THE THEORY OF THE α DEGREES IS UNDECIDABLE

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Abstract. Let α be an arbitrary Σ_1-admissible ordinal. For each n, there is a formula $\varphi_n(\vec{x},\vec{y})$ such that for any relation R on a finite set F with n elements, there are α-degrees $\vec{p}$ such that the relation defined by $\varphi_n(\vec{x},\vec{p})$ is isomorphic to R. Consequently, the theory of α-degrees is undecidable.

1. Introduction

Let α be an admissible ordinal, i.e. $L_\alpha \models \Sigma_1$-replacement. There is a well-developed theory of the α-recursively enumerable sets and degrees (cf. Sacks [9]). In particular, it is known that the Σ_1-theory of the α-recursively enumerable degrees is decidable (Lerman [5]) and that the theory of the ω_1^{CK}-recursively enumerable degrees is recursively isomorphic to that of $\langle L_{\omega_1^{CK}}, \in\rangle$ (Greenberg, Shore and Slaman [3]). The latter is an exact characterization of the complexity of the theory of the ω_1^{CK}-recursively enumerable degrees.

Less is known about the α-degrees in general. Let $\mathfrak{D}_\alpha = \langle \mathfrak{D}_\alpha, \leq\rangle$ denote the upper semilattice of α-degrees under the partial ordering of α-reducibility $\leq$. MacIntyre [6] showed that for countable α, one could construct an α-degree $\mathbf{a}$ and embed every countable distributive lattice with greatest and least elements into $\mathfrak{D}_\alpha$ above $\mathbf{a}$ as a segment. The undecidability of the theory of $\mathfrak{D}_\alpha$, for countable admissible α, follows as a consequence.

The uncountable case is still a mystery. The main difficulty has been with the minimal α-degree problem: "Is there an $\mathbf{a} > \mathbf{0}$ such that no $\mathbf{b}$ satisfies $\mathbf{0} < \mathbf{b} < \mathbf{a}$?" This problem, first posed almost 40 years ago, still remains open.

A set M of minimal Turing degree was exhibited by Spector [14]. Spector's construction of M ensures that if Φ is a Turing reduction and $\Phi(M)$ total, then M is a path through a recursive perfect tree T such that either Φ does not assume incompatible values on T or Φ defines an injective map on the paths in T. According to Spector's analysis of these two cases, $\Phi(M)$ is recursive in M or M is recursive in $\Phi(M)$, respectively. Spector's set of minimal degree M is a path on some such tree for each Φ.

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We note that the above split into cases corresponds to a dichotomy between Σ_2 and Π_2 possibilities. Defining a set by a recursion in which each step is determined by such a dichotomy is well beyond the direct scope of Σ_1-admissibility. The ordinal height α could well be exhausted executing less than α-many steps of the recursion within L_α.

What is known is as follows. MacIntyre [6] showed that minimal α-degrees exist for all countable α, and Shore [10] showed this for all Σ_2-admissible α. Countability of α or Σ_2-admissibility of α provide different means to adapt Spector's argument to the α-degrees. When α is countable, one can build M as the intersection of countable collection of non-empty closed subsets of 2^α. When α is Σ_2-admissible, one can prove that the collections of perfect subsets of 2^α which appear at the bounded points in the construction have perfect intersections.

To circumvent the difficulties of the minimal α-degree problem, we take a different approach. Namely, we import the Slaman and Woodin [13] coding technique. Slaman and Woodin's Coding Theorem states that every countable relation on the Turing degrees is definable from finitely many parameters, uniformly in the arity of the relation. From it, one can deduce Simpson's Theorem [11] that the second order theory of arithmetic is recursively isomorphic to the first order theory of the Turing degrees. In particular, $\mathrm{Th}(\mathfrak{D}_\omega)$ is not decidable. Unlike other approaches to undecidability, the proof of the Coding Theorem uses variations on Cohen forcing. It therefore is more amenable to implementation relative to an arbitrary α.

In this paper, we establish a finite version of the Coding Theorem for all α. Recursively in the jump of a Cohen 1-generic regular, hyperregular α-degree, every finite relation is representable in $\mathfrak{D}_\alpha$ with parameters, uniformly in the arity of the relation. As a consequence, every finite, finite-signature first order structure can be uniformly interpreted within the α-degrees. In particular, finite bipartite graphs can be uniformly interpreted. By the hereditary undecidability of the theory of finite bipartite graphs, one concludes that $\mathrm{Th}(\mathfrak{D}_\alpha)$ is undecidable.

There are a number of technical difficulties to address in attempting to code finite n-ary relations R into $\mathfrak{D}_\alpha$. Most of these arise in the overall organization of our construction of generic sets. As becomes quickly familiar to every α-recursion theorist, we are asking the Σ_1-admissible ordinal α to perform work that is essentially Σ_2 or higher. Such problems are endemic in the study of the α-recursively enumerable sets and are well understood there. We will apply the α-finite injury method of Sacks and Simpson [8] to build generic predicates B on L_α and then again to build coding predicates which are generic over $L_\alpha[B]$.

Interestingly, it is not the problem of finding generic objects that limits us to interpreting only finite structures within $\mathfrak{D}_\alpha$. The argument in the Turing degree proof of Coding Theorem rests on the observation that if p_1 and p_2 are two finite binary strings, then there is a sequence of such strings $r_1, \dots, r_n$ such that $r_1 = p_1$, $r_n = p_2$, and each r_i disagrees with r_{i+1} at exactly one bit. For arbitrary α, a sequence of one-point differences $(r_i : i \leq \beta)$ might start with p_1 and reach p_2 with β an infinite limit ordinal, leaving the Coding Theorem argument with nothing to rest upon. Our approach, given in detail in Section 3, circumvents this difficulty by directly exploiting the condition that the relation to be coded is itself finite. The reader will observe that, technical details aside, the process of transiting from one string to another in a finite number of steps in the Turing case is now replaced by effecting a finite number of changes of forcing conditions. The difficulty with infinite sequences of conditions for $\alpha > \omega$ has not gone away. We view this situation as not entirely satisfactory. It is not clear if this is due to a limitation of the method deployed, or reflects a deeper fact about the definability of subsets of $\mathfrak{D}_\alpha$.

In the next section, we revisit the α-finite injury method to construct Cohen 1-generic sets relative to a regular, hyperregular set with an appropriate Σ_1-fine structure property. In Section 3, we introduce the Cohen-like partial order to uniformly code finite structures of fixed arity from parameters within $\mathfrak{D}_\alpha$ in Theorem 3.1. As a corollary, we conclude the undecidability of $\mathrm{Th}(\mathfrak{D}_\alpha)$ in Theorem 3.9. We end the paper with a selection of open questions.

For background material on α-recursion theory, the reader is referred to [9] whose notations we follow closely.

2. Cohen forcing over L_α

Recall that $X \subset L_\alpha$ is α-finite if and only if $X \in L_\alpha$.

Definition 2.1. *Let $B \subset \alpha$. Then B is regular if $B \upharpoonright \gamma$ is α-finite for all $\gamma < \alpha$. B is hyperregular if $L_\alpha[B]$ is an admissible structure.*

An element of $L_\alpha[B]$ is said to be α-B-finite. If B is regular, then every α-B-finite set is α-finite. We will use this fact implicitly throughout this paper. The key properties to be exploited with regard to a regular, hyperregular set are (i) "weakly α-recursive in" and "α-recursive in" coincide, and (ii) every function weakly α-recursive in B maps an α-finite set onto an α-finite set. Both of these are explained in [9].

Definition 2.2. *Let $B \subset \alpha$ be regular and hyperregular. The $\Sigma_1(B)$ projectum of α, denoted α^*_B, is the least $\gamma \leq \alpha$ for which there is an injection from α into γ that is α-recursive in B.*

Observe that if $\alpha^*_B < \alpha$, then it is an $L_\alpha[B]$-cardinal. In fact, by [8], it is the greatest cardinal in $L_\alpha[B]$.

If $B = \emptyset$, then Jensen's fine structure theory [4] says that α^* is the least ordinal for which there is an α-B-recursively enumerable subset which is not α-finite. While this is not true in general for arbitrary B, we will show that the equivalence is preserved for sets which are sufficiently generic with respect to suitable notions of forcing. In particular, the equivalence holds for Cohen generic subsets of α (cf. Theorem 2.6 and the remarks that follow).

This property provides a setting in which to implement the α-finite injury method of Sacks and Simpson [8], relative to a regular, hyperregular 1-generic set B, which we will then use to construct $\Sigma_1(B)$-generic sets G_1, G_2 (Theorem 2.9) for the Coding Theorem that is established in the next section.

