Global Properties of the Turing Degrees and the Turing Jump

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Abstract

We present a summary of the lectures delivered to the Institute for Mathematical Sciences, Singapore, during the 2005 Summer School in Mathematical Logic. The lectures covered topics on the global structure of the Turing degrees $\mathscr{D}$, the countability of its automorphism group, and the definability of the Turing jump within $\mathscr{D}$.

1 Introduction

This note summarizes the tutorial delivered to the Institute for Mathematical Sciences, Singapore, during the 2005 Summer School in Mathematical Logic on the structure of the Turing degrees. The tutorial gave a survey on the global structure of the Turing degrees $\mathscr{D}$, the countability of its automorphism group, and the definability of the Turing jump within $\mathscr{D}$.

There is a glaring open problem in this area: Is there a nontrivial automorphism of the Turing degrees? Though such an automorphism was announced in Cooper (1999), the construction given in that paper is yet to be independently verified. In this paper, we regard the problem as not yet solved. The Slaman-Woodin Bi-interpretability Conjecture 5.10, which still seems plausible, is that there is no such automorphism.

Interestingly, we can assemble a considerable amount of information about $Aut(\mathscr{D})$, the automorphism group of $\mathscr{D}$, without knowing whether it is trivial. For example, we can prove that it is countable and every element is arithmetically definable. Further, restrictions on $Aut(\mathscr{D})$ lead us to interesting conclusions concerning definability in $\mathscr{D}$.

Even so, the progress that can be made without settling the Bi-interpretability Conjecture only makes the fact that it is open more glaring. With these notes goes the hope that they will spark further interest in this area and eventually a solution to the problems that they leave open.

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1.1 Style

In the following text, we will state the results to be proven in logical order. We will summarize the proofs when a few words can convey the reasoning behind them. When that fails, we will try to make the theorem plausible. A complete discussion, including proofs omitted here, can be obtained in the forthcoming paper Slaman and Woodin (2005).

2 The coding lemma and the first order theory of the Turing degrees

Definition 2.1
- $\mathscr{D}$ denotes the partial order of the Turing degrees. $a+b$ denotes the join of two degrees. ($A \oplus B$ denotes the recursive join of two sets.)
- A subset $\mathscr{I}$ of $\mathscr{D}$ is an *ideal* if and only if $\mathscr{I}$ is closed under $\leq_T$ ($x \in \mathscr{I}$ and $y \leq_T x$ implies $y \in \mathscr{I}$) and closed under $+$ ($x \in \mathscr{I}$ and $y \in \mathscr{I}$ implies $x+y \in \mathscr{I}$). A *jump ideal* is closed under the Turing jump ($a \mapsto a'$) as well.

Early work on $\mathscr{D}$ concentrated on its naturally order-theoretic properties. For example, Kleene and Post (1954) showed that every finite partial order is isomorphic to a suborder of $\mathscr{D}$. Sacks (1961) extended this embedding theorem from finite to countable partial orders. Spector (1956) constructed a minimal nonzero degree, which began a long investigation into the structure of the initial segments of $\mathscr{D}$. Some of the high points in that investigation are Lachlan (1968), every countable distributive lattice with a least element is embeddable as an initial segment, Lerman (1971), every finite lattice with a least element is embeddable as an initial segment, and Lachlan and Lebeuf (1976), every countable lattice with a least element is embeddable as an initial segment.

Our focus will be on the logical properties of $\mathscr{D}$. We can view the results in the previous paragraph as steps in deciding the low-level fragments of the first order theory of $\mathscr{D}$. By the Kleene-Post theorem, the existential theory of $\mathscr{D}$ is decidable. An existential statement is true in $\mathscr{D}$ if and only if it is true in some finite partial order of size the length of the sentence. Lerman's theorem can be combined with a strengthening of the Kleene-Post theorem to show that the $\exists\forall$-theory of $\mathscr{D}$ is decidable; see Lerman (1983) or Shore (1978).

One can also come to conclusions concerning undecidability. The theory of distributive lattices is undecidable, Lachlan's theorem yields an interpretation of that theory into the first order theory of $\mathscr{D}$, and so the first order theory of $\mathscr{D}$ is undecidable.

The exact degree of the theory of $\mathscr{D}$ was calculated by Simpson (1977). Simpson showed that the theory of $\mathscr{D}$ is recursively isomorphic to the second order theory of arithmetic. We will obtain a proof of Simpson's theorem as a corollary to the machinery we develop here.

2.1 The coding lemma

Definition 2.2 A countable n-place relation $\mathscr{R}$ on $\mathscr{D}$ is a countable subset of the n-fold Cartesian product of $\mathscr{D}$ with itself. In other words, $\mathscr{R}$ is a countable subset of the set of length n sequences of elements of $\mathscr{D}$.

Theorem 2.3 (The Coding Lemma, Slaman and Woodin (1986)) *For every n there is a first order formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that for every countable n-place relation*

$\mathscr{R}$ on $\mathscr{D}$ there is a sequence of degrees $\overrightarrow{p} = (p_1, \dots, p_m)$ such that for all sequences of degrees $\overrightarrow{d} = (d_1, \dots, d_n)$,

$$\overrightarrow{d} \in \mathscr{R} \iff \mathscr{D} \models \varphi(\overrightarrow{d}, \overrightarrow{p}).$$

By the coding lemma, quantifiers over countable relations on $\mathscr{D}$ can be interpreted in the first order language of $\mathscr{D}$ by quantifying over the parameters used to define these relations. Consequently, the first order theory of $\mathscr{D}$ can interpret all of countable mathematics.

