Curavature and the Evolution of Fronts

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ABSTRACT

The evolution of a front propagating along its normal vector field with curvature dependent speed is considered. We define an "energy-like" quantity of the moving front, the total variation, and prove a general result relating the growth/decay of this energy to the speed. We then study a front moving with speed $1-\varepsilon K$, where ε is a constant and K is the curvature, and show that the curvature term plays a smoothing role in the solution similar to that of viscosity in Burgers equation. For ε=0, the solution is seen to blow up, differentiability is lost, and an entropy condition can be formulated to provide an explicit construction for a weak solution beyond blow up time. We numerically solve the equations of motion and show that the solution converges to the constructed weak solution as the curvature smoothing term vanishes. Corners that develop in the propagating front swallow variation in the solution, providing a discontinuous stabilizing mechanism. Finally, we discuss the difficulties involved in numerically solving such problems and describe a possible remedy is described.

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I. Introduction

In this paper we study the evolution of a front propagating along its normal vector field with speed function dependent on the local curvature. We define an "energy-like" quantity of the moving front, the total variation, and prove a general result relating the growth/decay of this energy to the speed. We then study the special case of a front moving with speed $1-\varepsilon K$, where ε is a constant and K is the curvature, and show that the curvature term plays a smoothing role in the solution similar to that of viscosity in Burgers equation. For $\varepsilon=0$, in which case the front moves at constant speed, we show that the solution blows up, differentiability is lost, and an entropy condition can be formulated to provide an explicit construction for a weak solution beyond blow up time. We then solve numerically the equations of motion and show that as $\varepsilon\to 0$ and the curvature smoothing term vanishes, the solution converges to our constructed weak solution. We show that corners that develop in the propagating front swallow variation in the solution, providing a discontinuous stabilizing mechanism. Finally, we discuss the difficulties involved in numerically solving such problems, and describe one remedy.

First, we briefly describe the physical motivation behind our interest.

Crystal Growth

A relatively straightforward example of the mathematical issues pertinent to our study is the growth of a solid immersed in a supercooled liquid, discussed extensively in Langer [9]. To illustrate, we imagine a solid ice crystal placed in a bath of water that has been supercooled below its freezing point. We limit ourselves to two dimensions and neglect a variety of effects such as convective heat transport, solid impurities and crystalline anisotropy. Let $\gamma(t)$ be the closed curve representing the boundary between ice and water at time t, with ice inside the curve. The diffusion equation for the temperature T holds both inside and outside $\gamma(t)$, namely, $\partial T_{s(l)}/\partial t = c_{s(l)}\,\nabla^2 T_{s(l)}$, where ∇^2 is the two-dimensional Laplacian, $c_{s(l)}$ is the thermal diffusion coefficient and $T_{s(l)}$ is the temperature in the solid(s)/liquid(l) region. Conservation of heat flux across the boundary interface

must include the heat required to go from solid to liquid, thus $c_s\ \partial T_s/\partial n - c_l\ \partial T_l/\partial n = L\ V_\gamma$, where $\partial/\partial n$ is differentiation in the normal direction, L is the latent heat of formation and V_γ is the velocity of the boundary $\gamma(t)$ along its normal vector field. Finally, the thermodynamic boundary condition, which includes the effects of surface tension, is given by the Gibbs-Thomson relation, see Turnbull [20]; at each point $\bar{x}$ of the boundary $\gamma(t)$, we require that

$$T(\bar{x}) = T_M(1 - \varepsilon K(\bar{x}))$$

where T_M is the melting temperature, ε is a constant and $K(\bar{x})$ is the curvature of the boundary $\gamma(t)$ at $\bar{x}$. Thus, if points of negative curvature are concave towards the solid region, they yield a higher temperature than those with positive curvature and this variation in the solidification rate along the boundary as a function of curvature provides intricate growth patterns.