Definition 2.3. *An ordinal $\gamma < \alpha$ is $\Sigma_1(B)$-stable if $L_\gamma[B \cap \gamma]$ is a Σ_1 elementary substructure of $L_\alpha[B]$.*

Lemma 2.4. *Let $B \subset \alpha$ be regular and hyperregular. If $\alpha = \alpha^*_B$ has a largest $L_\alpha[B]$-cardinal, then α is the limit of $\Sigma_1(B)$-stable ordinals.*

Proof. Suppose that α has a largest $L_\alpha[B]$-cardinal $\aleph$. Let $\gamma > \aleph$ and set $\mathfrak{M}$ to be the canonical $\Sigma_1(B)$ Skolem hull of $L_\gamma[B \cap \gamma]$ in $L_\alpha[B]$, obtained by choosing the least witness for each $\Sigma_1(B)$-formula (if it exists). Then $\mathfrak{M}$ is closed under ordinals since if $\delta > \aleph$ is in $\mathfrak{M}$, then there is a bijection in $L_\alpha[B]$ between the two, and the least index of the α-finite function that witnesses this bijection is in $\mathfrak{M}$. But this entails that every ordinal less than δ is in the hull. Thus $\mathfrak{M} = L_{\hat{\gamma}}[B \cap \hat{\gamma}]$ for some $\hat{\gamma} \leq \alpha$.

If $\exists y \varphi$ is $\Sigma_1(B)$ with parameters less than γ and has a solution for y in $L_\alpha[B]$, then this fact is known in $L_\alpha[B]$ at a stage before α. The least such witness is a member of $L_{\hat{\gamma}}[B \cap \hat{\gamma}]$. Since the parameters in $\exists y \varphi$ are less than γ, there is a one-one weakly B-recursive injection π of $L_{\hat{\gamma}}[B \cap \hat{\gamma}]$ into $L_\gamma[B \cap \gamma]$ obtained by matching a witness with the corresponding $\Sigma_1(B)$ formula that includes some parameters less than γ. Since B is hyperregular, the injection is α-recursive in B. As $\gamma < \alpha$ and $\alpha^*_B = \alpha$, $\hat{\gamma}$ must be less than α, else one obtains an α-B-recursive injection from α into γ, contradicting the assumption that $\alpha^*_B = \alpha$. Now $L_{\hat{\gamma}}[B \cap \hat{\gamma}]$ is a $\Sigma_1(B)$ substructure of $L_\alpha[B]$. Thus α is a limit of $\Sigma_1(B)$ stable ordinals. □

The proof of the next lemma is essentially that of Lemma 2.3 in [8] where $B = \emptyset$.

Lemma 2.5. *Let B be regular and hyperregular. Let $\aleph$ be a regular cardinal in $L_\alpha[B]$. Let $\delta < \aleph$. If $\{I_\nu | \nu < \delta\}$ is a simultaneous α-B-recursive set such that each I_ν is α-finite and has $L_\alpha[B]$-cardinality less than $\aleph$, then $\bigcup_{\nu<\delta} I_\nu$ is α-finite and has $L_\alpha[B]$-cardinality less than $\aleph$.*

Proof. Suppose for the sake of contradiction that the $L_\alpha[B]$-cardinality of $\bigcup_{\nu<\delta} I_\nu$ is at least $\aleph$. Then from $\bigcup_{\nu<\delta} I_\nu$ one may enumerate an α-B-recursive sequence S from X of length $\aleph$. Since B is hyperregular, this sequence is α-finite. Now each $S \cap I_\nu$ is α-finite of length less than $\aleph$. This implies that $\aleph$ is the limit of an α-finite union of δ many α-finite sequences each of length less than $\aleph$, contradicting its regularity.

Thus let $\gamma < \aleph$ be the length of the sequence of elements enumerated in X. Let $g(x)$ be the xth element in the enumeration. Then g is α-B-recursive and so its image, which is X, is α-finite by the hyperregularity of B. Furthermore, since $\gamma < \aleph$, the cardinality of X in $L_\alpha[B]$ is less than $\aleph$. □

An α-finite injury construction is an α-recursive priority construction of one or more sets such that the number of times a requirement is injured during the construction is α-finite. Sacks and Simpson [8] developed the α-finite injury method to solve Post's problem for all admissible ordinals. We apply this method to obtain a regular, hyperregular set B recursive in $\emptyset'$ that is 1-generic, with its j-sections, for $j < \omega$, also mutually (Cohen) 1-generic (see Theorem 2.6). [9] gives a different proof of this fact. For each $n < \omega$, a second application of the α-finite injury method relative to B, but based on a more sophisticated notion of forcing $\mathcal{P}$, yields two regular, hyperregular sets G_1, G_2 (α-recursive in B') that are sufficiently $\mathcal{P}$-generic relative to B (Theorem 2.9). The pair $\{G_1, G_2\}$ will form the basic building block for Theorem 3.1.

We begin with the unrelativized case and define an α-recursive partially ordered set $\langle \mathcal{P}, \leq \rangle$. Members of $\mathcal{P}$ are α-finite functions $p : \omega \times \delta \to \{0, 1\}$ for $\delta < \alpha$. Denote by p^j the jth section of p, i.e. $p^j(x) = p(j, x)$ for each x such that (j, x) is in the domain of p. We will abbreviate and use the symbol $\mathcal{P}$ to refer to both the domain of the order and the order itself. Of course, $\mathcal{P}$ is α-recursively isomorphic to the ordering of α-finite maps from ordinals less than α into 2, but we will be interested in the finite sections of the generic set and so we have put them in the foreground.

Our language includes constant symbols B and B_j for each $j < \omega$. We define $p \Vdash \varphi$, for $p \in \mathcal{P}$ and φ a sentence in the expanded language by induction:

(1) If φ is $\Delta_0(B, B_0, \ldots, B_n)$ ($n < \omega$), then $p \Vdash \varphi$ if and only if $L_\alpha \models \varphi$ when B and B_j, $j < n$, are interpreted respectively as the sets (whose characteristic functions are) p and p^j.
(2) If φ is $\exists y \psi$, then $p \Vdash \varphi$ if and only if for some a, $p \Vdash \psi(\underline{a})$;
(3) $p \Vdash \neg\varphi$ if and only if no $q \leq p$ satisfies $q \Vdash \varphi$.

$B \subset \alpha$ is 1-generic if for all $\exists y \varphi$ where φ is $\Delta_0(B)$, there is a condition p which is an initial segment of B (hence p^j is an initial segment of B_j for appropriate j), such that $p \Vdash \exists y \varphi$ or $p \Vdash \neg\exists y \varphi$. Notice that for such a φ, the relation $p \Vdash \exists y \varphi$ is α-r.e.

Theorem 2.6. *Let $\mathcal{P}$ be the notion of forcing defined above. There exist $\emptyset'$-recursive sets B and $B_j, j < \omega$ ($B_j = \{x | B(j, x) = 1\}$), that are 1-generic with respect to $\mathcal{P}$. Furthermore,*

(1) *B is regular and hyperregular. Hence, so is each section B_j of B.*
(2) *α^*_B is the least ordinal γ for which there is an α-B-recursively enumerable subset that is not α-finite.*
(3) *(Independence of the sections of B) Fix $i, i_1, \ldots, i_m$ to be distinct elements of ω. For any $X and Y \leq_\alpha B_i$, $1 \leq k \leq m$, and Z is a join of sets from $\{B_{i_1}, \ldots B_{i_m}\}$, $X \oplus B_{i_k} \leq_\alpha Y \oplus Z$ if and only if $X \leq_\alpha Y$ and B_{i_k} occurs the join to produce Z.*

Proof. Let $\hat{R}_e$ be the subset of $\mathcal{P}$ each of whose elements force the eth Σ_1 sentence in the expanded language (under an effective enumeration). Let $p < B$ designate that p is (the characteristic function of) an initial segment of B. Define "B meets $\hat{R}_e$" to mean there is a $p \in \hat{R}_e$ such that $p < B$. The eth requirement states the following.

$$R_e: \text{ Either } B \text{ meets } \hat{R}_e \text{ or there is an initial segment of } B \text{ with no extension in } \hat{R}_e.$$

We split the construction into three cases, according to the value of α^* relative to α and the properties associated with α. Each case will determine an order of priorities for the requirements R_e to be satisfied. Regularity of B follows from its 1-genericity, while hyperregularity, which requires special steps to achieve, will guarantee that (ii) holds. We present Case 1 in greater detail, and only sketch the construction as well as verification of other cases, pointing out the necessary modifications to be made.

2.1. **Case 1, $\alpha^* < \alpha$.** Let $\pi : \alpha \to \alpha^*$ be an injection that is α-recursive. Then R_d has higher priority than R_e if and only if $\pi(d) < \pi(e)$. To make B hyperregular, our construction will depend on the cofinality of α^*. First suppose that α^* is regular in L_α. For $d < \alpha$, let φ_d be the dth Σ_1 sentence in the expanded language.

Begin with $B^0 = \emptyset$. Suppose B^δ is defined. We say that R_e *requires attention at* $\delta + 1$ if, in $\delta + 1$ steps of computation, no initial segment of B^δ meets $\hat{R}_e$ and there is a $p \in \hat{R}_e$ which extends every $q \leq B^\delta$ that (i) is the least string in $\hat{R}_d$ for some d with $\pi(d) < \pi(e)$, and (to ensure hyperregularity of B) (ii) is the least string forcing $\exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta')$ for some $\pi(d) < \pi(e)$ and some $\beta < \alpha^*$. Choose the requirement with the highest priority requiring attention (call it $e_{\delta+1}$) and let $B^{\delta+1}$ extend B^δ be the least string satisfying (i) for $e_{\delta+1}$ and (ii). If no such e exists, let $B^{\delta+1} = B^\delta$ and go to the next sage.