For example, the isomorphism type of the standard model of arithmetic $\mathbb{N}$ is characterized in countable terms. There is a finitely axiomitized theory T such that for any countable model $\mathfrak{M}$ of T, either there is an infinite decreasing sequence in $\mathfrak{M}$ (a countably expressed property) or $\mathfrak{M}$ is isomorphic to $\mathbb{N}$. Similarly, second order quantifiers over a copy of $\mathbb{N}$ are just quantifiers over countable sets. Hence, there is an interpretation of second order arithmetic in the first order theory of $\mathscr{D}$ and Simpson's theorem follows.

One can push the application of the coding lemma further in this direction. Rather than interpreting the second order theory of arithmetic, one can interpret the subsets of the natural numbers, work with them individually, and associate them with the degrees that they represent, to the extent that it is possible to do so. This is a principal theme in what follows.

Finally, we will have a more metamathematical use of the coding lemma. Every first order structure has a countable elementary substructure, typically a Skolem hull of the original. In models of set theory, this fact becomes a reflection property. For example, if a sentence φ is true in L, then it is true in some countable initial segment of L. In the Turing degrees, we will show that global properties of $\mathscr{D}$ can be reflected to countable jump ideals.

In retrospect, the coding lemma should be expected. One can construct sets and directly control everything that is arithmetically definable from them. Consider the Friedberg (1957) jump inversion theorem. Given a set $X \geq_T 0'$, Friedberg constructs another set A such that $A' \equiv_T X$. Though the theory of forcing was only introduced later, Friedberg uses the ingredients Cohen forcing to decide atomic facts about A'. He alternates between meeting the dense sets associated with building a generic real and steps to code atomic facts about X.

One can think of the parameters as sets generically engineered to distinguish between elements of the relation and elements of it's complement. The interactions to be controlled have bounded complexity in the arithmetic hierarchy. Consequently, one can obtain coding parameters which are uniformly arithmetic in any presentation of the relation. A sharp analysis of the coding methods in Slaman and Woodin (2005) gives the following.

Theorem 2.4 *Suppose that there is a presentation of the countable relation $\mathscr{R}$ which is recursive in the set R. There are parameters $\overrightarrow{p}$ which code $\mathscr{R}$ in $\mathscr{D}$ such that the elements of $\overrightarrow{p}$ are recursive in R'.*

Similarly, since the coding only involves arithmetic properties of the parameters, the relation $\mathscr{R}$ is arithmetically definable from the parameters which code it.

Theorem 2.5 (Decoding Theorem) *Suppose that $\overrightarrow{p}$ is a sequence of degrees which lie below y and $\overrightarrow{p}$ codes the relation $\mathscr{R}$. Letting Y be a representative of y, $\mathscr{R}$ has a presentation which is $\Sigma^0_5(Y)$.*

In the particular case of coding a model of arithmetic with a unary predicate, one can do much better than Σ^0_5.

Theorem 2.6 *For any degree x and representative X of x, there are parameters $\overrightarrow{p}$ such that the following conditions hold.*

- *$\overrightarrow{p}$ codes an isomorphic copy of $\mathbb{N}$ with a unary predicate for X.*
- *$\overrightarrow{p}$ is recursive in $x + 0'$.*

In the other direction, suppose that $\overrightarrow{p}$ is a sequence of degrees below y, and $\overrightarrow{p}$ codes an isomorphic copy of $\mathbb{N}$ together with a unary predicate U. As a direct application of the decoding theorem, for Y a representative of y, U is $\Sigma^0_5(Y)$.

3 Properties of automorphisms of $\mathscr{D}$

3.1 Results of Nerode and Shore

We can apply the coding and decoding theorems to obtain some early results of Nerode and Shore on the global properties of $\mathscr{D}$.

Theorem 3.1 (Nerode and Shore (1980)) *Suppose that $\pi : \mathscr{D} \tilde{\to} \mathscr{D}$. For every degree x, if x is greater than $\pi^{-1}(0')$ then $\pi(x)$ is arithmetic in x.*

Proof: Let Y be a representative of $\pi(x)$. Since $x \geq_T \pi^{-1}(0')$, $Y \geq_T 0'$. By Theorem 2.6, there are parameters $\overrightarrow{p}$ which are recursive in Y such that $\overrightarrow{p}$ codes Y. But then, $\pi(\overrightarrow{p})$ is recursive in x and still codes Y. By the decoding theorem, Y is $\Sigma^0_5(X)$. ■

Theorem 3.2 (Nerode and Shore (1980)) *Suppose $\pi : \mathscr{D} \tilde{\to} \mathscr{D}$ is an automorphism of $\mathscr{D}$ and $x \geq_T \pi^{-1}(0')^{(5)} + \pi^{-1}(\pi(0')^{(5)})$. Then, $\pi(x) = x$. Consequently, π is the identity on a cone.*

Proof: Given x above $\pi^{-1}(0')^{(5)}$, fix y_1 and y_2 so that $y_1 \vee y_2 = x$; $\pi(y_1)$ and $\pi(y_2)$ are greater than $0'$; and $y_1^{(5)}$ and $y_2^{(5)}$ are recursive in x. By Theorem 2.6, each $\pi(y_i)$ can compute a sequence of parameters which codes one of its representatives. The preimages of these parameters are recursive in Y_i. By the decoding theorem, representatives of each $\pi(y_i)$ is $\Sigma^0_5(Y_i)$ and hence recursive in X. Thus, $x \geq_T \pi(y_1) \vee \pi(y_2) = \pi(x)$. By symmetry, $\pi(x) \geq_T x$. ■

The Nerode-Shore theorems pose a challenge. Given an automorphism, where is the base of the cone on which it is the identity? In the 1980's, Jockusch and Shore produced a remarkable sequence of papers on *REA*-operators, with the conclusion that every automorphism of $\mathscr{D}$ is fixed on the cone above the degrees of the arithmetic sets. See Jockusch and Shore (1984). We go a step further and show that every automorphism of $\mathscr{D}$ is fixed on the cone above $0''$.