The above equations have been studied extensively, with much work aimed at analyzing the stability/instability of the interface to small perturbations and ascertaining the existence of stable, that is, morphologically invariant states. Using linear stability theory, Mullins and Sekerka [13] showed that a growing sphere was unstable to perturbations greater than a critical size; the effect of surface diffusion on the precipitate (solid) surface was included in the stability study of Nichols and Mullins [14]. A full discussion of the theory of interface stability may be found in Pamphlin [16]; others examples of stability studies in crystal growth may be found in [9] and [10].

Flame Propagation

Much work surrounding the stability/instability of a flame is aimed at understanding the "turbulization" or wrinkling of a flame front and its interaction with the hydrodynamic flow field. Once again, liner stability theory has played a major role in many of these investigations. The pioneering work in this field is the analysis of a plane flame front by Landau [8]. The flame front is idealized as a surface of discontinuity, i.e., a closed curve $\gamma(t)$, separating regions of constant steady-state velocity, density and temperature. In Landau's model, the flame speed V_γ of the curve along its normal vector field is constant. By ignoring all but hydrodynamic effects, flames are shown to be unstable to perturbations in velocity and pressure around a mean state. Since this conclusion is physically unreasonable, Markstein [11] postulated that the flame speed depended on the curvature so that

$$V_\gamma = V^o(1 - \varepsilon K(\gamma))$$

where V^o is the constant speed of a flat flame, ε is a constant and $K(\gamma)$ is the curvature. The motivation behind such an assumption, loosely speaking, is that parts of the flame front which bend in towards the hot burnt region are subject to greater heat and hence burn faster, slower flame speeds are thus implied for fingers reaching out into cool gas. Using linear stability analysis of this model, Markstein demonstrated the stabilizing effect of curvature. Since then, there have been numerous investigations of flame stability for a variety of combustion models; a comprehensive though now outdated account may be found in Markstein [12], here, we also mention the work of Sivashinsky [19], Frankel and Sivashinsky [4] and Zeldovich [21].

Although linear stability analysis is a powerful tool for analyzing front evolution questions, there are some limitations. Most notably, such an analysis assumes that the solution or perturbation remains smooth, thus ruling out the possibility of discontinuities in the solution as a stabilizing or destabilizing effect. As Markstein points out [12], results are valid only in the limit as the amplitude of the perturbations goes to zero, and there may be steady-state regions of linear instability.

Furthermore, there are phenomena so fundamentally non-linear that they do not submit to a linearized analysis and thus the sympathetic response across all modes to a finite amplitude perturbation cannot be captured.

Motivated by the above discussion, in this paper we take a geometrical approach to the idealized problem of the motion of a closed curve $\gamma(t)$ along its normal vector field with speed a function of curvature. Our goal is to analyze the evolution of the front and study the issues of stability/instability, breakdown of solution and long-time steady states.

II. General Results

A. Equations of Motion

Starting with a simple, smooth, closed initial curve $\gamma(0)$ in R^2, let $\gamma(t)$ be the one parameter family of curves, where $t \in [0,\infty)$ is time, generated by moving the initial curve along its normal vector field with speed F a given function of the curvature. Let $\vec{x}(s,t)$ be the position vector which, at time t, parameterizes $\gamma(t)$ by s, $0 \le s \le S$, $\vec{x}(0,t) = \vec{x}(S,t)$. We assume the curve is parameterized so that the interior is on the left as we travel along the curve in the direction of increasing s. With $K(s,t)$ as the curvature at $\vec{x}(s,t)$, the equations of motion are

$$\vec{n}(s,t) \cdot \frac{\partial \vec{x}(s,t)}{\partial t} = F(K(s,t)) \tag{2.1}$$

$$\vec{x}(s,0) = \gamma(0) \qquad \text{prescribed}$$

$$s \in [0,S] \qquad t \in [0,\infty)$$

where $\vec{n}(s,t)$ is the unit normal vector at $\vec{x}(s,t)$.