If δ is a limit ordinal and B^γ is defined for all $\gamma < \delta$, let z_δ be the least $z = (j, x)$ (for some j, x) such that $\lim_{\gamma \to \delta} B^\gamma(z)$ is not defined. Let $B^{<\delta}(z) = \lim_{\gamma \to \delta} B^\gamma(z)$ for $z < z_\delta$. We say that R_e *requires attention at* δ if (i) and (ii) in the previous paragraph holds with respect to $B^{<\delta}$. Then B^δ is the least string that satisfies the requirement of highest priority that requires attention.

Next assume that α^* is singular. The construction is similar to the previous case except that in ensuring hyperregularity we consider, for each d, only formulas of the type $\exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta')$ for β less than the least L_α-cardinal greater than $\pi(d)$. Then proceed as before.

This completes the construction when $\alpha > \alpha^*$.

We first establish Theorem 2.6 for $\alpha > \alpha^*$. The requirement R_e is said to be injured at stage δ if either (a) δ is a successor ordinal and $B^{\delta-1}$ has an initial segment that belongs to $\hat{R}_e$ but B^δ does not, or (b) δ is a limit ordinal and B^γ has an initial segment that belongs to $\hat{R}_e$ for all sufficiently large $\gamma < \delta$ but B^δ does not.

Claim 2.7. *Every requirement is injured α-finitely many times.*

Proof. We verify Claim 2.7 for each requirement by induction on $\rho < \alpha^*$, where ρ is its priority. First notice that for each e, the strategy to achieve hyperregularity for φ_e is carried out by accepting, and not by injuring, existing computations using B as oracle at any given stage. Injury occurs only in attempting to satisfy (i) for a requirement of higher priority. Thus a computation of the form $B^\delta \Vdash \exists y \forall \beta' < \beta \exists z < y \varphi_e(\beta')$ is destroyed only if mandated by the need to satisfy a requirement R_d with $\pi(d) < \pi(e)$ for the sake of (i).

Let $\kappa > \rho = \pi(e)$ be a regular cardinal in L_α less than or equal to α^*. The assumption that $\alpha > \alpha^*$ implies that α^* is the ordinal which is the greatest cardinal in L_α. Now, a strategy requires attention at most one time after each injury. Consequently, if $\pi(e) < \kappa$ (a regular cardinal) and R_e is injured less than κ many times, then so is the requirement whose priority comes right after R_e's. Let $I_{\pi(e),\delta}$ be the set of stages less than or equal to δ where R_e is injured. Let $I_{\pi(e)} = \bigcup_\delta I_{\pi(e),\delta}$. For $\nu < \rho$ and $\delta < \alpha$, define $I_{\nu,\delta} = \emptyset$ if $\nu \notin \{\pi(0), \dots, \pi(\delta)\}$, and equal to $I_{\pi(e),\delta}$ if $\nu = \pi(e)$ where $e \leq \delta$. Since for each $\nu < \rho$, if I_ν has L_α-cardinality less than κ then so does $I_{\nu+1}$, Lemma 2.5 implies that $\bigcup_{\nu<\rho} I_\nu$ also has L_α-cardinality less than κ. The construction then ensures that the same holds for I_ρ whenever $\rho < \kappa$. We conclude that for every $\rho < \alpha^*$ (a regular α-cardinal or a limit of such cardinals), there is a stage δ where no requirement of priority higher than ρ is injured after δ. This proves Claim 2.7. $\square$

Our task now is to show that, as a consequence, given $\rho < \alpha^*$, there is an initial segment p of B such that for any requirement of priority higher than or equal to ρ, either p meets $\hat{R}_d$ or no extension of p meets $\hat{R}_d$, so that B is 1-generic. In addition, we will show that B is hyperregular. This is shown in the next lemma.

Lemma 2.8. *Let $\alpha^* < \alpha$ and $\rho = \pi(e)$. There is a δ_ρ such that B^δ extends B^{δ_ρ} for all $\delta > \delta_\rho$, and for all d with $\pi(d) < \rho$,*

(1) *B^{δ_ρ} meets $\hat{R}_d$, or no extension of B^{δ_ρ} meets $\hat{R}_d$, and*
(2) *B is hyperregular with respect to φ_d. By this we mean that one of the following holds:*
   (a) *α^* is regular and $B^{\delta_\rho} \Vdash \exists y \forall \beta' < \alpha^* \exists z < y \varphi_d(\beta')$;*
   (b) *α^* is regular and there is a least $\beta < \alpha^*$ such that $B^{\delta_\rho} \Vdash \neg \exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta')$;*
   (c) *α^* is singular and $B^{\delta_\rho} \Vdash \exists y \forall \beta' < \beta(d) \exists z < y \varphi_d(\beta')$, where $\beta(d)$ is the least L_α-cardinal after $\pi(d)$;*
   (d) *α^* is singular and there is a least β less than $\beta(d)$ such that $B^{\delta_\rho} \Vdash \neg \exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta')$.*

Proof. We argue that for each $\rho = \pi(e)$, there is a uniform bound on the conditions p preserved because $p \Vdash \exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta')$, where $\beta < \alpha^*$, $\pi(d) < \rho$ and φ_d is Σ_1.

Fix $\rho = \pi(e)$. Assume that δ_ρ^- is a stage after which no $\pi(d) < \rho$ is injured. We claim that there is a $\delta_\rho \geq \delta_\rho^-$ that satisfies (1) and (2).

First assume that α^* is regular in L_α. Then the construction dictates that for all $\delta > \delta_\rho^-$, the shortest conditions p such that $p \Vdash \exists y \forall \beta' < \beta \exists z < y\varphi_d(\beta')$, $p < B^\delta$, $\pi(d) < \pi(e)$ and $\beta < \alpha^*$, are permanently preserved (see the earlier discussion). This means that α-recursively, one may enumerate the set

$$X_1 = \{(\pi(d), p, \beta) | \exists \delta > \delta_\rho^- [\pi(d) < \rho \,\&\, p \Vdash \exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta') \,\&\, p \leq B^\delta]\}.$$

Fix d. Suppose that for each $\beta < \alpha^*$ there is a δ and a $p \leq B^\delta$ such that $p \Vdash \exists y \forall \beta' < \beta \exists z < y \exists z < y \varphi_d(\beta')$. Choosing the least such δ for each β and applying admissibility one computes α-recursively a $\delta(d)$ and a $p \leq B^{\delta(d)}$ such that $p \Vdash \exists y \forall \beta' < \alpha^* \exists z < y \varphi_d(\beta')$. It then follows that the set

$$X_2 = \{\pi(d) < \rho | \exists \delta > \delta_\rho^- \exists p \leq B^\delta (p \Vdash \exists y \forall \beta' < \alpha^* \exists z < y \varphi_d(\beta'))\}$$

is a Σ_1 subset of ρ and so α-finite. Thus there is a least $\delta \geq \delta_\rho^-$ for which a $p \leq B^\delta$ exists satisfying for all $d \in X_2$, $p \Vdash \exists y \forall \beta' < \alpha^* \exists z < y \varphi_d(\beta')$. Denote the least such δ by $\delta(X_2)$ and the corresponding p by $p(X_2)$.

Since X_2 is α-finite, the set of $\pi(d)$'s less than ρ not in X_2 is α-finite as well. Call this set X_3. Then for each $d \in X_3$, $\{\beta | \exists \delta > \delta(X_2) \exists p \leq B^\delta (p \Vdash \exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta')\}$ is uniformly α-r.e. (in parameter d) and has α-cardinality less than the regular cardinal α^* according to the construction. By Lemma 2.5, the set

$$X_4 = \{\beta | \exists d \in X_3 \exists \delta > \delta_\rho^- \exists p \leq B^\delta (p \Vdash \exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta'))\}$$

is α-finite. By admissibility, there is a least $\delta > \delta_\rho^-$, denoted $\delta(X_3)$, and a $p(X_3) \leq B^{\delta(X_3)}$ such that for each $\pi(d) \in X_3$, for each $\beta < \alpha^*$, if $p \Vdash \exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta')$ where $p \leq B^{\delta'}$ for some p and δ', then $\beta \in X_4$ and $p(X_3) \Vdash \exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta')$. Since least computations verifying $p \Vdash \exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta')$, for $\pi(d) < \rho$ and $\beta < \alpha^*$, are always preserved after δ_ρ^-, we may assume that $p(X_3)$ extends $p(X_2)$.