Our final observation on directly applying the coding lemma is due to Odifreddi and Shore. In brief, the local action of a global automorphism of $\mathscr{D}$ is locally definable.

Theorem 3.3 (Odifreddi and Shore (1991)) *Suppose that π is an automorphism of $\mathscr{D}$ and that $\mathscr{I}$ is an ideal in $\mathscr{D}$ which includes $0'$ such that π restricts to an automorphism of $\mathscr{I}$. For any real I, if there is a presentation of $\mathscr{I}$ which is recursive in I then the restriction of π to $\mathscr{I}$ has a presentation which is arithmetic in I.*

Proof: Code a counting of $\mathscr{I}$ by parameters $\overrightarrow{p}$ which are arithmetic in I. The action of π on $\mathscr{I}$ is determined by the action of π on $\overrightarrow{p}$. Since $0' \in \mathscr{I}$, the Nerode and Shore Theorem implies that $\pi(\overrightarrow{p})$ is arithmetic in I. ■

4 Slaman and Woodin analysis of $Aut(\mathscr{D})$

Until indicated otherwise, we will follow the Slaman and Woodin (2005) analysis of $Aut(\mathscr{D})$.

If $\mathscr{J}$ is an ideal in the Turing degrees (such as $\mathscr{D}$), $\mathscr{I}$ is a countable subideal of $\mathscr{J}$ such that $0' \in \mathscr{I}$ and $\mathscr{J}$ codes a counting of $\mathscr{I}$, and ρ is an automorphism of $\mathscr{J}$ that restricts to an automorphism of $\mathscr{I}$, then $\rho \upharpoonright \mathscr{I}$ is definable from the action of ρ on the parameters which code the counting. Now, suppose that $\rho \upharpoonright \mathscr{I}$ is an automorphism of $\mathscr{I}$ such that for any counting of $\mathscr{I}$, $\rho \upharpoonright \mathscr{I}$ can be extended to an automorphism of an ideal which includes that counting. Then $\rho \upharpoonright \mathscr{I}$ would be definable from that counting. But, if $\rho \upharpoonright \mathscr{I}$ is definable from every counting of $\mathscr{I}$, then $\rho \upharpoonright \mathscr{I}$ is definable from $\mathscr{I}$ itself. If we can apply this analysis to a countable reflection of $\mathscr{D}$, then any automorphism of $\mathscr{D}$ would be definable from the reals, that is be an element of $L[\mathbb{R}]$. We follow just this line of reasoning in this section.

4.1 Persistent automorphisms

Definition 4.1 An automorphism ρ of a countable ideal $\mathscr{I}$ is *persistent* if and only if for every degree x there is a countable ideal $\mathscr{I}_1$ such that the following conditions hold.

- $x \in \mathscr{I}_1$ and $\mathscr{I} \subseteq \mathscr{I}_1$.
- There is an automorphism ρ_1 of $\mathscr{I}_1$ such that the restriction of ρ_1 to $\mathscr{I}$ is equal to ρ.

We will show that ρ is persistent if and only if ρ extends to a automorphism of $\mathscr{D}$. One direction of the equivalence is obvious.

Theorem 4.2 *Suppose that $\pi : \mathscr{D} \tilde{\to} \mathscr{D}$. For any countable ideal $\mathscr{I}$, if π restricts to an automorphism $\pi \upharpoonright \mathscr{I}$ of $\mathscr{I}$ then $\pi \upharpoonright \mathscr{I}$ is persistent.*

Thus, if $\mathscr{D}$ is not rigid, then there is a nontrivial persistent automorphism of some countable ideal in $\mathscr{D}$.

Theorem 4.3 *Suppose that $\rho : \mathscr{I} \tilde{\to} \mathscr{I}$, that $\mathscr{J}$ is a jump ideal contained in $\mathscr{I}$, and that $\rho(0') \vee \rho^{-1}(0') \in \mathscr{J}$. Then $\rho \upharpoonright \mathscr{J}$ is an automorphism of $\mathscr{J}$.*

Proof: The theorem follows from the effective coding and decoding theorems. If $x \in \mathscr{J}$, then $\rho(x)$ is arithmetic in $x \vee \rho^{-1}(0')$, which is also in $\mathscr{J}$. ■

Corollary 4.4 *Suppose that $\mathscr{I}$ is a countable ideal such that $0'$ is an element of $\mathscr{I}$ and suppose that ρ is a persistent automorphism of $\mathscr{I}$. For any countable jump ideal $\mathscr{J}$ extending $\mathscr{I}$, ρ extends to an automorphism of $\mathscr{J}$.*

To prove the corollary, extend ρ to an automorphism of a countable ideal containing some upper bound of $\mathscr{J}$ and apply Theorem 4.3.