Written in terms of the coordinates $\vec{x}(s,t) = (x(s,t),y(s,t))$, an equivalent formulation which will prove useful later is

$$x_t = F\left(\frac{y_{ss}x_s - x_{ss}y_s}{(x_s^2 + y_s^2)^{3/2}}\right)\frac{y_s}{(x_s^2 + y_s^2)^{1/2}} \tag{2.2}$$

$$y_t = -F\left(\frac{y_{ss}x_s - x_{ss}y_s}{(x_s^2 + y_s^2)^{3/2}}\right)\frac{x_s}{(x_s^2 + y_s^2)^{1/2}} \tag{2.3}$$

$$(x(s,0),y(s,0)) = \gamma(0) \qquad 0 \le s \le S\,.$$

Here, the curvature $K(s,t)$ is expressed in terms of partials of x and y. One can easily check that $(x_t,y_t)\cdot(x_s,y_s) = 0$ (motion along normals) and that $(x_t^2 + y_t^2)^{1/2} = F(K)$ (speed function).

In the above equations of motion, s cannot be arclength. If we let $\alpha(s)$ correspond to arclength, then

$$d\alpha = g(s,t)ds \tag{2.4}$$

where $g(s,t) = (x_s^2 + y_s^2)^{1/2}$. Using the easily derived identities

$$x_s y_{ts} + y_s y_{ts} = \frac{F(K)\,K}{x_s^2 + y_s^2} \tag{2.5}$$

$$x_s y_{ts} - y_s x_{ts} = -\frac{\partial F(K)}{\partial s}\,(x_s^2 + y_s^2)^{1/2} \tag{2.6}$$

one can produce an evolution equation for the metric g, namely

$$g_t(s,t) = g(s,t)\,K(s,t)\,F(K(s,t))\,. \tag{2.7}$$

and an evolution equation for the curvature K, namely

$$K_t(s,t) = -\Big[F_s(K(s,t))\,g^{-1}(s,t)\Big]_s\,g^{-1}(s,t) - K^2(s,t)\,F(K(s,t)) \tag{2.8}$$

These two equations form the foundation for our investigations.

We point out here that the case in which the speed of the front is exactly equal to the curvature ($V_\gamma = K(\gamma)$) occurs in the modeling of grain boundaries in metals and has been studied extensively by Brakke [1]. In that work, the curvature is the mean curvature of the moving surface and the sign is chosen so that the surface moves inward when the curvature is positive. Recently, it has been shown (Huisken [7]) that a convex surface remains smooth as it collapses to a single point; to the best of our knowledge, the question of smoothness vs. singularities in the moving surface for a

non-convex surface remains open.

B. Decay of Total Variation - Smoothing of Solution

In this section, we show how the change in the total variation of the solution, which measures the "energy" in the propagating front, depends on the speed function $F(K)$, and point out the mechanism behind the formation of a singularity.

Let $Var(t)$ be the total variation of the front at time t, defined as

$$Var(t) = \int_0^S |K(s,t)| \, g(s,t) \, ds \, . \tag{2.9}$$

The following proposition relates the change in total variation in time to the speed function $F(K)$.

Proposition 1 Consider a front moving along its normal vector field with speed $F(K)$, as in Equation (2.1). Assume that the initial curve $\gamma(0)$ is non-convex, so that $K(s,0)$ changes sign, and assume K is zero at a finite number of points. Assume that F is twice differentiable, and that $K(s,t)$ is twice differentiable for $0 \le s \le S$ and $0 \le t \le T$. Then

1) if $F_K \le 0$ ($F_K \ge 0$) wherever $K = 0$, then

$$\frac{dVar(t)}{dt} \le 0 \quad (\frac{dVar(t)}{dt} \ge 0)$$

2) if $F_K < 0$ ($F_K > 0$) and $K_s \neq 0$ wherever $K = 0$, then

$$\frac{dVar(t)}{dt} < 0 \quad (\frac{dVar(t)}{dt} > 0)$$

$$\text{for} \quad 0 \le t \le T \, .$$

Remarks: Proposition 1 states that if $F_K < 0$ wherever $K = 0$, then the total variation decreases as the front moves and the front "smooths out" - the energy of the front dissipates. We have assumed that the front remains smooth in the interval $0 \le t \le T$ (the curvature is assumed to be twice differentiable). In section III we discuss what happens if the front ceases to be smooth and develops a corner.