After stage $\delta(X_3)$, the strategy to make B meet $\hat{R}_d$ where $\pi(d) < \rho$ will succeed whenever there is an opportunity to do so, since there will be no more conditions p to preserve for the sake of $p \Vdash \exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta')$, and no more injury caused by requirements with priority higher than ρ. We conclude that there is a stage greater than or equal to $\delta^(X_3)$ after which no requirement of priority higher than ρ requires attention due to (i) of the construction. Thus, B is 1-generic.

Given $\pi(d) \in X_2$, we have, since $p(X_3)$ extends $p(X_2)$, $p(X_3) \Vdash \exists y \forall \beta' < \alpha^* \exists z < y \varphi_d(\beta')$. On the other hand, suppose that $\pi(d) \in X_3$ and let β_d be the least β such that $p(X_3) \nVdash \exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta')$. Then there is a (least) requirement $R_{d'}$ which enumerates all strings p extending $p(X_3)$ such that $p \Vdash \exists y \forall \beta' < \beta_d \exists z < y \varphi_d(\beta')$. If there is a stage δ after $\delta(X_3)$ where such a p is enumerated as a substring of B^δ, then the construction ensures that this computation is preserved and this would contradict the choice of β_d. In particular, $R_{d'}$ is not met after stage $\delta(X_3)$. Since this is true for each $\pi(d) \in X_3$, and $\{\pi(d')|\pi(d) \in X_3\}$ has L_α-cardinality less than or equal to ρ, there is a stage after which all these requirements (whose priorities are contained in a proper initial segment of α^*) will no longer require attention. Then there is a least stage $\delta_\rho \geq \delta(X_3)$ such that for all $\delta \geq \delta_\rho$, no extension of B^{δ_ρ} meets $\hat{R}_{d'}$, for $\pi(d) \in X_3$. This implies that 2(a) or 2(b) of Lemma 2.8 holds.

The hyperregularity of B now follows immediately from 2(a) and 2(b). In fact, if Φ^B is total on an α-finite set X, then it may be considered to be total on a cardinal less than or equal to α^*. By 2(a) and 2(b), the restriction of Φ^B to that cardinal is α-finite.

Next, suppose that α^* is a singular cardinal in L_α. The proof of (1) is similar to the case when α^* is regular. For 2(c) and 2(d), we replace, in the definition of X_2, the cardinal α^* by the least cardinal after $\pi(d)$, which is a regular cardinal in L_α. In defining X_3, we also require that β be less than the next cardinal after $\pi(d)$. Arguing along the same line, we conclude that either 2(c) or 2(d) is true.

To show that B is hyperregular, let Φ^B be total on an α-finite set X which we may identify with an cardinal less than or equal to α^*. Let κ be the cofinality of α^* in L_α. First suppose that the cardinality of X is α^* and let $g : \kappa \to \alpha^*$ be an α-finite cofinality map. If for all $\gamma < \kappa$, $\Phi^B \restriction g(\gamma)$ has α-finite image, then there is a Θ so that for each $\gamma < \kappa$, $\Theta^B(\gamma)$ is the code of the α-finite set $\Phi^B[g(\gamma)]$. Then 2 (c), (d) implies that $\Theta^B[\kappa]$, hence $\Phi^B[\alpha^*]$, is α-finite. On the other hand, let γ_0 be the least γ such that $\Phi^B[g(\gamma)]$ is not α-finite. By 2(c) and (d), this is not possible since there is an initial segment p of B such that $p \Vdash$ "$\exists y(\Phi^p[g(\gamma_0)]$ is total in y steps of computation." Now suppose that X has cardinality κ less than α^*. Again we may assume that the domain of Φ is κ, and then 2 (c) and (d) guarantee that Φ maps κ onto an α-finite set using B as oracle. This completes the proof of Lemma 2.8. □

To prove (ii) of Theorem 2.6, we first claim that $\alpha^*_B = \alpha^*$. Clearly $\alpha^*_B \leq \alpha^*$. Let $\pi_B : \alpha \to \beta$ be an α-B-recursive injection and suppose $\beta < \alpha^*$. By the hyperregularity of B and the assumption that $\alpha^* < \alpha$, the map $\pi_B \restriction \alpha^*$ is α-finite. But this gives an α-finite injection of α^* into β, which is impossible. Thus $\alpha^*_B = \alpha$. Now if $\beta < \alpha^*$ and $X \subset \beta$ is α-B-r.e., then as before any ordinal that effectively enumerates X must be less than α^*. But again by the hyperregularity of B, X is α-finite. This proves Theorem 2.6 (ii).

Modulo the fact that every requirement attains its final priority from some stage onward, an argument similar to that of Theorem 2.5.6 in [12], exploiting the mutual Cohen 1-genericity of $\{B_j|j < \omega\}$, may now be used to establish (iii) of Theorem 2.6. We leave the details to the reader (the same point applies to Case 2 and Case 3 below which we will therefore omit mentioning in the sequel).

2.2. **Case 2, $\alpha^* = \alpha$ is a limit of cardinals in L_α.** Here we stipulate that R_d has higher priority than R_e if and only if $d < e$. Repeat the steps in Case 1 when α^* is singular, but treat the function π as the identity map. Then by Lemma 2.5, R_e is injured less than κ many times, where κ is the least L_α-cardinal greater than e. With this, Theorem 2.6 (i) follows from an application of 1 and 2(c), 2(d) of Lemma 2.8. Part (ii) is again a consequence of the hyperregularity of B, just like the case for $\alpha < \alpha^*$.

2.3. **Case 3, $\alpha^* = \alpha$ and there is a largest L_α-cardinal $\aleph$.** By Lemma 2.4, α is a limit of Σ_1-stable ordinals $> \aleph$. Let $\lambda \leq \alpha$ be the order type of the collection $\{\beta_\gamma\}_{1\leq\gamma<\lambda}$ of these Σ_1-stable ordinals. The map π will be defined as an injection from α onto $\aleph \cdot \lambda$, given in terms of α-finite bijective maps $\pi_\gamma : \beta_\gamma \to \aleph$. Let $\beta_0 = 0$. There is a uniformly α-recursive approximations of β_γ such that for all $\delta < \alpha$, $\beta_{\delta,\gamma}$ is the γth Σ_1-stable ordinal "at stage δ" (meaning $\beta_{\delta,\gamma}$ appears to be the γth Σ_1-stable ordinal after δ steps of calculation). In fact it is not difficult to see that given $\gamma_0 < \lambda$ and $\delta_0 > \beta_{\gamma_0}$, $\beta_{\delta,\gamma} = \beta_\gamma$ for each $\gamma < \gamma_0$ and $\delta > \delta_0$. Let $\pi_{\gamma,\delta}$ be the least bijection between $\beta_{\delta,\gamma}$ and $\aleph$. For each $e, \delta < \alpha$, let $\pi'(\delta, e) = \aleph\cdot\rho+\pi_{\delta,\gamma}(e)$, where γ is the least Σ_1-stable greater than e at stage δ, and the order type of Σ_1-stable ordinals not exceeding e, at stage δ, is ρ. Then $\pi(e)$ is defined to be $\lim_{\delta\to\alpha}\pi'(\delta, e)$. The limit exists for each e since it depends on the sequence $\beta_{\delta,\gamma}$ which does have a limit. In fact, the property of Σ_1-stability guarantees that for each $\rho < \aleph \cdot \lambda$, the set $X = \pi^{-1}[\rho]$ is α-finite and there is a δ_0 such that for all $\delta > \delta_0$, $\pi'(\delta, e) = \pi(e)$ for all $e \in X$.

The γth block of requirements at stage δ refers to all ordinals e such that $\beta_{\delta,\gamma} \leq e < \beta_{\delta,\gamma+1}$. Two requirements e and e' are in the same block at stage δ if there is a γ so that $\beta_{\delta,\gamma} \leq e < \beta_{\delta,\gamma+1}$ and $\beta_{\delta,\gamma} \leq e' < \beta_{\delta,\gamma+1}$. We say that R_e has higher priority than $R_{e'}$ at stage δ if either e is in block γ and e' is in block γ' at stage δ and $\gamma < \gamma'$, or $\gamma = \gamma'$ and $\pi_{\delta,\gamma}(e) < \pi_{\delta,\gamma}(e')$. The values $\pi_{\delta,\gamma}(e)$ and $\pi(e)$ prescribe the priority of e at stage δ and in the limit respectively.

The ground is now set for implementing a construction similar to that given earlier. Define B^δ, for $\delta \geq 0$, as in Case 1 except that we replace α^* by $\aleph$ and observe the current rules governing the order of priorities. Thus at stage δ, we compute $\pi_{\gamma,\delta}$ for all $\gamma < \lambda$ if $\lambda < \alpha$, and all $\gamma < \delta$ if $\lambda = \alpha$, and compare the priorities of requirements according to the arrangement above. A requirement R_e may be injured at stage δ if one of the following holds: (i) The block to which it belongs changes at δ; (ii) the block does not change but a requirement of higher priority requires attention and is satisfied. Fix $\rho < \aleph \cdot \lambda$. There is a stage δ after which no requirement of priority higher than ρ changes its priority (as discussed above). Then δ does not exceed the next Σ_1-stable ordinal $\beta(\rho)$ after $\pi^{-1}[\rho]$. Suppose that for each $\rho' < \rho$, the set of stages where a requirement of priority higher than ρ' is injured is α-finite and bounded below $\beta(\rho)$. Then this in turn implies, by the construction, that all requirements of priority higher than *or equal to* ρ will not be injured after $\beta(\rho)$. Thus for all $\rho < \aleph \cdot \lambda$, the set of requirements of priority higher than ρ is bounded below $\beta(\rho)$ in the stages where the requirements are injured.