Theorem 4.5 *Suppose that $\mathscr{I}$ is a countable ideal in $\mathscr{D}$ such that $0'$ is an element of $\mathscr{I}$. Suppose that there is a presentation of $\mathscr{I}$ which is recursive in I. Finally, suppose that $\mathscr{J}$ is a jump ideal which includes the degree of I and ρ is an automorphism of $\mathscr{J}$ that restricts to an automorphism of $\mathscr{I}$. Then, the restriction $\rho \upharpoonright \mathscr{I}$ of ρ to $\mathscr{I}$ has a presentation which is arithmetic in I.*

Proof: There is a code $\overrightarrow{p}$ for a counting of $\mathscr{I}$ which is arithmetic in I and hence an element of the subideal of $\mathscr{J}$ consisting of the degrees of sets which are arithmetic in I. By Theorem 4.3, ρ restricts to an automorphism of this subideal. Consequently, $\rho(\overrightarrow{p})$ is arithmetic in I. Now, apply the Odifreddi-Shore argument of Theorem 3.3. ■

Corollary 4.6 *Suppose that $\mathscr{I}$ is a countable ideal and $0'$ is an element of $\mathscr{I}$. If ρ is a persistent automorphism of $\mathscr{I}$, then ρ is arithmetically definable in any presentation of $\mathscr{I}$.*

Consequently, persistent automorphisms of $\mathscr{I}$ are locally presented and there are at most countably many of them.

4.2 Persistently extending persistent automorphisms

Theorem 4.7 *Suppose that $\mathscr{I}$ is a countable ideal and $0'$ is an element of $\mathscr{I}$. Suppose that ρ is a persistent automorphism of $\mathscr{I}$. For any countable jump ideal $\mathscr{J}$ which extends $\mathscr{I}$, ρ extends to a persistent automorphism of $\mathscr{J}$.*

Proof: Suppose that $\mathscr{J}$ were a countable jump ideal such that there is no persistent automorphism of $\mathscr{J}$ which extends ρ. Let J compute a presentation of $\mathscr{J}$. Choose x_e so that the eth arithmetic in J extension of ρ to $\mathscr{J}$ cannot be extended further to include x_e. Let x bound the x_e's. By its persistence, extend ρ to an automorphism ρ_1 of the jump ideal generated by x. Then, $\rho_1 \upharpoonright \mathscr{J}$ is arithmetic in J, contradiction. ■

We now draw some conclusions about the complexity of ρ's being persistent.

Theorem 4.8 *The property* I is a representation of a countable ideal $\mathscr{I}$, $0' \in \mathscr{I}$, and R is a presentation of a persistent automorphism ρ of $\mathscr{I}$ *is* Π^1_1.

Proof: ρ is persistent if and only if for every presentation J of a countable jump ideal $\mathscr{J}$ extending $\mathscr{I}$, there is an arithmetic in J extension of ρ to $\mathscr{J}$. This property is Π^1_1. ■

Corollary 4.9 *The properties* R is a presentation of a persistent automorphism *and* There is a countable map $\rho : \mathscr{I} \xrightarrow{\sim} \mathscr{I}$ such that $0' \in \mathscr{I}$, ρ is persistent, and ρ is not equal to the identity *are absolute between well-founded models of ZFC.*

Proof: These properties are Π^1_1 and Σ^1_2, respectively. The corollary then follows from Shoenfield (1961) Absoluteness. ■

4.3 Persistence and Reflection

Let T be the fragment of *ZFC* in which we include only the instances of replacement and comprehension in which the defining formula is Σ_1.

Definition 4.10 Suppose that $\mathfrak{M} = (M, \in^{\mathfrak{M}})$ is a model of T.

1. $\mathfrak{M}$ is an *ω-model* if and only if $\mathbb{N}^{\mathfrak{M}}$ is isomorphic to the standard model of arithmetic.
2. $\mathfrak{M}$ is *well-founded* if and only if the binary relation $\in^{\mathfrak{M}}$ is well-founded. That is to say that there is no infinite sequence $(m_i : i \in \mathbb{N})$ of elements of $\mathfrak{M}$ such that for all i, $m_{i+1} \in^{\mathfrak{M}} m_i$.

Theorem 4.11 *Suppose that $\mathfrak{M}$ is an ω-model of T. Let $\mathscr{I}$ be an element of $\mathfrak{M}$ such that*

$$\mathfrak{M} \models \mathscr{I} \text{ is a countable ideal in } \mathscr{D} \text{ such that } 0' \in \mathscr{I}.$$

Then, every persistent automorphism of $\mathscr{I}$ is also an element of $\mathfrak{M}$.

Proof: $\mathfrak{M}$ is closed under arithmetic definability. ∎

Corollary 4.12 *Suppose that $\mathfrak{M}$ is an ω-model of T and that ρ and $\mathscr{I}$ are elements of $\mathfrak{M}$ such that $0' \in \mathscr{I}$, $\rho : \mathscr{I} \xrightarrow{\sim} \mathscr{I}$, and $\mathscr{I}$ is countable in $\mathfrak{M}$. Then,*

$$\rho \text{ is persistent} \Longrightarrow \mathfrak{M} \models \rho \text{ is persistent.}$$

Proof: Persistent automorphisms extend to larger countable jump ideals persistently. Hence, these extensions belong to $\mathfrak{M}$. ∎

4.4 Generic persistence

We now extend the notion of persistence to uncountable ideals. In what follows, V is the universe of sets and G is a V-generic filter for some partial order in V.

Definition 4.13 Suppose that $\mathscr{I}$ is an ideal in $\mathscr{D}$ and ρ is an automorphism of $\mathscr{I}$. We say that ρ is *generically persistent* if there is a generic extension $V[G]$ of V in which $\mathscr{I}$ is countable and ρ is persistent.