We point out that in the special case that $\gamma(t)$ is convex for all t, Proposition 1 is trivial, since $Var(t)=\int_0^S Kgds=2\pi$.

Proof of Proposition 1:

From Equations (2.7-2.8), we have

$$g_t = g\ K\ F \tag{2.10}$$

$$K_t = -g^{-1}\,[g^{-1}\,F_s]_s - K^2\,F\,. \tag{2.11}$$

Let $s_1(t)$, $s_2(t)$, $s_3(t)$, ... $s_n(t)$ be the values of s for which K changes sign at time t. That is, $K(s_i(t),t) = 0$ and K changes sign going from $s<s_i(t)$ to $s>s_i(t)$. Assume $K>0$ in $(s_1(t), s_2(t))$, $(s_3(t), s_4(t))$, ... $(s_{n-1}(t), s_n(t))$ and $K<0$ in $(s_2(t), s_3(t))$, $(s_4(t), s_5(t))$, ... $(s_n(t)< s_1(t)+S)$. (Here, when we write $s_1(t)+S$, we use the fact that the parameterization is of period S.) Then

$$Var(t) = \int_0^S |K|\ g\ ds$$

$$= \int_{s_1(t)}^{s_2(t)} Kgds + \int_{s_3(t)}^{s_4(t)} Kgds + \cdots + \int_{s_{n-1}(t)}^{s_n(t)} Kgds$$

$$- \int_{s_2(t)}^{s_3(t)} Kgds - \int_{s_4(t)}^{s_5(t)} Kgds - \cdots - \int_{s_n(t)}^{s_1(t)+S} Kgds\,.$$

We wish to evaluate $\frac{dVar(t)}{dt}$. For simplicity, we shall assume K changes sign only at two points, $s_1(t)$ and $s_2(t)$, and write

$$Var(t) = \int_{s_1(t)}^{s_2(t)} Kgds - \int_{s_2(t)}^{s_1(t)+S} Kgds$$

Then

$$\frac{dVar(t)}{dt} = \int_{s_1(t)}^{s_2(t)} (Kg)_t\,ds - \int_{s_2(t)}^{s_1(t)+S} (Kg)_t\,ds$$

$$+ K(s_2(t),t)\, g(s_2(t),t)\, s_2'(t) - K(s_1(t),t)\, g(s_1(t),t) s_1'(t))$$

$$- K(s_1(t)+S,t)\, g(s_1(t)+S,t)\, (s_1(t)+S)' + K(s_2(t),t)\, g(s_2(t),t) s_2'(t)$$

where both the subscript t and the prime refer to differentiation with respect to t. By assumption, $K(s_1(t),t) = K(s_2(t),t) = K(s_1(t)+S,t) = 0$, thus

$$\frac{dVar(t)}{dt} = \int_{s_1(t)}^{s_2(t)} (K_t g + K g_t)\, ds - \int_{s_2(t)}^{s_1(t)+S} (K_t g + K g_t)\, ds \tag{2.12}$$

Using Equations (2.10-2.11), we have

$$\frac{dVar(t)}{dt} = \tag{2.13}$$

$$\int_{s_1(t)}^{s_2(t)} (-(g^{-1}F_s)_s - K^2 F g + g K^2 F) ds - \int_{s_2(t)}^{s_1(t)+S} (-(g^{-1}F_s)_s - K^2 F g + g K^2 F)\, ds$$

$$= -\int_{s_1(t)}^{s_2(t)} (g^{-1}F_s)_s\, ds + \int_{s_2(t)}^{s_1(t)+S} (g^{-1}F_s)_s\, ds$$

$$= -\left[g^{-1}F_s \,\middle|_{s_2(t)} - g^{-1}F_s \,\middle|_{s_1(t)} \right] + \left[g^{-1}F_s \,\middle|_{s_1(t)+S} - g^{-1}F_s \,\middle|_{s_2(t)} \right]$$

$$= -2\, (g^{-1}F_K K_s) \,\middle|_{s_2(t)} + 2\, (g^{-1}F_K K_s) \,\middle|_{s_1(t)} \tag{2.14}$$

