Letter 5

The term model as a model of set theory

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§1 Introduction

We are currently working in the theory **ZFC** + **V**=**L** + "There is an inaccessible cardinal". We let θ be an inaccessible cardinal. Our goal is to prove the consistency of **NFU**$*$

Let $\alpha < \theta$ be a lbfp. In the previous letter, we have associated to α a term language $\mathcal{L}$.

A term model [say $\mathcal{M}$] is given by putting an equivalence relation [say $\equiv$] on the closed terms of $\mathcal{L}$ which satisfies the analogues of the usual axioms of equality. [This is fully discussed in letter 4.] The goal of this letter is to describe a condition on $\mathcal{M}$ [being *well-instantiated*] which allows us to make $\mathcal{M}$ into a model of set-theory.

To be quite pedantic, we will construct a model of a certain version of set-theory, T [to be described precisely in a moment] whose underlying set is the collection of equivalence classes of the equivalence relation $\equiv$. But we shall, in the future, abuse notation, and refer to the model thus created as $\mathcal{M}$ also.

1. 1 The theory T

The theory T will have the following axioms:

1. The extensionality, foundation, infinity, union, power set, choice and pairing axioms.
2. The Σ_1 replacement axiom. This says that if $\phi(x, y)$ is a Σ_1 relation which is "functional" [that is, if $\phi(x, y_1)$ and $\phi(x, y_2)$ then $y_1 = y_2$] and A is a set then $\{y \mid (\exists x \in A)\phi(x, y)\}$ is a set. [Notice we do *not* require that $\forall x \in A \exists y \phi(x, y)$.]
3. **V**=**L**.
4. There are arbitrarily large lbfp's.

This is a fairly strong set-theory. [Though it is much weaker than **ZFC**.] Certainly, almost all of mathematics as it is done by the "mathematician in the street" can be carried out in this theory.

1. 2 An addition to letter 4

In letter 4 we made intuitive remarks about the purposes of the various function symbols of the language $\mathcal{L}$. We wish to make the following additional remark:

Remark 1.1 It is our intention that the functions h_i will have the following properties.

1. If x is not an ordinal, then $h_i(x) = 0$.
2. If β is an ordinal, $h_i(\beta) \leq \beta$.

The purpose of this condition will be to insure that the ξ_i are cofinal in the ordinals of the term model $\mathcal{M}$

§2 Well-instantiated term models

Let $\mathcal{M}$ be a term model of $\mathcal{L}$. We are going to introduce the notion of $\mathcal{M}$ being *well-instantiated*. This will serve to insure that $\mathcal{M}$ does indeed conform to the intuitive remarks of the preceding letter 4.

2. 1

We begin with a series of preliminary definitons. Let $\alpha_1 < \alpha$. Let $n \in \omega$. We say that a closed term τ of $\mathcal{L}$ has *rank at most* (α_1, n) iff:

1. If $\bar{\gamma}$ occurs in τ, then $\gamma < \alpha_1$.
2. If h_i occurs in τ, then $i < n$.
3. If ξ_i occurs in τ, then $|i| \leq n$.

2. 2

An (α_1, n) instantiation model M consists of the following:

1. The underlying set of M is L_λ for some lbfp λ such that $\alpha_1 < \lambda$.
2. There are lbfps $\beta_{-n} \ldots \beta_n$ such that:

$$\alpha_1 < \beta_{-n} < \ldots < \beta_n < \lambda.$$

3. For $i < n$ there is a map $H_i : L_\lambda \mapsto L_\lambda$ whch satisfies the requirements discussed in Remark 1.1.

2. 3

Suppose that τ is a closed term of $\mathcal{L}$ of rank at most (α_1, n). We show how to associate a value to τ, τ^M, lying in L_λ.

For this it suffices to give the interpretation of all the function symbols and constants appearing in τ. We do this as follows:

1. If $\gamma < \alpha_1$, then the interpretation of $\bar{\gamma}$ is γ.
2. The interpretation of ξ_i is β_i (for $-n \leq i \leq n$).
3. The interpretation of $f_{n,i}$ is as discussed in letter 4.
4. Let $i < n$. Then the interpretation of h_i is given by H_i.

2. 4

Let $\alpha_1 < \alpha$. Let $n \in \omega$. Let M be an (α_1, n) instantiation model. Then M *instantiates* $\mathcal{M}$ if the following holds:

Let τ_1 and τ_2 be closed terms of $\mathcal{L}$ of rank at most (α_1, n). Then $\tau_1^M = \tau_2^M$ *iff* $\tau_1 \equiv \tau_2$ in the term model $\mathcal{M}$.