We can now prove an analog of Lemma 2.8. Fix $\rho < \aleph \cdot \lambda$. The first step is to argue that 2(a) or 2(b) of Lemma 2.8 holds, when α^* is replaced by $\aleph$. The only point to note here is that any computation of the type $p \Vdash \exists y \forall \beta' < \beta \exists z < y \varphi_d(\beta')$, where $\beta < \aleph$ and d has priority higher than ρ, is enumerated by stage $\beta(\pi(d))$. This ensures that after stage $\beta(\pi(d))$, no action is taken to preserve any computation of this type, and so a uniform bound on the preservation of such computations exists, for requirements of priority higher than ρ. This implies that a stage δ_ρ as prescribed in Lemma 2.8 exists. With this, it is straightforward to verify that B and B_j, $j < \omega$, are all 1-generic and hyperregular. Finally, part (ii) follows from the hyperregularity of B.

This completes the proof of Theorem 2.6. $\square$

For a Cohen-type notion of forcing, in which a condition's (strongly) forcing a Σ_1 formula is a Σ_1-property of the condition and the formula, Theorem 2.6 may be generalized to hold relative to a regular, hyperregular set B that satisfies the conclusions of that theorem. We single out the key properties we need for the Coding Theorem in the following. Let $\mathcal{P}$ be a Cohen-type notion of forcing with set variables B, B_j (for j less than some fixed n), and G_1, G_2.

Theorem 2.9. *Let B satisfy (i) and (ii) of Theorem 2.6. Let $n < \omega$ such that $\{B_j | j < n\}$ satisfies (iii) in the theorem. There exist (by abuse of notation) regular, hyperregular sets $G_1, G_2 \leq_\alpha B'$ which are $\Sigma_1(B)$-generic with respect to $\mathcal{P}$.*

The proof is similar to that of Theorem 2.6, except that the construction is carried out α-recursively in the regular, hyperregular set B that satisfies (i)—(iii) of the Theorem. The sets to be constructed are now G_1 and G_2, and they will again satisfy (i) and (ii) over $L_\alpha[B]$.

Remark. Theorem 2.6 (iii) implies that the α-degrees of B_j are pairwise incomparable. This is an instance of an antichain of α-degrees, a crucial fact to be exploited in the next section.

3. The Coding Theorem

Our main objective in this section is to establish the following:

Theorem 3.1. *(Coding Theorem) Let $\alpha > \omega$ be an admissible ordinal. Let B be a regular, hyperregular set satisfying (i)—(iii) of Theorem 2.6 with α-degree* $\mathbf{b}$. *For each $n < \omega$, there is a formula $\varphi_n(x_1, \ldots, x_n, y_1, \ldots, y_m)$*

in the language of $\langle \mathfrak{D}_\alpha, \leq\rangle$ such that if $\mathcal{R}$ is a finite n-ary relation on α-degrees below $\mathbf{b}$, then there exist degrees $\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_m}$ such that $\mathbf{g_i} < \mathbf{b}'$ for $i = 1, \ldots, m$, and for all $(\mathbf{a_1}, \ldots, \mathbf{a_n})$,

$$(\mathbf{a_1}, \ldots, \mathbf{a_n}) \in \mathcal{R} \longleftrightarrow \mathfrak{D}_\alpha \models \varphi_n(\mathbf{a_1}, \ldots, \mathbf{a_n}, \mathbf{g_1}, \ldots, \mathbf{g_m}).$$

Let $\mathbf{b}$ and B, $\{B_j|j < \omega\}$ be as in Theorem 2.6. Then (iii) implies that for each $n < \omega$, $\{B_j|j < n\}$ is an antichain. The first step is to show that the Coding Theorem holds for every finite antichain below B. While this appears to be a finite version of Theorem 3.1.5 of [12], a rather different argument is required in view of the general domain (an arbitrary admissible ordinal) in which the construction is carried out, as noted in Section 1.

Theorem 3.2. *Let $\mathbf{b} > \mathbf{0}$ be the α-degree of a regular, hyperregular set B satisfying (i)—(ii) of Theorem 2.6. There is a φ in the language of $\langle \mathfrak{D}_\alpha, \leq\rangle$ such that whenever $\{\mathbf{b_j}|j < n\}$ is a finite antichain below $\mathbf{b}$ there exist $\mathbf{g_1}, \mathbf{g_2} < \mathbf{b}'$ such that for all $\mathbf{y} \leq \mathbf{b}$,*

$$\mathbf{y} \in \{\mathbf{b_j}|j < n\} \leftrightarrow \mathfrak{D}_\alpha \models \varphi(\mathbf{y}, \mathbf{b}, \mathbf{g_1}, \mathbf{g_2}).$$

Fix $n < \omega$. For $j < n$, let B_j be a set with α-degree $\mathbf{b_j}$. Then B_j is regular and hyperregular. The key properties we need concerning B and $\{B_j|j < n\}$ are (ii) and (iii) of Theorem 2.6. These properties are degree invariant. In the style of Dekker and Myhill [1], we may assume that each B_j is the set of (codes of) initial segments of a regular set B_j^*, and thus is α-recursive in any of its its unbounded subsets. We introduce the following notion of forcing for which G_1 and G_2 will be 1-generic.

3.1. **Forcing conditions.** A forcing condition is a set p of the form $\langle p_1, p_2\rangle$ such that p_1 and p_2 are α-finite binary strings of the same length, satisfying the following:

(1) The domain of p_i, for $i = 1, 2$, is an initial segment of $n \times \alpha$ (henceforth the word "string" will refer to sets with such a property);
(2) A condition q extends a condition p if q_i extends p_i for $i = 1, 2$ and for all $j < n$, for each (j, x) that is in the domain of $q \setminus p$, $q_1(j, x) = q_2(j, x)$ whenever $x \in B_j$.

We assume an α-recursive bijection between $n \times \alpha$ and α to be given. If p is a condition, the points (j, x) in the domain of p where $x \in B_j$ are called the jth *coding location.* Note that the collection $\mathcal{P}$ of conditions is recursive but "q extends p" is a relation α-recursive in B. Further, p strongly forces a Σ_1-sentence in the same way that a Cohen condition does, consequently we may apply Theorem 2.9 to obtain two $\Sigma_1(B)$-generic sets G_1, G_2 with respect to this notion of forcing. Fix such sets G_1 and G_2.

We now prove Theorem 3.2 by showing that $\{\mathbf{b_j}|j < n\}$ is definable in $\langle \mathfrak{D}_\alpha, \leq\rangle$ with parameters $\mathbf{g_1}, \mathbf{g_2}, \mathbf{b}$, where $\mathbf{g_i}$ is the α-degree of G_i. The plan is to show that $\{\mathbf{b_j}|j < n\}$ is the set of minimal elements of the set

$$\{\mathbf{x} \leq \mathbf{b}|\exists \mathbf{z}((\mathbf{z} \leq (\mathbf{x} \vee \mathbf{g_1}) \;\&\; \mathbf{z} \leq (\mathbf{x} \vee \mathbf{g_2}) \;\&\; \mathbf{z} \not\leq \mathbf{x})\}.$$

This is done in two parts. Firstly, one proves in Lemma 3.7 that for each j, there is a set Z α-recursive in $B_j \oplus G_1$ and $B_j \oplus G_2$, but not α-recursive in B_j. Secondly for each $Y \leq_\alpha B$, and triple of reduction procedures $\Phi_{e_i}, i < 3$, the following which we denote as S_{e0,e_1,e_2} holds:

$$[\Phi_{e_0}^B = Y \ \& \ \Phi_{e_1}^{Y \oplus G_1} = \Phi_{e_2}^{Y \oplus G_2})] \rightarrow [\Phi_{e_1}^{Y \oplus G_1} \leq_\alpha Y \ \vee \ \exists j < n(B_j \leq_\alpha Y)].$$

On the surface, satisfying an S-requirement appears to demand at least $\Sigma_2(B)$-admissibility, reflecting the two quantifiers in its hypothesis. We will show that the α-finite injury construction of the previous section is sufficient to satisfy S_{e_0,e_1,e_2}.