By assumption, $K>0$ for $s_1(t)<s<s_2(t)$, hence

$$K_s \,\middle|_{s_1(t)} \geq 0 \quad \text{and} \quad K_s \,\middle|_{s_2(t)} \leq 0$$

Assume $F_K \geq 0$ at $K = 0$. Then, since $g^{-1} > 0$, both terms of the right hand side of Equation (2.14) are non-negative and $\frac{dVar(t)}{dt} \geq 0$. Conversely, if $F_K \leq 0$ at $K = 0$, then both terms are non-positive and $\frac{dVar(t)}{dt} \leq 0$. If F_K is strictly less or greater than zero and $K_s \neq 0$ at $s_1(t)$ and $s_2(t)$, then the energy inequalities are also strict. This completes the proof.

By examining the case $F_K \leq 0$ and $F_K \geq 0$, we have the following, which applies to a front moving along its normal vector field at constant speed.

Corollary If $F_K = 0$, (front moves at constant speed), then the total variation is constant.

Consider a speed function of the form

$$F(K) = 1 - \varepsilon K \tag{2.15}$$

where ε is a constant. (This expression might be the first two terms in an expansion For $\varepsilon = 0$, $F_K = 0$ and by the Corollary, $Var(t)$ is constant. Thus, the "energy" of the curve is constant for $\varepsilon = 0$, and decreases for $\varepsilon > 0$ through the diffusion term εK.

The reason for labelling this a diffusion term can be seen from examining the curvature evolution equation (2.8). With speed function $F(K) = 1 - \varepsilon K$, we have

$$K_t = \varepsilon K_{\alpha\alpha} + \varepsilon K^3 - K^2 \tag{2.16}$$

where here we have changed variables and taken the derivative of curvature with respect to arclength to eliminate the metric g. Equation (2.16) is a reaction-diffusion equation. The effect of the reaction term ($\varepsilon K^3 - K^2$), which can cause the solution to blow up, is mitigated by the diffusion term ($\varepsilon K_{\alpha\alpha}$), which smooths the solution. Indeed, if we consider the case ε=0, in which the front moves at constant speed, we have

$$K_t = -K^2 \tag{2.17}$$

which has solution

$$K(s,t) = \frac{K(s,0)}{1+K(s,0)t} \tag{2.18}$$

which clearly blows up if $(K(s,0))$ is anywhere negative.

What happens to the propagating front when the curvature evolution equation produces blow up in the curvature? A simple example shows that the front develops a "corner"; a point where the curve is no longer differentiable. Consider the initial curve

$$\gamma(0) = (s, s^2) \qquad -\infty < s < \infty \tag{2.19}$$

The above parabola is, of course, not a closed curve, but will demonstrate simply the singularity. The solution to the propagation equations with $F(K)=1$ and with the above initial data is

$$x(s,t) = \frac{2s}{(4s^2+1)^{1/2}}\ t + s \qquad y(s,t) = \frac{1}{(4s^2+1)^{1/2}}\ t + s^2 \tag{2.20-2.21}$$

A calculation shows that although the initial curve is smooth, at $t=1/2$, a section of the propagating front collapses to a point, the front loses its differentiability and a corner develops. Indeed if one continues the solution and plots Equations (2.20-2.21) for later times, the front "passes through itself".

We have thus shown that an energy-like quantity of the propagating front, the total variation, decreases for $\varepsilon>0$ and is constant for $\varepsilon=0$ for a non-convex initial curve. While it reasonable to expect that the propagating front remains smooth for $\varepsilon>0$, due to the presence of a diffusion-like term in (2.16), for $\varepsilon=0$ a section of the propagating front collapses to a point and loses differentiability. This situation is analogous to the development of shocks in hyperbolic conservation laws. Consider Burgers equation with viscosity, namely

$$u_t + u\, u_x = \varepsilon\, u_{xx} \tag{2.22}$$

It is well-known, see [6] that, for $\varepsilon=0$, shock discontinuities can develop in the solution, even for smooth initial data. A typical example is initial data

$$u(x,0) = \left\{ \begin{array}{ll} 1 & x<0 \\ 1-x & 0\le x\le 1 \\ 0 & 1\le x \end{array} \right\} \tag{2.23}$$

With $\varepsilon=0$, the characteristics for Equations (2.22-2.23) are straight lines (in the $x-t$ plane) along which the solution u is constant. Although the initial data is continuous, at $t=1$ the characteristics collide and a shock develops. Conversely, for $\varepsilon>0$ the "viscosity" term on the right hand side of Equation (2.22) diffuses the steepening fronts, and the solution remains smooth, see [6].

