if τ_1 is divergent, and $[\tau_1] \leq [\tau_2]$ [with repesect to the well-ordering introduced in section 2.2], then τ_2 is divergent. So we can fix $\tau_0 \in W_n$ which has as small as possible equivalence class among divergent τ 's. The divergent τ's are precisely those whose equivalence classes are $\geq [\tau_0]$.

We can find a refinement $F_{1.5}$ of F_1 with the following property:

Let $\eta < \theta$. Then the value of τ_0 in $F_{1.5}(\eta)$ is greater than the order type of the "Y" of $F_{1.5}(\eta)$.

We now define F_2, a refinement of $F_{1.5}$, as follows:

Let $\eta < \theta$. Let $F_{1.5}(\eta) = \langle M, Y \rangle$. Then $F_2(\eta) = \langle M', Y' \rangle$ where:

1. $Y' = Y$.
2. The structure M is a reduct of the structure M'. The only additional structure for M' is that a meaning is given to the function symbol h_n. It denotes the function with domain the underlying set of M which is 0 except on Y' and which gives the order isomorphism of Y' with its order-type.

2. 8

It is clear that there is an ordinal $\theta_0 < \theta$ such that if $\tau \in W_n$ is not divergent, then its value is less than θ_0. To be definite, we take θ_0 as small as possible with this property.

We are now in a position to define α_{n+1}. It is the least lbfp which is greater than both α_n and θ_0.

2. 9

We now employ the Erdös-Rado theorem to get the following:

There is a refinement F_3 of F_2 such that for any $\eta < \theta$, there is an equivalence relation $\equiv_\eta$ on the terms of rank at most $(\alpha_{n+1}, n+1)$ such that if we set

$$F_3(\eta) = \langle M, Y \rangle$$

then for any increasing $2n+3$-tuple, $\beta_0, \ldots, \beta_{2n+3}$ chosen from Y instantiates $\equiv_\eta$.

There are less than θ possibilities for $\equiv_\eta$. Hence there must be some fixed $\equiv'$ which occurs as $\equiv_\eta$ for θ different η's. We pick the least such as our $\equiv_{n+1}$. This completes our description of the sucessor step of the construction.

2. 10

That the term model constructed by the length ω construction just described meets the requirements of letter 6 will be left as an "exercise for the reader".

I am hoping that I have supplied sufficient detail for you to see your way through the proof of phase A of the converse direction. Of course, the hypothesis used in this proof can be considerably weakened. I shall take that issue up in phase B.

This ends letter 7.