From now on assume that the hypothesis of S_{e_0,e_1,e_2} holds. Since $\Phi_{e_1}^{Y \oplus G_1} = \Phi_{e_2}^{Y \oplus G_2}$, $\langle G_1, G_2 \rangle$ does not meet the Y-α-recursively enumerable set

$$\hat{R}_{\langle e_1,e_2 \rangle} = \{\langle p_1, p_2 \rangle | \exists x(\Phi_{e_1}^{Y \oplus p_1}(x) \downarrow \neq \Phi_{e_2}^{Y \oplus p_2}(x) \downarrow)\}.$$

By the $\Sigma_1(B)$-genericity of $\langle G_1, G_2 \rangle$, there is an initial segment of $\langle G_1, G_2 \rangle$ with no extension in $\mathcal{P}$ belonging to $\hat{R}_{\langle e_1,e_2 \rangle}$. To simplify notation, assume that the initial segment is $\langle \emptyset, \emptyset \rangle$.

We say that a pair of strings $\langle r, s \rangle$ split (Φ_e, Y) over p_1 if both r and s extend p_1 as Cohen conditions and there is an x such that $\Phi_e^{Y \oplus r}(x) \downarrow \neq \Phi_e^{Y \oplus s}(x) \downarrow$. If $q = \langle q_1, q_2 \rangle$ extends $p = \langle p_1, p_2 \rangle$, then q is said to split $(\Phi_{e_1}, \Phi_{e_2}, Y)$ over p if there is an x such that $\Phi_{e_1}^{Y \oplus q_1}(x) \downarrow \neq \Phi_{e_2}^{Y \oplus q_2}(x) \downarrow$. The first step is straightforward (we assume henceforth that whenever we consider requirements and their associated computations, we are at a stage where the requirement and those with higher priority will no longer be injured. This is justified by the α-finite injury construction.):

Lemma 3.3. *Let $p = \langle p_1, p_2 \rangle$ be a condition which is an initial segment of $\langle G_1, G_2 \rangle$. If no extension of p_1 splits (Φ_{e_1}, Y) over p_1, then $\Phi_{e_1}^{Y \oplus G_1} \leq_\alpha Y$.*

Proof. Since B is hyperregular and $Y \leq_\alpha B$, it is is sufficient to show that $\Phi_{e_1}^{Y \oplus G_1} \leq_{w\alpha} Y$. That is, it is is sufficient to show that there is an α-recursive in Y procedure to compute $\Phi_{e_1}^{Y \oplus G_1}$ pointwise. For this, let x be given. If no extension of p_1 splits (Φ_{e_1}, Y) over p_1, then for any r extending p_1 and any x, if $\Phi_{e_1}^{Y \oplus r}(x) \downarrow$ then it is equal to $\Phi_{e_1}^{Y \oplus G_1}(x)$. Thus, Y can compute $\Phi_{e_1}^{Y \oplus G_1}(x)$ by finding any such r. □

Thus, we may assume that for each initial segment $\langle p_1, p_2 \rangle$ of $\langle G_1, G_2 \rangle$, there is a pair $\langle r, s \rangle$ that splits (Φ_{e_1}, Y) over p_1. Our aim now is to show that Y computes B_j for some $j < n$, yielding S_{e_0,e_1,e_2}. Let $x_{r,s}$ be the least (Y-effectively enumerated) witness for $\langle r, s \rangle$ to split (Φ_{e_1}, Y) over p_1. We will simplify the notation by ignoring the dependence of $\langle r, s \rangle$ on p_1.

Given $\langle r, s \rangle$ that splits $\langle \Phi_{e_1}, Y \rangle$ over p_1, let $\sigma_{-1}^{r,s} = r$ and $\sigma_0^{r,s}$ be the string obtained by setting $\sigma_0^{r,s}(i, x) = s(i, x)$ if $i = 0$, and equal to $r(i, x)$ for $i \neq 0$. We claim that there is a pair $\langle r, s \rangle$ splitting $\langle \Phi_{e_1}, Y \rangle$ over p_1 for which an extension σ of $\sigma_0^{r,s}$ satisfying $\Phi_{e_1}^{Y \oplus \sigma}(x_{r,s}) \downarrow$ exists. If not, the 1-genericity of G_1 requires that there be an initial segment p_1^* of G_1 extending

p_1 so that no $\langle r, s\rangle$ splitting $\langle \Phi_{e_1}, Y\rangle$ over p_1^* includes a corresponding σ for $\sigma_0^{r,s}$ as stipulated above. Now since every initial segment of G_1 is extended by a pair $\langle r, s\rangle$ that splits $\langle \Phi_{e_1}, Y\rangle$ (hence over p_1), the construction in Theorem 2.9 ensures that G_1 must extend a $\sigma_0^{r,s}$ coming from an $\langle r, s\rangle$ that splits $\langle \Phi_{e_1}, Y\rangle$ over p_1^*. But this implies that $\Phi_{e_1}^{Y\oplus G_1}$ is partial, contradicting our assumption. Thus there exists a pair $\langle r, s\rangle$ splitting $\langle \Phi_{e_1}, Y\rangle$ over p_1 such that some σ extending $\sigma_0^{r,s}$ satisfies $\Phi_{e_1}^{Y\oplus\sigma}(x_{r,s})\downarrow$. Let $\hat\sigma_0^{r,s}$ be the least such σ. By induction, suppose that every initial segment of G_1 is extended by a pair $\langle r, s\rangle$ that splits $\langle \Phi_{e_1}, Y\rangle$ so that for each $k \le j-1$, $\hat\sigma_k^{r,s}$ is defined and $\Phi_{e_1}^{Y\oplus\hat\sigma_k^{r,s}}(x_{r,s})\downarrow$ holds, where $\hat\sigma^{r,s}$ extends the condition $\sigma^{r,s}$ for which $\sigma_k^{r,s}(i,x) = s(i,x)$ if $i = k$ and equal to $\hat\sigma_{k-1}^{r,s}(i,x)$ otherwise. Arguing as before, but now considering only pairs $\langle r, s\rangle$ splitting $\langle \Phi_{e_1}, Y\rangle$ over p_1 so that $\hat\sigma_k^{r,s}$ is defined for each $k \le j-1$, we conclude that every initial segment of G_1 is extended by a pair $\langle r, s\rangle$ splitting $\langle \Phi_{e_1}, Y\rangle$ so that $\hat\sigma_k^{r,s}$ is defined for $k \le j$, where $\hat\sigma_k^{r,s}$ now satisfies $\Phi_{e_1}^{Y\oplus\hat\sigma_k^{r,s}}(x_{r,s})\downarrow$ and extends a $\sigma_k^{r,s}$ such that $\sigma_k^{r,s}(i,x) = s(i,x)$ if $i = k$, and equal to $\hat\sigma_{k-1}^{r,s}(i,x)$ otherwise. It follows that every initial segment of G_1 is extended by a pair $\langle r, s\rangle$ splitting $\Phi_{e_1}, Y\rangle$ so that $\hat\sigma_k^{r,s}$ is defined for all $k < n$. From now on, we shall consider only pairs $\langle r, s\rangle$ that split $\langle \Phi_{e_1}, Y\rangle$ over p_1 with such a property. The collection of pairs extending p_1 with this property may be enumerated α-recursively in Y. Since $\sigma_{-1}^{r,s} = r$ and $\hat\sigma_{n-1}^{r,s}$ is compatible with s, the next lemma is immediate.

Lemma 3.4. *Let $p = \langle p_1, p_2\rangle < \langle G_1, G_2\rangle$. For each pair of strings $\langle r, s\rangle$ that splits (Φ_{e_1}, Y) over p_1 such that $\hat\sigma_j^{r,s}$ is defined for all $j < n$, there is one j where $\Phi_{e_1}^{Y\oplus\hat\sigma_{j-1}^{r,s}}(x_{r,s})\downarrow = k \ne \Phi_{e_1}^{Y\oplus\hat\sigma_j^{r,s}}(x_{r,s})\downarrow = 1-k$, where $k = 0$ or 1.*

We say that j is $\langle r, s\rangle$-critical if it is the least such that $\langle \sigma_{j-1}^{r,s}, \sigma_j^{r,s}\rangle$ satisfies the conclusion of Lemma 3.4. The pair $\langle \hat\sigma_{j-1}^{r,s}, \hat\sigma_j^{r,s}\rangle$ is then called the $\langle r, s\rangle$-critical pair. If j is $\langle r, s\rangle$-critical, then $\hat\sigma_{j-1}^{r,s}$ and $\hat\sigma_j^{r,s}$ disagree only on the jth section. The next lemma exploits the key feature of the notion of forcing introduced in this section, and shows how it leads to the desired outcome.

Lemma 3.5. *Let j_0 be the least $j < n$ such that unboundedly many initial segments of G_1 are extended by pairs $\langle r, s\rangle$ splitting $\langle \Phi_{e_1}, Y\rangle$ and j is $\langle r, s\rangle$-critical. Then $B_{j_0} \le_\alpha Y$.*

Proof. First of all, by Lemma 3.4, we may assume that every initial segment of G_1 is extended by a pair $\langle r, s\rangle$ splitting $\langle \Phi_{e_1}, Y\rangle$ such that r and s agree on the ith component whenever $i \ne j_0$ and such that j_0 is $\langle r, s\rangle$-critical. From now on, all pairs $\langle r, s\rangle$ considered are assumed to have the properties described here.