2. 5

We now define what it means for $\langle M, Y \rangle$ to be an (α_1, n) *pre-instantiation model.*

1. M is a first-order structure for the language of set-theory. The underlying set of M is L_λ for some lbfp λ such that $\alpha_1 < \lambda$.
2. Y is a subset of λ consisting of lbfps. The order-type of Y is a limit ordinal.
3. For $i < n$ there is a map $H_i : L_\lambda \mapsto L_\lambda$ whch satisfies the requirements discussed in Remark 1.1.

2. 6

Finally, we can say what it means for $\mathcal{M}$ to be well-instantiated:

For every limit ordinal $\alpha_1 < \alpha$, for every $n \in \omega$, there is a (α_1, n) pre-instantiation model $\langle M_1, Y \rangle$ such that if

$$\beta_{-n} < \ldots < \beta_n$$

are an increasing $2n+1$-tuple of elements of Y, then $\langle M_1, \beta_{-n}, \ldots, \beta_n \rangle$ instantiates $\mathcal{M}$.

§3 Making $\mathcal{M}$ into a model of set-theory.

Some terminology: I shall refer to the $f_{n,i}$'s as *local functions*. As I have already remarked this conflicts with my terminology in my proof that NFUA yields n-Mahlos.

We are going to associate a model of the theory T to the term model $\mathcal{M}$. We have already said that the underlying set of this model will be the set of $\equiv$ equivalences classes.

3. 1

Next, we have to define the ϵ-relation of the model. Let g_1 be the binary function [one of the $f_{2,i}$'s] that does the following: If $x \in y$, then $g_1(x, y) = 1$. Otherwise, $g_1(x, y) = 0$.

Let τ be a closed term of $\mathcal{L}$. We let $[\tau]$ be the equivalence class of τ with respect to the relation $\equiv$ of $\mathcal{M}$. The various function symbols of $\mathcal{L}$ determine, in an obvious way, functions which act on the set of equivalence classes. We use the same notation for these derived functionns as for the corresponding symbols.

Let τ_1 and τ_2 be closed terms. We put $[t_1]\epsilon_{\mathcal{M}}[\tau_2]$ iff $g_1([\tau_1], [\tau_2]) = 1$. [By abuse of language, for $\gamma < \alpha$, we write γ for $[\bar{\gamma}]$.]

3. 2

It will take us a while to establish that T holds in the model just described. Before taking the first step along the way, we need to nail down precisely what we mean by Σ_0-formulas. These are the class of formulas inductively defined by the following requirements:

1. If v and w are variables then "$v \in w$" and "$v = w$" are Σ_0 formulas.
2. If ψ is a Σ_0 formula, then so is $\neg\psi$.
3. If ψ_1 and ψ_2 are Σ_0 formulas so is $\psi_1 \vee \psi_2$.
4. If v and w are distinct variables and ψ is a Σ_0 formula, then so is $(\exists v \in w)\psi$.

It is no real loss of generality to require in clause 4 that neither v or w appears bound in ψ. We shall assume this in what follows.

3. 3

Lemma 3.1 Let $\mathcal{M}$ be well-instantiated. Let $\psi(v_0, \ldots, v_m)$ be a Σ_0 formula. Let α_1 be a limit ordinal less than α and let $n_1 \in \omega$.

We suppose that $\tau_0, \ldots, \tau_r$ are closed terms of $\mathcal{L}$ of rank at most (α_1, n). Let M be an (α_1, n_1) instantiation model for $\mathcal{M}$.

Then the following are equivalent:

1. $\mathcal{M} \models \psi([\tau_0], \ldots, [\tau_r])$.
2. $M \models \psi({\tau_0}^M, \ldots, {\tau_r}^M)$.

Proof: The proof proceeds by induction on the length of ψ. We shall only consider the case when ψ has the form $(\exists v \in v_0)\chi(v, v_0, v_1, \ldots, v_r)$. The other cases [when ψ is atomic or a boolean combination of shorter formulas] will be left to the reader.

We fix $\alpha_1 < \alpha$ and $n_1 \in \omega$. Let $\tau_0, \ldots, \tau_r$ be closed subterms of $\mathcal{L}$ of rank at most (α_1, n_1). Let M be an (α_1, n_1) instantiation model for $\mathcal{M}$.

This case reduces to two subclaims.

Claim 3.1.A Suppose that $M \models \psi({\tau_0}^M, \ldots, {\tau_r}^M)$. Then $\mathcal{M} \models \psi([\tau_0], \ldots, [\tau_r])$.

Proof: Let $g_2(x_0, \ldots, x_r)$ be a local function with the following property:

1. If $(\exists x \in a_0)\chi(x, a_0, \ldots, a_r)$ then $g_2(a_0, \ldots, a_r)$ is the L-least such x.
2. Otherwise, $g_2(a_0, \ldots, a_r) = 0$.

Let $\tau^\star$ be the term $g_2(\tau_0, \ldots, \tau_r)$. Then clearly $\tau^\star$ has rank at most (α_1, n_1).

