Gleason's Theorem for non-separable Hilbert spaces: Extended abstract

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1 Introduction: Statement of Results

The probelm of generalizing Gleason's theorem to the non separable case arose in correspondence with Paul Chernoff. I am very grateful to him for suggesting this charming problem to me.

Let $\mathcal{H}$ be a Hilbert space. The coefficient field K of $\mathcal{H}$ can be either the reals or the complexes. We let $\mathcal{P}(\mathcal{H})$ denote the collection of all closed subspaces of $\mathcal{H}$. A *Gleason measure* on $\mathcal{H}$ is a map $\mu : \mathcal{P}(\mathcal{H}) \to [0, 1]$ satisfying the following conditions:

- $\mu(\{0\}) = 0$; $\mu(\mathcal{H}) = 1$.
- Let $\langle M_i \mid i \in \omega \rangle$ be a pairwise orthogonal sequence of subspaces of $\mathcal{H}$ with sum M. Then
$$\mu(M) = \sum_{i=0}^{\infty} \mu(M_i)$$

We follow the general convention that if M [possibly decorated with subscripts or superscripts] is a closed subspace of $\mathcal{H}$ then P [with the same decoration] is the corresponding projection.

One way to get such a Gleason measure is as follows. Let A be a positive self-adjoint trace class operator whose trace is 1. Set

$$\mu(M) = \operatorname{trace}(AP) \tag{1}$$

Then μ is easily checked to be a Gleason measure.

In [2] the following remarkable theorem is proved:

Theorem 1.1 (Gleason). *Let $\mathcal{H}$ be separable and of dimension unequal to 2. Then every Gleason measure arises from precisely one positive self-adjoint operator, A, of trace 1 in the manner just described.*

As Gleason remarks in [2], the restrictions to dimensions other than 2 is essential to the validity of the theorem. In this paper, we completely analyze the generalization of Gleason's theorem to non-separable Hilbert spaces. We find that the naive generalization is false, but that a more refined generalization is valid. Before stating our precise results we need some preliminary definitions.

Let X be a set. Then a *measure* on X is a map ν from the power set of X, $\mathcal{P}(X)$, to the interval $[0, 1]$ such that:

- $\nu(\emptyset) = 0$; $\nu(X) = 1$.
- if $\langle A_i \mid i \in \omega\rangle$ is a pairwise disjoint sequence of subsets of X with union A, then

$$\nu(A) = \sum_{i=0}^{\infty} \nu(A_i)$$

The reader should note the analogy with our definition of a Gleasom measure. Note also, that contrary to the usual definitions in analysis, we require that our measure be defined on every subset of X.

A measure ν is *continuous* if it vanishes on one point sets. It is a consequence of Gödel's theorem that one cannot prove from the usual axioms of set-theory that there are continuous measures.

Whether or not a set X carries a continuous measure clearly depends only on the cardinality of X. A cardinal κ is *real-valued measurable* if there is a continuous measure ν defined on all the subsets of κ which is κ-additive in the sense that the union of fewer than κ sets of ν-measure zero again has ν-measure zero. It is a standard and easy fact that the existence of a real-valued measurable cardinal is equivalent to the existence of a continuous measure. Moreover the least cardinal to support a continuous measure is a real-valued measurable cardinal.

Definition 1.2. *A Gleason measure is* exotic *if it does not arise from a positive trace class operator of trace 1 (in the manner described above).*

Theorem 1.3. *Let $\mathcal{H}$ be a non-separable Hilbert space of dimension λ. Then the following are equivalent:*

- *There is a real-valued measurable cardinal κ with $\kappa \leq \lambda$.*
- *There is an exotic Gleason measure on $\mathcal{H}$.*

Hence the following are equivalent:

- *There is a real-valued measurable cardinal.*
- *There is an exotic Gleason measure.*

Remark: This theorem is independently due to Dvurecenskij [1].

1.1 Our main theorem describes how to construct all the Gleason measures on a non-separable Hilbert space. (The analogous result for separable Hilbert spaces of dimension unequal to 2 follows from Theorem 1.1.) We also have a result which gives a "complete system of invariants" for a Gleason measure which we formulate in section 1.5.

Let $\mathcal{H}$ be a non-separable Hilbert space and let Y be an orthonormal basis of $\mathcal{H}$. Let μ be a measure on Y. To the data Y and μ we shall associate a Gleason measure ν on $\mathcal{H}$ as follows.

Let M be a closed subspace of $\mathcal{H}$, and let P be the projection associated to M. Then

$$\nu(M) = \int_Y (Py, y)\, d\mu(y) \tag{2}$$

Theorem 1.4 (Main Theorem)**.** *Let $\mathcal{H}$ be a non-separable Hilbert space and let ν be a Gleason measure on $\mathcal{H}$. Then there is an orthonormal basis Y of $\mathcal{H}$, and a measure μ on Y such that ν is defined from this data as in equation 2.*

The proof of the main theorem involves a substantial amount of set theory. We do wish to emphasize that all our results are provable in ZFC and that they do not require any extra "large cardinal" hypotheses.

Here is a very brief outline of how the proof of our main theorem proceeds. We are given a Gleason measure ν on a non-separable Hilbert space $\mathcal{H}$. From ν we shall define [in a non-canonical fashion] an ordinary measure μ_0 [defined on the subsets of X_0 say] with a technical property called "purity". Let $\mathcal{B}$ be the measure algebra of the measure μ_0. In the Boolean valued universe, $V^{\mathcal{B}}$, we define a Gleason measure, ν_1, on the Hilbert space ℓ^2. In $V^{\mathcal{B}}$ all the theorems of ZFC are available. In particular we can apply the usual version of Gleason's theorem to ν_1 getting a positive trace class operator of trace 1 in $V^{\mathcal{B}}$, A_1.