Theorem 4.14 *Suppose that $\rho : \mathscr{I} \xrightarrow{\sim} \mathscr{I}$ is generically persistent. If $V[G]$ is a generic extension of V in which $\mathscr{I}$ is countable, then ρ is persistent in $V[G]$.*

Proof: Generics for any two forcings can be realized simultaneously. By absoluteness, persistence is evaluated consistently in the two generic extensions. ∎

Theorem 4.15 *Suppose that $\pi : \mathscr{D} \xrightarrow{\sim} \mathscr{D}$. Then, π is generically persistent.*

Proof: If not, then the failure of π to be generically persistent would reflect to a countable well-founded model $\mathfrak{M}$. One could then add a generic counting of $\mathscr{D}^{\mathfrak{M}}$ to $\mathfrak{M}$ and obtain $\mathfrak{M}[G]$ in which $\pi \restriction \mathscr{D}^{\mathfrak{M}}$ is not persistent. This would contradict the persistence of $\pi \restriction \mathscr{D}^{\mathfrak{M}}$ and Corollary 4.12. ∎

Theorem 4.16 *Suppose that $V[G]$ is a generic extension of V. Suppose that π is an element of $V[G]$ which maps the Turing degrees in V automorphically to itself (that is, $\pi : \mathscr{D}^V \tilde{\to} \mathscr{D}^V$). If π is generically persistent in $V[G]$, then π is an element of $L(\mathbb{R}^V)$. That is, π is constructible from the set of reals in V.*

Proof: π is generically persistent, so π is arithmetically definable relative to any $V[G]$-generic counting of $\mathscr{D}^V$. Consequently, π must belong to the ground model for such countings, namely $L(\mathbb{R}^V)$. ∎

Theorem 4.17 *Suppose that $\mathscr{I}$ is a countable ideal in $\mathscr{D}$, $0'$ is an element of $\mathscr{I}$, and $\rho : \mathscr{I} \tilde{\to} \mathscr{I}$ is persistent. Then ρ can be extended to an automorphism $\pi : \mathscr{D} \tilde{\to} \mathscr{D}$.*

Proof: ρ can be persistently extended to $\mathscr{D}^V$ in a generic extension of V. By Theorem 4.16, this extension belongs to $L[\mathbb{R}^V]$ and hence to V. ∎

Corollary 4.18 *The statement* There is a non-trivial automorphism of the Turing degrees *is equivalent to a Σ^1_2 statement. It is therefore absolute between well-founded models of ZFC.*

Theorem 4.19 *Let π be an automorphism of $\mathscr{D}$. Suppose that $V[G]$ is a generic extension of V. Then, there is an extension of π in $V[G]$ to an automorphism of $\mathscr{D}^{V[G]}$, the Turing degrees in $V[G]$.*

Proof: There is a persistent extension π_1 of π in any generic extension of $V[G]$ in which $\mathscr{D}^{V[G]}$ is countable. This π_1 belongs to $V[G]$. ∎

4.5 Definability of automorphisms of $\mathscr{D}$

Definition 4.20 Given two functions $\tau : \mathscr{D} \to \mathscr{D}$ and $t : 2^\omega \to 2^\omega$, we say that t *represents* τ if for every degree x and every set X in x, the Turing degree of $t(X)$ is equal $\tau(x)$.

We will analyze the behavior of an automorphism of $\mathscr{D}$ in terms of the action of its extensions on the degrees of the generic reals.

Theorem 4.21 *Suppose that $\pi : \mathscr{D} \tilde{\to} \mathscr{D}$. There is a countable family $\vec{D}$ of dense open subsets of $2^{<\omega}$ such that π is represented by a continuous function f on the set of $\vec{D}$-generic reals.*

Proof: The proof has several steps, which we will sketch. We use $\Pi(Z)$ to denote a representative of $\pi(degree(Z))$.

1. Let $V[\mathscr{G}]$ be a generic extension of V obtained by adding ω_1-many Cohen reals and let π_1 be an extension of π to $\mathscr{D}^{V[\mathscr{G}]}$.

2. Since $\pi_1 \in L[\mathbb{R}^{V[\mathscr{G}]}]$, fix $X \in \mathbb{R}^{V[\mathscr{G}]}$ so that π_1 is ordinal definable from X in $L[\mathbb{R}^{V[\mathscr{G}]}]$. Work in $V[X]$ and note that $V[\mathscr{G}]$ is a generic extension of $V[X]$ obtained by adding ω_1-many Cohen reals. (The forcing factors.)

3. Consider a set G, of degree g, which is Cohen generic over $V[X]$. $\pi_1(g)$ is arithmetically definable relative to g and $\pi^{-1}(0')$. We can find an e and a k such that it is forced that $\pi_1(g)$ is represented by $\{e\}((G \oplus \Pi^{-1}(\emptyset'))^{(k)})$. Since G is Cohen generic, we can assume that e has the form $\{e\}(G \oplus \Pi^{-1}(\emptyset')^{(k)})$. Thus, π_1 is continuously represented on the set of $V[X]$-generic reals.

4. We make an aside to exploit a phenomenon first observed by Jockusch and Posner (1981). For any $\overrightarrow{D}$, the $\overrightarrow{D}$-generic degrees generate $\mathscr{D}$ under meet and join. We fix a mechanism by which this coding can be realized.

Let G and Y be given. Let G_{even} and G_{odd} be the even and odd parts of G. Construct $\mathbb{C}(Y,G)$ by injecting the values of Y into G_{even} at those places where G_{odd} is not zero. That is, we shuffle the bitstreams of G_{even} and Y like a deck of playing cards and use G_{odd} to determine the points at which the cards in the Y half of the deck are inserted between the cards in the G_{even} half of the deck.