Moreover, by the hypothesis of this claim and the definition of g_2 we have

$$M \models [\chi(\tau^\star, \tau_0, \ldots, \tau_r) \wedge \tau^\star \in \tau_0].$$

Using our inductive hypothesis, we conclude that

$$\mathcal{M} \models [\chi([\tau^\star], [\tau_0], \ldots, [\tau_r]) \wedge [\tau^\star] \in [\tau_0]].$$

Hence $\mathcal{M} \models \psi([\tau_0], \ldots, [\tau_r])$. $\square_{\text{Claim}}$

Claim 3.1.B Suppose that $\mathcal{M} \models \psi([\tau_0], \ldots, [\tau_r])$. Then $M \models \psi({\tau_0}^M, \ldots, {\tau_r}^M)$.

Proof: Let $\tau^\star$ be a closed term such that

$$\mathcal{M} \models [\chi(\tau^\star, \tau_0, \ldots, \tau_r) \wedge \tau^\star \in \tau_0].$$

We choose $\alpha_2 < \alpha$ and $n_2 \in \omega$ such that $\tau^\star, \tau_0, \ldots, \tau_n$ are closed terms of rank at most (α_2, n_2). Let M_2 be an (α_2, n_2) instantiation model for $\mathcal{M}$.

Let $g_3(x_0, \ldots, x_r)$ be a local function with the following properties:

1. $g_3(x_0,\ldots,x_r) \in \{0,1\}$ for any choice of $x_0,\ldots,x_r$.
2. $g_3(x_0,\ldots,x_r) = 1$ *iff* $\psi(x_0,\ldots,x_r)$.

Let $\tau^{\#}$ be the closed term $g_3(\tau_0,\ldots,\tau_r)$. Then $\tau^{\#}$ has rank at most (α_1,n_1). By our inductive hypothesis, $M_2 \models [\chi(\tau^{\#M_2},\tau_0^{M_2},\ldots,\tau_r^{M_2}) \wedge \tau^{\#M_2} \in \tau_0^{M_2}]$. Hence $M_2 \models \psi(\tau_0^{M_2},\ldots,\tau_r^{M_2})$. It follows that in M_2, $g_3((\tau_0^{M_2},\ldots,\tau_r^{M_2}) = 1$.

Since M_2 is an (α_2,n_2) instantiation model for $\mathcal{M}$, we have $g_3(\tau_0^{M_2},\ldots,\tau_r^{M_2}) \equiv \bar{1}$ in $\mathcal{M}$.

Since M is an (α_1,n_1) instantiation model for $\mathcal{M}$ we have, in M,

$$g_3(\tau_0^{M},\ldots,\tau_r^{M}) = 1.$$

Hence, $M \models \psi(\tau_0^{M},\ldots,\tau_r^{M})$. $\square_{\text{Claim}}$

$\square_{\text{Lemma}}$

3. 4

The proofs of the following lemmas will be left to the reader. Their proofs employ both the statement of Lemma 3.1 and ideas used in its proof. [Using local functions as Skolem functions; checking assertions by using auxiliary models [like M_2 in the proof of Claim 3.1.B.]]

Lemma 3.2 *The axioms of foundation, extensionality, pairing, union, power set and infinity hold in $\mathcal{M}$.*

3. 5

It is well-known that there is a sentence σ such that:

1. If λ is an uncountable cardinal, then L_λ models σ.
2. If x is a transitive set such that σ holds in x, then x has the form L_λ for some limit ordinal λ.

Remark. I have stated a version of this result which suffices for our applications and which is easy to prove. In fact, one can arrange that the transitive models of σ are precisely the sets L_λ for some ordinal $\lambda > 0$. But this takes considerably more delicate arguments.

When I say **V=L** in the following lemma, I mean: every set x is a member of a transitive set y such that y models σ.

Lemma 3.3 **V=L** *holds in $\mathcal{M}$.*

Corollary 3.4 *The axiom of choice holds in $\mathcal{M}$.*

3. 6

In checking that an ordinal β is a lbfp, it suffices to see that a certain sentence [which we will not construct in detail] holds in the power set of L_β. Using this and Remark 1.1, the following is easy to prove:

Lemma 3.5 *The elements* $[\xi_i]$ *(for* $i \in \omega$*) are lbfp's in* $\mathcal{M}$ *and are cofinal in the ordinals of* $\mathcal{M}$.

3. 7

Using the previous lemmas and the fact that for β, β' bfp's with $\beta < \beta'$, Σ_1 formulas are absolute between L_β and $L_{\beta'}$, it is easy to prove the following lemma:

Lemma 3.6 Σ_1*-replacement holds in* $\mathcal{M}$.

We have proved:

Theorem 3.7 $\mathcal{M}$ *is a model of* T.

This ends letter 5.