In the case of Burgers equation without viscosity ($\varepsilon=0$), an entropy condition is used to select the proper way of continuing the solution past the point when the shock develops, resulting in a globally defined weak solution which is the limiting solution of Equation (2.22) as $\varepsilon\to 0$. Continuing this analogy, in the next section we study in detail the propagating front for the case $\varepsilon=0$.

III. Limiting Case - The Formation of Cusps

A.) Breakdown

In this section, we discuss the limiting case $\varepsilon = 0$ in Equation (2.15), in which case the front moves at a constant speed. The equations of motion become simply

$$x_t = \frac{y_s}{(x_s^2+y_s^2)^{1/2}} \qquad y_t = -\frac{x_s}{(x_s^2+y_s^2)^{1/2}} \tag{3.1}$$

$$(x(s,0),y(s,0)) = \gamma(0); \qquad 0 \le s \le S \tag{3.2}$$

Written in vector notation, the above is known as the Eikonal equation. In this and the following section, we shall always assume that the initial curve $\gamma(0)$ is C^2. We begin by noting that an exact solution to (3.1,3.2) can be obtained, namely

$$x(s,t) = \frac{\beta_s}{(\alpha_s^2+\beta_s^2)^{1/2}}\, t + \alpha(s) \qquad y(s,t) = -\frac{\alpha_s}{(\alpha_s^2+\beta_s^2)^{1/2}}\, t + \beta(s) \tag{3.3}$$

where

$$(\alpha(s),\beta(s)) = (x(s,0),y(s,0)) = \gamma(0) \quad 0 \le s \le S \tag{3.4}$$

The above solution parameterizes by s and t the straight lines normal to the initial curve; in the language of the Eikonal equation, these normals are the geometric rays of optics theory, see [5].

If the initial curve is convex, the solution to the propagation equations (3.1,3.2) can be obtained in two ways; either by using the exact solution (Equations (3.3,3.4)) or by relying on a Huyghens principle construction, which says that the solution at time t corresponds to the envelope generated by the set of all disks of radius t centered on the initial curve, see [6]. These two

constructions produce the same curve, since given a point outside a convex initial curve, there is a unique normal to the curve passing through that point. As normals cannot intersect, no corners can develop and the propagating front remains smooth. The front is also reversible; if we know the position of the front at time t, we may solve the evolution equations backwards in time (or reverse the geometric construction) to reconstruct the initial curve.

B) The Entropy Condition

Suppose the initial curve is non-convex. Since the propagating front develops a corner and passes through itself, the Huyghens principle construction does not give the same solution as Eqn. (3.3,3.4). We now make use of an "entropy condition" that allows us to determine the boundary of the propagating front. For this discussion, we shall consider the front as a flame separating a burnt region inside from an unburnt region outside; each point is transformed from unburnt to burnt when touched by the propagating front. The normals to the initial curve will be called "ignition curves" and correspond to curves along which the temperature jumps.

Our entropy condition may be stated very simply: Once a particle burns, it remains burnt. Thus, let $\phi(x,y,t)$ be the indicator function of the propagating front; $\phi(x,y,t) = 1$ if the particle at (x,y) is burnt at time t and zero otherwise. The propagating front satisfies the entropy condition if when $\phi(x,y,t^*) = 1$ for some t^*, then $\phi(x,y,t) = 1$ for all $t>t^*$. Thus, the boundary of the set of all points where $\phi = 1$ gives the position of the front at time t.