Lemma 4.22 *If G_{odd} is infinite, then $\mathbb{C}(Y,G) \oplus G \equiv_T Y \oplus G$.*

Lemma 4.23 *For any dense open subset of $2^{<\omega}$, D, there is a dense open set D^*, such that for all D^*-generic G and all Y, $\mathbb{C}(Y,G)$ is D-generic. In particular, for all G, Y, and Z, if G is generic over $V[Z]$, then so is $\mathbb{C}(Y,G)$.*

Definition 4.24 For $Y \in 2^{\omega}$, let (Y) denote the set $\{Z : Z \leq_T Y\}$.

Let Y be given with Turing degree y, and let G_1 and G_2 be mutually Cohen generic over $V[X \oplus Y]$. We can write the ideal generated by Y as the meet of joins of generic ideals.

$$\begin{aligned}(\mathbb{C}(Y,G_1) \oplus G_1) \cap (\mathbb{C}(Y,G_2) \oplus G_2) &= (Y \oplus G_1) \cap (Y \oplus G_2) \\ &= (Y)\end{aligned}$$

Thus, as Jockusch and Posner observed, the degrees of the generic sets generate the Turing degrees under meet and join.

5. The previous equality is preserved by π_1, as represented on generic reals.

$$\begin{aligned}&\{Z : \text{the degree of } Z \text{ belongs to } \left(\pi_1(y)\right)\} = \\ &\qquad \left(\{e\}(\mathbb{C}(Y,G_1) \oplus \Pi^{-1}(\emptyset')^{(k)}) \oplus \{e\}(G_1 \oplus \Pi^{-1}(\emptyset')^{(k)})\right) \\ &\qquad\qquad \bigcap \left(\{e\}(\mathbb{C}(Y,G_2) \oplus \Pi^{-1}(\emptyset')^{(k)}) \oplus \{e\}(G_2 \oplus \Pi^{-1}(\emptyset')^{(k)})\right)\end{aligned}$$

When Y is also generic:

$$\begin{aligned}&\left(\{e\}(Y \oplus \Pi^{-1}(\emptyset')^{(k)})\right) = \\ &\qquad \left(\{e\}(\mathbb{C}(Y,G_1) \oplus \Pi^{-1}(\emptyset')^{(k)}) \oplus \{e\}(G_1 \oplus \Pi^{-1}(\emptyset')^{(k)})\right) \\ &\qquad\qquad \bigcap \left(\{e\}(\mathbb{C}(Y,G_2) \oplus \Pi^{-1}(\emptyset')^{(k)}) \oplus \{e\}(G_2 \oplus \Pi^{-1}(\emptyset')^{(k)})\right)\end{aligned}$$

This exhibits the desired representation of π on generic reals Y. ■

Sharper results will follow, but we can obtain some preliminary information concerning the definability of π from what we already know.

Corollary 4.25 *Suppose $\pi : \mathscr{D} \tilde{\to} \mathscr{D}$. Then π has a Borel representation; in fact, π has a representation that is arithmetic in the real parameter $\Pi^{-1}(\emptyset')$.*

Next in this line is the proof that for any Z and any sufficiently generic real G, $(G \oplus Z)'' \geq_T \Pi(Z)$. The proof uses two facts. First, for any countable collection of dense open sets $\overrightarrow{D}$, there is another collection $\overrightarrow{D}^*$ such that if G^* is $\overrightarrow{D}^*$-generic then there is a $\overrightarrow{D}$-generic G with $\Pi(G^*) \geq_T G$. Second, for any set X, there is an X-recursive partial order P such that the degrees of any sufficiently generic sets for P form parameters to code X. The coding is sufficiently effective that X is recursive in the double-jump of any upper bound on these parameters. We leave the details of the proof to Slaman and Woodin (2005).

Theorem 4.26 *Suppose $\pi : \mathscr{D} \tilde{\to} \mathscr{D}$. For every $z \in \mathscr{D}$, $z'' \geq_T \pi(z)$.*

Proof: Let Z be given. Fix $\overrightarrow{D}$ so that for any $\overrightarrow{D}$-generic G, $\Pi(Z) \oplus G$ can compute a generic set for the partial order to produce parameters which code $\Pi(Z)$. Fix $\overrightarrow{D}^*$ so that if G^* is $\overrightarrow{D}^*$-generic, then $\Pi(G^*)$ computes a $\overrightarrow{D}$-generic. Let G^* be $\overrightarrow{D}^*$-generic and let G be a $\overrightarrow{D}$-generic recursive in $\Pi(G^*)$. The coding is preserved by π, so we may conclude that $degree(\Pi(Z)) \leq_T \pi^{-1}(degree(\Pi(Z) \oplus G))''$. Hence $\Pi(Z) \leq_T (Z \oplus G^*)''$, and so $\Pi(Z) \leq_T Z'' \oplus G^*$. G^* was any sufficiently generic, so $\Pi(Z) \leq_T Z''$. ■

Corollary 4.27 *Suppose $\pi : \mathscr{D} \tilde{\to} \mathscr{D}$. For any 2-generic set G,*

$$degree(G) \vee 0'' \geq_T \pi(degree(G)).$$

Theorem 4.28 *Suppose that $\pi : \mathscr{D} \tilde{\to} \mathscr{D}$.*

- *For all $x \in \mathscr{D}$, $x \vee 0'' \geq_T \pi(x)$.*
- *For all $x \in \mathscr{D}$, if $x \geq_T 0''$ then $x = \pi(x)$.*

