Letter 6

Sufficient conditions for a model of **NFU**$*$

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§1 Introduction

We are currently working in the theory **ZFC** + **V**=**L** + "There is an inaccessible cardinal". We let θ be an inaccessible cardinal. Our goal is to prove the consistency of **NFU**$*$

Let $\alpha < \theta$ be a lbfp. In letter 4, we have associated to α a term language $\mathcal{L}$.

This letter has the following goals:

1. To show that if $\mathcal{M}$ is a well-instantiated term model then $\mathcal{M}$ yields a model of **NFU** via "the usual construction".
2. To give sufficient conditions on $\mathcal{M}$ that this model of **NFU** will be a model of **NFU**$*$.

Subsequent letters [perhaps only one] will show that we can construct a term-model meeting these requirements

§2 The automorphism j

We define a map j on the closed terms of $\mathcal{L}$ as follows:

1. $j(\xi_i) = \xi_{i+1}$.
2. If $\gamma < \alpha$ then $j(\bar{\gamma}) = \bar{\gamma}$.
3. Suppose that f is an n-ary function symbol of $\mathcal{L}$ and that the closed term τ has the form $f(\tau_1, \ldots, \tau_n)$. Then $j(\tau) = f(j(\tau_1), \ldots, j(\tau_n))$.

That j is a bijection is clear. [It is easy to write down an analogous definition of j^{-1}.]

2. 1

Lemma 2.1 Let τ_1 and τ_2 be closed terms of $\mathcal{L}$. Then the following are equivalent:

1. $\mathcal{M} \models \tau_1 = \tau_2$.
2. $\mathcal{M} \models j(\tau_1) = j(\tau_2)$.

Proof: Choose $n < \omega$ and $\alpha_1 < \alpha$ large enough that $\tau_1, \tau_2, j(\tau_1)$ and $j(\tau_2)$ are all terms of rank at most (α_1, n). Let $\langle M^\star, Y\rangle$ be an (α_1, n) pre-instantiation model for $\mathcal{M}$ as provided by the definition of "well-instantiated". So in particular, if $\beta_0, \ldots, \beta_{2n+1}$ are the first $2n+2$ members of Y, then the models $M_0 = \langle M^\star, \beta_0, \ldots, \beta_{2n}\rangle$ and $M_1 = \langle, M^\star, \beta_1, \ldots, \beta_{2n+1}\rangle$ instantiate $\mathcal{M}$.

We have $j(\tau_i)^{M_0} = \tau_i^{M_1}$ (for $i = 0, 1$). Thus $\mathcal{M} \models j(\tau_0) = j(\tau_1) \leftrightarrow M_0 \models j(\tau_0) = j(\tau_1) \leftrightarrow M_1 \models \tau_0 = \tau_1 \leftrightarrow \mathcal{M} \models \tau_0 = \tau_1$. $\square_{\text{Lemma}}$

2. 2

Lemma 2.2 Let τ_1 and τ_2 be closed terms of $\mathcal{L}$. Then the following are equivalent:

1. $\mathcal{M} \models \tau_1 \in \tau_2$.
2. $\mathcal{M} \models j(\tau_1) \in j(\tau_2)$.

Proof: Entirely analogous to the proof of the preceding lemma. $\square_{\text{Lemma}}$

2. 3

We now know that j induces an automorphism of $\mathcal{M}$ viewed as a model of the set theory T. Evidently, $j([\xi_i]) = [\xi_{i+1}]$. Thus we can use j to construct a model of **NFU** in the usual way.

We say that an element of $\mathcal{M}$ is Cantorian if it's fixed by j.

Lemma 2.3 The following is a sufficient condition that the model, call it Q, of **NFU** derived from $\mathcal{M}$, j is in fact a model of **NFU**$*$:

If η is a Cantorian ordinal of $\mathcal{M}$, then either:

1. $\eta < \alpha$.
2. OR there is a non-Cantorian ordinal $\eta' < \eta$.

Proof: The conditions imply that the set of strongly Cantorian ordinals of Q can be identified with the ordinals less than α. It is clear that Q contains for each $\eta < \alpha$ all the constructible subsets of η. [This follows from the way that Q was constructed offline.] But since we are working in the theory $V = L$, this means that *all* the subsets of η are represented in the model Q. The crucial new axiom of **NFU**$*$ has been verified. $\square_{\text{Lemma}}$

This ends letter 6.