Fix $\langle r, s\rangle$. Given τ with length less than $\mathrm{lth}(r \cap s)$, let $\sigma_\tau^{r,s}$ be the string obtained by setting $\sigma_\tau^{r,s}(i,x) = r(i,x)$ if $i \ne j_0$ or if $i = j_0$ and x corresponds to an extension of τ, and equal to $s(i,x)$ otherwise. Thus $\sigma_\tau^{r,s}$ agrees with r at points on the ith coordinate when $i \neq j_0$ and on the points in the j_0th coordinate corresponding to initial segments of τ, and agrees with s otherwise. If $\tau^{j_0} \in B_{j_0}$ (recall, there is a set $B^*_{j_0}$ such that B_{j_0} is the set of codes of its initial segments), then we say that τ is j_0-correct for $\langle r,s\rangle$. Let $\hat{\sigma}_\tau^{r,s}$ be the least (Y recursively enumerated) string σ extending $\sigma_\tau^{r,s}$ such that $\Phi_{e_1}^{Y\oplus\sigma}(x_{r,s})\downarrow$ holds, if such a σ exists. We first claim that every initial segment of G_1 is extended by an $\langle r,s\rangle$ such that for some j_0-correct τ for $\langle r,s\rangle$ with $\mathrm{lth}(\tau) < \mathrm{lth}(r\cap s)$, $\hat{\sigma}_\tau^{r,s}$ is defined. If the claim is false, then there is an initial segment of G_1 beyond which no such $\langle r,s\rangle$ exists. For simplicity of notation, assume that p_1 is such an initial segment. Then the collection

$$\{\sigma_\tau^{r,s} \geq p_1 | (\mathrm{lth}(r\cap s) > \mathrm{lth}(\tau) \ \&\ \tau \text{ is } j_0\text{-correct for } \langle r,s\rangle)\}$$

contains elements extending arbitrary initial segments of G_1 with no extension σ of $\sigma_\tau^{r,s}$ satisfying $\Phi_{e_1}^{Y\oplus\sigma}(x_{r,s})\downarrow$. Then as in the remarks preceding Lemma 3.4, the $\Sigma_1(B)$-genericity of G_1 mandates that G_1 extend at least one $\sigma_\tau^{r,s}$ in the set. However, this would imply that $\Phi_{e_1}^{Y\oplus G_1}$ is partial, which is again a contradiction.

Now suppose that τ is j_0-correct for $\langle r,s\rangle$ and $\mathrm{lth}(\tau) < \mathrm{lth}(r\cap s)$, with the additional property that $\hat{\sigma}_\tau^{r,s}$ is defined. Extend s if necessary to an $\hat{s}$ so that $\hat{s}$ and $\hat{\sigma}_\tau^{r,s}$ agree above $\mathrm{lth}(\tau\cap r)$. Since τ is j_0-correct and $\mathrm{lth}(\tau) > \mathrm{lth}(r\cap s)$, $\hat{\sigma}_\tau^{r,s}$ and $\hat{s}$ agree at all coding locations. Thus, by our assumption, $\Phi_{e_1}^{Y\oplus\hat{\sigma}_\tau^{r,s}}(x_{r,s}) = \Phi_{e_1}^{Y\oplus\hat{s}}(x_{r,s})$.

On the other hand, if τ is not j_0-correct for $\langle r,s\rangle$, let x_0 be the least x such that $\tau \upharpoonright x$ is not j_0-correct. Then no extension of $\tau\upharpoonright x_0$ is j_0-correct for $\langle r,s\rangle$. In going from r to $\sigma_\tau^{r,s}$, the only possible changes to values at coding locations are in the j_0th section and only there for coding locations greater than $lth(r\cap s)$. If x_0 were less than $lth(r\cap s)$, then $\sigma_\tau^{r,s}$ and r would agree at all coding locations. In this case, by our assumption, $\Phi_{e_1}^{Y\oplus\hat{\sigma}_\tau^{r,s}}(x_{r,s}) = \Phi_{e_1}^{Y\oplus\hat{r}}(x_{r,s})$.

The above analysis gives an algorithm to compute B_{j_0} from Y: Given a z, look for a τ and a j_0-critical pair $\langle r,s\rangle$ such that $\mathrm{lth}(r\cap s) > z$, $\hat{\sigma}_\tau^{r,s}$ is defined, and $\Phi_{e_1}^{Y\oplus\hat{\sigma}_\tau^{r,s}}(x_{r,s}) = \Phi_{e_1}^{Y\oplus s}(x_{r,s})$. Then $\tau\upharpoonright z \in B_{j_0}$. □

This leads to the following:

Lemma 3.6. *If $\Phi_{e_0}^B = Y$ and $Z = \Phi_{e_1}^{Y\oplus G_1} = \Phi_{e_2}^{Y\oplus G_2}$, then either*

(i) $\Phi_{e_1}^{Y\oplus G_1} \leq_\alpha Y$, *or*
(ii) $Y \geq_\alpha B_j$ *for some* $j < n$

Proof. Suppose $Z = \Phi_{e_1}^{Y\oplus G_1} = \Phi_{e_2}^{Y\oplus G_2}$. By the observation made before Lemma 3.3, there is an initial segment of $\langle G_1, G_2\rangle$ over which no extension splits $(\Phi_{e_1}, \Phi_{e_2}, Y)$. Without loss of generality, we assume that this already happens at $\langle\emptyset,\emptyset\rangle$.

There are now two cases to consider:

Case 1. There is an initial segment $p = \langle p_1, p_2\rangle$ of $\langle G_1, G_2\rangle$ which is not extended by any pair of incompatible strings splitting (Φ_{e_1}, Y).

Then by Lemma 3.3, $\Phi_{e_1}^{Y\oplus G_1} \leq_\alpha Y$.

Case 2. For any $p = \langle p_1, p_2\rangle$ that is an initial segment of $\langle G_1, G_2\rangle$, there exist incompatible extensions r and s of p_1 and an $x_{r,s}$ such that $\Phi_{e_1}^{Y\oplus r}(x_{r,s})\downarrow = k \neq \Phi_{e_2}^{Y\oplus s}(x_{r,s})\downarrow = 1-k$, for $k = 0, 1$.

Then by Lemmas 3.4 and 3.5 there is a $j < n$ such that $B_j \leq_\alpha Y$. □

Lemma 3.7. *For each j, there is a C_j such that $C_j \leq_\alpha B_j \oplus G_1$ and $C_j \leq_\alpha B_j \oplus G_2$ but $C_j \not\leq_\alpha B_j$.*

Proof. Let $C_j = \{(j,x) | G_1(j,x) = 1 = B_j(x)\}$. Since G_1 and G_2 agree on the jth coding location (by the definition of extension for forcing conditions), C_j is α-recursive in both $B_j \oplus G_1$ and $B_j \oplus G_2$. We claim that $C_j \not\leq_\alpha B_j$. Assume for the sake of contradiction that $\Phi^{B_j} = C_j$. Then the set

$$D = \{\langle p_1, p_2\rangle | \exists x[(j,x) \in C_j \ \&\ \Phi^{B_j}(x) \neq p_1(j,x)]\}$$

is a dense $\Sigma_1(B_j)$—hence $\Sigma_1(B)$—set. Since G_1 is 1-generic relative to B, it meets D and so there is an x such that $\Phi^{B_j}(x) \neq G_1((j,x) = C_j(x)$. Thus B_j does not compute C_j. □

The proof of Theorem 3.2 is now immediate: Given $B >_\alpha \emptyset$ which is regular and hyperregular satisfying (i)—(ii) of Theorem 2.6, let $\mathbf{b}$ be its α-degree. If $\{\mathbf{b_j} | j < n\}$ is a finite antichain below $\mathbf{b}$ with representatives $\{B_j | j < n\}$ given by (iii) of Theorem 2.6, let $\mathbf{g_1}, \mathbf{g_2}$ be the α-degrees of $\Sigma_1(B)$-generic sets G_1, G_2 with respect to the notion of forcing introduced in this section and guaranteed by Theorem 2.9. By Lemma 3.7, if $\mathbf{y} \leq \mathbf{b}$, $\mathbf{z} \leq \mathbf{y} \vee \mathbf{g_1}$ and $\mathbf{z} \leq \mathbf{y} \vee \mathbf{g_2}$, then either $\mathbf{z} \leq \mathbf{y}$ or $\mathbf{b_j} \leq \mathbf{y}$ for some $j < n$. Together with the requirement that

$$\exists \mathbf{z}(\mathbf{z} \leq (\mathbf{y} \vee \mathbf{g_1}) \ \&\ \mathbf{z} \leq (\mathbf{y} \vee \mathbf{g_2}) \ \&\ \neg(\mathbf{z} \leq \mathbf{y})),$$

we see that the only $\mathbf{y}$'s satisfying these conditions are the $\mathbf{b_j}$'s. Thus $\{\mathbf{b_j} | j < n\}$ is definable with parameters $\mathbf{b}, \mathbf{g_1}$ and $\mathbf{g_2}$.