This entropy condition has a geometric interpretation similar to that for shocks in Burgers equation; it guarantees that ignition curves always reach back to the initial front. The entropy condition states that if two ignition curves cross at a particular point, whichever one arrives first will ignite the particle located there. We first show that the late-arriving ignition curve can be eliminated beyond the intersection point; each point it reaches must have already been ignited. To see that this is so, suppose that ignition curve A of length l_A, starting at point A, collides at point P with ignition curve B of length l_B which started at point B. Assume $l_A<l_B$. It is clear that any point Q on

ignition curve B past P is closer to A than it is to B, thus ignition curve B cannot be responsible for igniting it. Hence, we may eliminate ignition curve B beyond the intersection point P. We next show that if an ignition curve is eliminated, it must be eliminated by ignition curve of equal length, and hence by the above, both can have no effect beyond the intersection point.

Proposition 2 Consider the ignition curves described by Equations (3.3,3.4) and suppose, by invoking the entropy condition, we eliminate those parts of ignition curves that reach previously burnt points. Then, if an ignition curve is eliminated at time t_1, it must be eliminated by an ignition curve of equal length, and therefore both pass through previously burnt regions for $t \geq t_1$.

Proof Suppose ignition curve A leaving point A on the initial curve is eliminated at time t_A. Then, for $0 \leq t < t_A$, ignition curve A passes through unburnt points, and for $t \geq t_A$, passes through burnt points. Let ignition curve B leaving point B be the one that eliminates A; thus there exists a t_B, with $t_B \leq t_A$, where A reaches the intersection points at time t_A and B reaches at time t_B. We show that $t_B = t_A$. Suppose not. Then $t_B < t_A$, and all points on A reached between time $(t_A + t_B)/2$ and t_A are closer to B than they are to A and are thus burnt before A reaches them. Hence, the curve leaving A must be eliminated before t_A, contradicting the hypothesis. Thus $t_A = t_B$, and both can be eliminated at the same time; this completes the proof.

With the above construction, we can describe the motion of the propagating front. We extend ignition curves from the initial curve according to Equations (3.3,3.4) and move the front along these curves until there is a collision. Eliminate the curves that collide and continue moving the front along the remaining ignition curves, all the while eliminating those that collide. Although as time progresses, the front will be parameterized by a smaller and smaller subset of the original parameterization $[0,S]$ and any point of the front can be traced back to the initial curve. Conversely, there will be points along the initial curve whose "effect" will be totally eliminated at some later time.

With the above construction, the choice of the phrase "entropy condition" becomes clear. Once ignition curves collide, corners form and the entropy condition is invoked; the position of the front contains no information about the discarded curves. Information about the initial data is "swallowed up" at the cusp and the solution becomes irreversible, hence the name "entropy". As variation in the solution is swallowed by the propagating corners, the total energy in the curve decreases, allowing the front to flatten out into a circle (see Prop. 3). Thus, in analogy with shocks, corners form in response to collisions from different propagating pieces of the initial data, and serve the role of swallowing up variation on the system and decreasing the "energy" in the system.

C) Asymptotic States

The proposition below states that corners eventually swallow up all the variation until the moving front smooths into a circle. Thus, although corners initially correspond to a sharpening of the front and a singularity in the curvature, they also serve as a smoothing mechanism. We shall only outline the proof here; complete details may be found in [17].

Proposition 3 Let $\gamma(0) = (\alpha(s),\beta(s))$, $s \in [0,S]$ be a simple, closed, piecewise C^2, positively oriented initial curve. Let $\gamma(t)$ be the entropy-satisfying solution constructed from the ignition curves given in Equations (3.3,3.4). Then, as $t \to \infty$, $\gamma(t)$ approaches a circle. That is, let $\gamma(\bar{t})$ be the front rescaled at each time so that the total length is 1. Then, given ε, there exists t_o such that for all $t > t_o$,

1) $\gamma(\bar{t})$ is outside a circle of radius $(1/(2\pi) - \varepsilon)$ and inside a circle of radius $(1/(2\pi) + \varepsilon)$.

2) $|2\pi - K(\gamma(\bar{t}))| < \varepsilon$, where $K(\gamma(\bar{t}))$ is the curvature of the rescaled front.