Proof: A degree above $0''$ can be written as a join of 2-generic degrees. ■

Theorem 4.29 *Suppose that $\pi : \mathscr{D} \tilde{\to} \mathscr{D}$.*

- *There is a recursive functional $\{e\}$ such that for all G, if G is 5-generic, then $\pi(degree(G))$ is represented by $\{e\}(G, \emptyset'')$.*
- *There is an arithmetic function $F : 2^\omega \to 2^\omega$ such that for all $X \in 2^\omega$, $\pi(degree(X))$ is represented by $F(X)$.*

Proof: Replay the proof of Theorem 4.21, using the new information that $\pi(degree(G))$ is recursive in G''. Conclude that there is a fixed reduction which works for all 5-generic G's. Since the 5-generics generate $\mathscr{D}$, the representation on 5-generics propagates to an arithmetic representation everywhere. ■

Theorem 4.30 *$Aut(\mathscr{D})$ is countable.*

Theorem 4.31 *If g is 5-generic and $\pi : \mathscr{D} \tilde{\to} \mathscr{D}$, then π is determined by its action on g.*

Proof: If G is 5-generic, then $\{e\}(G, \emptyset'') \equiv_T G$ if and only if the same is true for all 5-generics. ■

4.6 Invariance of the double jump

The efficient coding that lies behind the proof of Theorem 4.26 can be sharpened not just to produce parameters that code z but rather to produce parameters that code z''.

Theorem 4.32 *For every $Z \subseteq \omega$, there is a countable family of dense open sets $\overrightarrow{D}$ such that such that for all $\overrightarrow{D}$-generic G, $\pi(degree(Z \oplus G))'' \geq_T degree(Z'')$*

Theorem 4.33 *Suppose that $\pi : \mathscr{D} \tilde{\to} \mathscr{D}$. For all $z \in \mathscr{D}$, $z'' = \pi(z)''$.*

Theorem 4.34 *The relation $y = x''$ is invariant under π.*

Proof: Suppose that $y = x''$. Since $y \geq_T 0''$, $\pi(y) = y$. By the previous theorem, $x'' = \pi(x)''$. Consequently, $\pi(y) = \pi(x)''$. By the same argument applied to π^{-1}, if $\pi(y) = \pi(x)''$ then $y = x''$. ■

5 Definability in $\mathscr{D}$

5.1 Bi-interpretability

Definition 5.1 An *assignment* of reals consists of

- A countable ω-model $\mathfrak{M}$ of T ($T = \Sigma_1$-*ZFC*).
- A function f and a countable ideal $\mathscr{I}$ in $\mathscr{D}$ such that $f : \mathscr{D}^{\mathfrak{M}} \to \mathscr{I}$ surjectively and for all x and y in $\mathscr{D}^{\mathfrak{M}}$, $\mathfrak{M} \models x \geq_T y$ if and only if $f(x) \geq_T f(y)$ in $\mathscr{I}$.

An assignment of reals to an ideal $\mathscr{I}$ is a representation of an isomorphism between the Turing degrees of the reals in an ω-model $\mathfrak{M}$ and the elements of $\mathscr{I}$. We can work with countable assignments within $\mathscr{D}$, via the coding lemma, and we can investigate which assignments extend, just as we did with persistent automorphism.

Definition 5.2 For assignments $(\mathfrak{M}_0, f_0, \mathscr{I}_0)$ and $(\mathfrak{M}_1, f_1, \mathscr{I}_1)$, $(\mathfrak{M}_1, f_1, \mathscr{I}_1)$ *extends* $(\mathfrak{M}_0, f_0, \mathscr{I}_0)$ if and only if

- $\mathscr{D}^{\mathfrak{M}_0} \subseteq \mathscr{D}^{\mathfrak{M}_1}$,
- $\mathscr{I}_0 \subseteq \mathscr{I}_1$,
- and $f_1 \restriction \mathscr{D}^{\mathfrak{M}_0} = f_0$.

Definition 5.3 An assignment $(\mathfrak{M}_0, f_0, \mathscr{I}_0)$ is *extendable* if

$$\forall z_1 \exists (\mathfrak{M}_1, f_1, \mathscr{I}_1) \left[\begin{array}{l} (\mathfrak{M}_1, f_1, \mathscr{I}_1) \text{ extends } (\mathfrak{M}_0, f_0, \mathscr{I}_0)),\ z_1 \in \mathscr{I}_1, \text{ and} \\ \quad \forall z_2 \exists (\mathfrak{M}_2, f_2, \mathscr{I}_2) \\ \quad \left(\begin{array}{l} (\mathfrak{M}_2, f_2, \mathscr{I}_2) \text{ extends } (\mathfrak{M}_1, f_1, \mathscr{I}_1),\ z_2 \in \mathscr{I}_2, \text{ and} \\ \forall z_3 \exists (\mathfrak{M}_3, f_3, \mathscr{I}_3) \left[\begin{array}{c} (\mathfrak{M}_3, f_3, \mathscr{I}_3) \text{ extends} \\ (\mathfrak{M}_2, f_2, \mathscr{I}_2) \text{ and } z_3 \in \mathscr{I}_3 \end{array} \right] \end{array} \right) \end{array} \right]$$

Theorem 5.4 *If $(\mathfrak{M}, f, \mathscr{I})$ is an extendable assignment, then there is a $\pi : \mathscr{D} \tilde{\to} \mathscr{D}$ such that for all $x \in \mathscr{D}^{\mathfrak{M}}$, $\pi(x) = f(x)$.*

Proof: One can compare ideals $\mathcal{D}^{\mathfrak{M}}$ and $\mathcal{I}$ by considering the sets that are coded within them. Sets coded in the range $\mathcal{I}$ belong to the domain $\mathfrak{M}$. Sets in the domain which together with $0'$ can only code elements of $\mathfrak{M}$ must belong to $\mathfrak{M}$.