We now apply the machinery of Boolean valued ultraproducts. A_1 determines a map from X_0 to the trace class operators on ℓ^2 back in V. With a little massaging, this gives a map from X_0 to the trace class operators on $\mathcal{H}$. Let this final map be noted $x \longrightarrow A_x$.

For M a closed subspace of $\mathcal{H}$ let P be the corresponding projection. It will turn out that we have the integral representation:

$$\nu(M) = \int_{X_0} \text{trace}(PA_x)\, d\mu_0(x) \tag{3}$$

However the proof of the validity of this representation requires a delicate argument.

1.2 Exotic Gleason measures In this section, we briefly list some consequences of Theorem 1.3.

1.2.1 First some applications of Gödel's theorems:

1. If ZFC is consistent, then ZFC does not prove the existence of an exotic Gleason measure.

2. If ZFC is arithmetically sound [i.e. ZFC proves no false arithmetic statement] then ZFC does not prove the assertion: If ZFC is consistent, then so is the theory ZFC + "There is an exotic Gleason measure".

1.2.2 To get the consistency of the existence of an exotic Gleason measure we need a strong large cardinal consistency assertion. Suppose then that the theory ZFC + "There is a measurable cardinal" is consistent. Then the following assertions are consistent with ZFC:

1. There is an exotic Gleason measure on a Hilbert space of dimension $c = 2^{\aleph_0}$.

2. The continuum hypothesis holds but there is an exotic Gleason measure on a Hilbert space of strongly inaccessible dimension.

1.3 Finally, the following results say that the dimension of a Hilbert space that carries an exotic Gleason measure must be very large.

We shall need the notation of the $\beth_\alpha$'s:

1. $\beth_0 = \aleph_0$;

2. $\beth_{\alpha+1} = 2^{\beth_\alpha}$;

3. For λ a limit ordinal $\beth_\lambda = \sup\{\beth_\alpha \mid \alpha < \lambda\}$.

Then we have the following results:

1. If there is an exotic Gleason measure on a Hilbert space of dimension at most c then there is a weakly inaccessible cardinal less than c. In particular c is greater than the least κ such that $\kappa = \aleph_\kappa$.

2. Suppose that there is no exotic Gleason measure on any space of dimension at most c. [By the preceding remark, this holds if the continuum hypothesis is true, or even if c is at most the least κ such that $\kappa = \aleph_\kappa$.] Then if a Hilbert space $\mathcal{H}$ supports an exotic Gleason measure, its dimension is at least as large as the first strongly inaccessible cardinal. In particular, it is larger than the least κ such that $\kappa = \beth_\kappa$.

1.4 Purity If f is a function and A is a subset of its domain then we let $f[A]$ denote the *direct image*:

$$f[A] = \{f(x) \mid x \in A\}$$

Let μ be a measure on the set X. A partial equivalence f for μ is a function such that:

1. f has domain a subset A of X.

2. f has range a subset B of X.

3. f is a bijection.

4. Let A' be any subset of A and let $B' = f[A']$. Then $\mu(A') = 0$ iff $\mu(B') = 0$.

A partial equivalence f is *trivial* if for almost all x in its domain $f(x) = x$. The measure μ is *pure* if every partial equivalence f for μ is trivial.

1.5 Uniqueness The purpose of this section is to give a different integral representation of a Gleason measure for which we will be able to provide a canonical choice.

Let $\mathcal{H}$ be a Hilbert space whose dimension is uncountable. Let $\mathcal{T}$ be the set of positive trace class operators on $\mathcal{H}$ of trace 1.

Suppose now that μ is a measure on $\mathcal{T}$. Then we associate to μ a Gleason measure on $\mathcal{H}$ by the following prescription:

$$\nu(M) = \int_{\mathcal{T}} \operatorname{trace}(PA)\, d\mu(A) \tag{4}$$

Theorem 1.5. *Every Gleason measure on $\mathcal{H}$ arises from a measure on $\mathcal{T}$ in this way.*

If $A \in \mathcal{T}$, then the *support* of A is the orthogonal complement of the nullspace of A. The support of A is a subspace of $\mathcal{H}$ of at most countable dimension.

If μ is a measure on $\mathcal{T}$ we say that μ is *separated* if there is a set Y of μ measure 1 such that if A_1 and A_2 are distinct members of Y then the supports of A_1 and A_2 are orthogonal.

Theorem 1.6 (Uniqueness)**.** *If ν is a Gleason measure on $\mathcal{H}$, then ν arises from precisely one pure separated measure on $\mathcal{T}$ by the prescription of 4.*

1.6 Comments

1. The theorems announced in this abstract have been "in my desk drawer" for some time. For example, I talked about these results at U. C. L. A. in May 1992.

2. This abstract has been extracted (at the request of Paul Chernoff) from an incomplete draft of a manuscript which will eventually present complete proofs of the results referred to.

References

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[2] A. M. Gleason. Measures on the closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics*, 6:885–893, 1957.

[3] Robert M. Solovay. Real-valued measurable cardinals. In Dana S. Scott, editor, *Axiomatic Set Theory*, volume XIII, Part I of *Proceedings of Symposia In Pure Mathematics*, pages 397–428, Providence, Rhode Island, 1971. American Mathematical Society.

[4] S. Ulam. Zur masstheorie in der allgemeinen mengenlehre. *Fund. Math.*, 16:140–150, 1930.