To code finite m-ary relations, we proceed as Lemma 3.1.11 and Theorem 3.1.2 in [12]:

Lemma 3.8. *Let B be 1-generic and satisfy (i)—(iii) of Theorem 2.6. Let $\mathcal{C} = \{\mathbf{c_0}, \ldots, \mathbf{c_{n-1}}\}$ be a finite collection of α-degrees below the α-degree of B. Then $\mathcal{C}$ is definable in $\mathfrak{D}_\alpha$ with parameters.*

Proof. Let $\hat{\mathcal{B}} = \{B_j | j < n\}$ be mutually Cohen 1-generic satisfying Theorem 2.6 (iii). Then $\mathcal{B} = \{\mathbf{b_0}, \ldots, \mathbf{b_{n-1}}\}$, where $\mathbf{b_j}$ is the α-degree of B_j, forms an antichain and so is definable by parameters $\mathbf{b}, \mathbf{g_1}, \mathbf{g_2}$ according to Theorem 3.2. Let $\psi : \mathcal{C} \to \mathcal{B}$ be a bijection so that $\psi(\mathbf{c_j}) = \mathbf{b_j}$. Then $\hat{\mathcal{C}} = \{\mathbf{c_j} \vee \mathbf{b_j} | j < n\}$ forms an antichain according to Theorem 2.6 (iii) and is therefore definable (Theorem 3.2). Then $\mathbf{c} \in \mathcal{C}$ if and only if there exist $\mathbf{x} \in \hat{\mathcal{C}}$ and $\mathbf{y} \in \mathcal{B}$ such that $\mathbf{x} = \mathbf{c} \vee \mathbf{y}$. □

Theorem 3.1 may now be proved as follows. Let $\mathbf{b}$ be as above. Let R be an n-ary finite relation on $\mathfrak{D}_\alpha$ below $\mathbf{b}$. For $j < n$, let

$$R_j = \{\mathbf{a} | \exists(\mathbf{x_0}, \ldots, \mathbf{x_{n-1}}) \in R \ \&\ \mathbf{x_j} = \mathbf{a}\}.$$

Then each R_j is definable according to Lemma 3.8 by a formula φ_j. For each j, let $\|R_j\| = r_j$ and let

$$\{\mathbf{b}_{0,0}, \ldots, \mathbf{b}_{0,r_1-1}, \mathbf{b}_{1,0}, \ldots, \mathbf{b}_{1,r_1-1}, \ldots, \mathbf{b}_{n-1,0}, \ldots, \mathbf{b}_{n-1,r_{n-1}-1}\}$$

be α-degrees of mutually Cohen 1-generic sets $B_{j,i}$ $(i < r_j)$, each of which is α-recursive in B. Let ψ_j be a bijection between R_j and $\{\mathbf{b}_{j,0}, \ldots, \mathbf{b}_{j,r_j-1}\}$. Since R_j and $\{\mathbf{b}_{j,0}, \ldots, \mathbf{b}_{j,r_j-1}\}$ are definable, it follows that ψ_j is definable as well. This implies that

$$S = \{\bigvee_{j<n} \mathbf{d}_j | \mathbf{d}_j \in \{\mathbf{b}_{j,0}, \ldots, \mathbf{b}_{j,r_j-1}\} \ \&\ (\psi_0^{-1}(\mathbf{d}_0), \ldots, \psi_{n-1}^{-1}(\mathbf{d}_{n-1})) \in R\}$$

is definable from parameters below $\mathbf{b}'$ and by mutual 1-genericity of the $\mathbf{b_{i,j}}$'s, different combinations of the joints determine different α-degrees. We can then define R as the set of all n-tuples $(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1})$ such that $\mathbf{x}_j \in R_j$ for all j and $(\psi_j(\mathbf{x}_0), \ldots, \psi_n(\mathbf{x}_{n-1})) \in S$. This completes the proof of Theorem 3.1.

Observe that in the proof of Theorem 3.2 there is a fixed formula φ_n which defines any finite antichain of size n of α-degrees below a regular, hyperregular 1-generic degree $\mathbf{b}$, using parameters $\mathbf{b}, \mathbf{g_1}$ and $\mathbf{g_2}$. Different antichains are definable using φ_n via different parameters. Similarly, according to the definition of the set S in the proof of the above theorem, for each $n < \omega$ there is a formula θ_n such that if $L = \{\mathbf{b_0}, \ldots, \mathbf{b_{n-1}}\}$ and $R = \{\hat{\mathbf{b}}_\mathbf{0}, \ldots, \hat{\mathbf{b}}_\mathbf{n-1}\}$ are disjoint mutually Cohen 1-generic below $\mathbf{b}$, and $E \subset L \times R$ is a binary relation, then there exists a pair of parameters $(\mathbf{e_1}, \mathbf{e_2})$ such that $E(\mathbf{a}, \hat{\mathbf{a}})$ if and only if $\theta_n(\mathbf{a}, \hat{\mathbf{a}}, \mathbf{b}, \mathbf{e_1}, \mathbf{e_2})$. Furthermore, θ_n is defined uniformly in n. This uniformity of definition leads to the following undecidability result.

Corollary 3.9. *The theory of* $\mathfrak{D}_\alpha$ *is undecidable.*

Proof. By the above Remark, there is a uniform interpretation into the α-degrees of finite structures with one binary relation. Now the collection of bipartite graphs (i.e. structures with left domain L and right domain R as well as an associated edge relations E) is an elementary class. Hence the class of bi-partite graphs which are interpreted in the α-degrees is definable over $\langle \mathfrak{D}_\alpha, \leq \rangle$. By [7] the theory of finite bi-partite graphs is hereditarily undecidable. Hence the theory of the α-degrees is undecidable. $\square$

4. Conclusion

4.1. **The α degrees.** The structure $\mathfrak{D}_\alpha$ is naturally interpreted within the second order structure with universe L_α. Syntactically, the first order theory of $\mathfrak{D}_\alpha$, which we write as $T(\mathfrak{D}_\alpha)$, is naturally reduced to the second order theory of L_α, which we write as $T_2(L_\alpha)$. That is, $T_2(L_\alpha) \geq_m T(\mathfrak{D}_\alpha)$. In the case of $\alpha = \omega$, Simpson's Theorem asserts the converse, $T(\mathfrak{D}_\omega) \geq_m T_2(L_\omega)$. Therefore, the first order theory of the Turing degrees is recursively isomorphic to the second order theory of arithmetic.

Theorem 3.9 provides a weak lower bound on the complexity of $T(\mathfrak{D}_\alpha)$. Namely, there is a recursive reduction of the Σ_1-theory of first order arithmetic to $T(\mathfrak{D}_\alpha)$. Of course, there is a considerable distance between the Σ_1-theory of first order arithmetic and the second order theory of L_α. Our first question asks whether $T(\mathfrak{D}_\alpha)$ always achieves its largest possible value.

Question 4.1. *Suppose that α is Σ_1-admissible. Is $T(\mathfrak{D}_\alpha) \geq_m T_2(L_\alpha)$?*

The answers may depend upon whether α is countable or upon ambient set theoretic assumptions.

Secondly, one can ask about the global properties of D_α, beyond those expressed within its first order theory. To what extent does $\mathfrak{D}_\alpha$ reflect α?

Question 4.2. *Do there exist distinct α_1 and α_2 such that $\mathfrak{D}_{\alpha_1}$ can be mapped elementarily to $\mathfrak{D}_{\alpha_2}$, even isomorphically?*

Even for cardinals α, the qualitative features of $\mathfrak{D}_\alpha$ vary greatly depending on whether α is singular or regular and on the set theoretic assumptions in which $\mathfrak{D}_\alpha$ is considered. Consider the cases $\mathfrak{D}_{\omega_1}$ and $\mathfrak{D}_{\omega_{\omega_1}}$ within the context of $V = L$. Under these conditions, every subset of ω_1 is regular and hyperregular. Thus, constructions as in the previous sections can be applied relative to every subset of ω_1. For example, for every non-zero X in $\mathfrak{D}_{\omega_1}$, there is a G in $\mathfrak{D}_{\omega_1}$ which is incomparable with X. Superficially, $\mathfrak{D}_{\omega_1}$ resembles $\mathfrak{D}_\omega$. In contrast, Friedman [2] shows that the elements of $\mathfrak{D}_{\omega_{\omega_1}}$ above the complete ω_{ω_1}-recursively enumerable degree are well-ordered with successor given by the ω_{ω_1}-jump. Consequently, $\mathfrak{D}_{\omega_{\omega_1}}$ does not resemble $\mathfrak{D}_\omega$ at all. Perhaps the difference can be used to advantage.

Question 4.3. *Assume $V = L$. Does there exist a Σ_1-admissible α such that $\mathfrak{D}_\alpha$ has no non-trivial automorphism?*

Finally, one can investigate the above questions in the setting of the α-recursively enumerable degrees.

Question 4.4. *Let α be Σ_1-admissible.*

(1) *Is there a characterization of the theory of the α-recursively enumerable degrees in terms of the theory of L_α?*
(2) *Is there a non-trivial automorphism of the α-recursively enumerable degrees?*

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