One shows that if $(\mathfrak{M}, f, \mathcal{I})$ is an extendable assignment, then $f : \mathcal{D}^{\mathfrak{M}} \to \mathcal{I}$ extends to a persistent automorphism of a larger ideal. Hence, it extends to an automorphism of $\mathcal{D}$. ■

Theorem 5.5 *If g is the Turing degree of an arithmetically definable 5-generic set, then the relation $R(\vec{c}, d)$ given by*

$$R(\vec{c}, d) \iff \vec{c} \text{ codes a real } D \text{ and } D \text{ has degree } d$$

is definable in $\mathcal{D}$ from g.

This is the internal realization of the previous result that every automorphism is determined by its action on g.

Corollary 5.6 *Suppose that R is a relation on $\mathcal{D}$. The following conditions are equivalent.*

- *R is induced by a projective, degree invariant relation R_{2^ω} on 2^ω.*
- *R is definable in $\mathcal{D}$ using parameters.*

Proof: $\vec{x}$ satisfies R if and only if there is a correct assignment of representatives to degrees such that $f(degree(\vec{Y})) = \vec{x}$, and $\vec{Y}$ satisfies R_{2^ω}. (The correctness of the assignment is defined using the arithmetic 5-generic of the previous theorem.) ■

Theorem 5.7 *Suppose that R is a relation on $\mathcal{D}$. The following conditions are equivalent.*

- *R is induced by a relation R_{2^ω} on 2^ω such that the following conditions hold.*
  - *R_{2^ω} is definable in second order arithmetic and degree invariant.*
  - *R_{2^ω} is preserved by $Aut(\mathcal{D})$.*
- *R is definable in $\mathcal{D}$.*

Proof: $\vec{x}$ satisfies R if and only if there is an extendable assignment such that $f(degree(\vec{Y})) = \vec{x}$ and $\vec{Y}$ satisfies R_{2^ω}. ■

Definition 5.8 $\mathcal{D}$ is *bi-interpretable with second order arithmetic* if and only if the relation on $\vec{c}$ and d given by

$$R(\vec{c}, d) \iff \vec{c} \text{ codes a real } D \text{ and } D \text{ has degree } d$$

is definable in $\mathcal{D}$.

Theorem 5.9 *The following are equivalent.*

- *$\mathcal{D}$ is bi-interpretable with second order arithmetic.*
- *$\mathcal{D}$ is rigid.*

Conjecture 5.10 (Slaman and Woodin (2005)) *$\mathcal{D}$ is bi-interpretable with second order arithmetic.*

The Bi-interpretability Conjecture, if true, reduces all the logical questions that one could ask of $\mathcal{D}$ to the exact same questions about second order arithmetic. The structures would be logically identical, though presented in different first order languages.

6 The Turing Jump

Theorem 6.1 *The function $x \mapsto x''$ is definable in $\mathcal{D}$.*

Proof: We have already shown that the relation $y = x''$ is invariant under all automorphisms of $\mathcal{D}$. It is clearly degree invariant and definable in second order arithmetic. Therefore, it is definable in $\mathcal{D}$. ■

We now turn to showing that $x \mapsto x'$ is definable in $\mathcal{D}$. This is an account of work appearing in Shore and Slaman (1999).

We show that (Δ^0_2), the ideal of Δ^0_2 degrees, is definable in $\mathcal{D}$. Our definition is based on the following join theorem for the double-jump.

Theorem 6.2 (Shore and Slaman, 1999) *For $A \in 2^\omega$, the following conditions are equivalent.*

- *A is not recursive in $0'$.*
- *There is a $G \in 2^\omega$ such that $A \oplus G \geq_T G''$.*

Theorem 6.2 is an extension of the Posner and Robinson (1981) Theorem that for every nonrecursive degree A there is a G such that $A \oplus G \geq_T G'$. The proof uses a notion of forcing introduced by Kumabe and Slaman.

By Theorem 6.2, (Δ^0_2) is definable in terms of order, join, and the double jump. Consequently, it is definable in $\mathcal{D}$.

Theorem 6.3 *The functions $a \mapsto (\Delta^0_2(a))$ and $a \mapsto a'$ are definable in $\mathcal{D}$.*

Proof: By relativizing the previous theorem. For each degree a and each d greater than or equal to a, d is not Δ^0_2 relative to a if and only if there is an x greater than or equal to a such that $d + x \geq_T x''$. Again, the double jump is definable in $\mathcal{D}$, and this equivalence provides first order definitions as required. ■

Recently, Shore (2007) has produced an alternate proof of Theorem 6.3. Shore's proof replaces the definition of the double jump obtained from the analysis that we have given for $Aut(\mathcal{D})$ with one that is more arithmetically based. He then argues as here that the definability of the double jump implies the definability of the Turing jump.

6.1 Recursive enumerability

We have already stated the Bi-interpretability Conjecture, which we regard as the central question concerning the global structure of the Turing degrees. As we have seen with the Turing jump, specialized definability results are possible even without settling the conjecture. We suggest the following.

Question 6.4 *Is the relation y* is recursively enumerable relative to *x definable in $\mathcal{D}$?*

A positive answer would follow from a proof of the Bi-interpretability Conjecture. Conceivably, it could also lead to a proof.

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