Proof The proof consists of showing that complicated curves can be trapped between simpler curves which evolve into circles. First, a C^2 curve with everywhere positive curvature is considered. Since such a curve remains convex as it evolves, the exact solution can be used to show that the solution tends to a circle under the above definition. Then, by appropriately defining what happens

at corners, it can be shown that convex, piecewise C^2 curves can be trapped between explicitly constructed arbitrarily close smooth C^2 curves which by the above tend to circles. Finally, given a non-convex initial curve, its convex hull is piecewise C^2 and hence must evolve into a circle. It can then be shown that ignition curves leaving those parts of the initial curve not touching the convex hull are eliminated under the entropy condition, thus the evolution of the initial curve is eventually completely determined by its convex hull, completing the proof.

Thus, an initial curve must flatten out as it moves at constant speed and become circular. It is the development of the singularity in the curvature and the ensuing propagation of corners which stabilize and flatten the front. Linear stability analysis assumes that the solution is smooth and therefore cannot detect the stabilizing influence of the discontinuity. Thus, for example, the complete instability of flames which Landau predicted using linear stability analysis ignores this discontinuous stabilizing mechanism.

IV. Numerical Results

A.) Convergence as $\varepsilon \to 0$.

In this section we show numerically that the solution to the propagation equation for $F(K)=1-\varepsilon K$ converges to the constructed weak solution as $\varepsilon \to 0$. We studied the motion of a cosine wave propagating at speed $1-\varepsilon K$ for various values of ε. Let $\gamma(0) = (\alpha(s),\beta(s)) = (s,-(\cos s+1))$; since the initial curve is periodic with period 2π, we shall only consider the section $-\pi \le s \le \pi$. The solution which satisfies the entropy condition is

$$x(s,t) = \begin{cases} \dfrac{\sin(s)}{(1+\sin^2(s))^{1/2}}\ t + s & -\pi \le s \le -g^{-1}(t),\ \ g^{-1}(t) \le s \le \pi \\ 0 & -g^{-1}(t) < s < g^{-1}(t) \end{cases} \tag{4.1}$$

$$y(s,t) = \begin{cases} \dfrac{-1}{(1+\sin^2(s))^{1/2}}\ t - (\cos s+1) & -\pi \le s \le -g^{-1}(t),\ \ g^{-1}(t) \le s \le \pi \\ 0 & -g^{-1}(t) < s < g^{-1}(t) \end{cases} \tag{4.2}$$

where $g(s) = |s|(1+\sec^2(s))^{1/2}$. A calculation shows that a corner forms at $t=1$.

We then numerically solved Equations (2.2-2.3) with speed $F(K) = 1-\varepsilon K$ for various values of ε. We first attempted to use the following simple numerical technique. A set of marker particles were placed along the front, and centered finite difference approximations were used for the spatial derivatives; this yielded a set of coupled ordinary differential equations for the motion of the marker particles. The time derivatives were then approximated by Heun's method. Thus, if we let (x_i^n, y_i^n) be the position of the *ith* marker particle at time n, we have the scheme

$$x_i^{n+1} = x_i^n + \frac{\Delta t}{2}(u_i + \bar{u_i}) \qquad y_i^{n+1} = y_i^n + \frac{\Delta t}{2}(v_i + \bar{v_i})$$

where

$$u_i = \left[1-\varepsilon K_i\right] \left[\frac{y_{i+1}-y_{i-1}}{(x_{i+1}-x_{i-1})^2+(y_{i+1}-y_{i-1})^2)^{1/2}}\right]$$

$$v_i = \lceil 1-\varepsilon K_i \rceil \left[-\frac{x_{i+1}-x_{i-1}}{(x_{i+1}-x_{i-1})^2+(y_{i+1}-y_{i-1})^2)^{1/2}} \right]$$

$$K_i = 4 \left[\frac{(y_{i+1}-2y_i+y_{i-1})(x_{i+1}-x_{i-1})-(x_{i+1}-2x_i+x_{i-1})(y_{i+1}-y_{i-1})}{((x_{i+1}-x_{i-1})^2+(y_{i+1}-y_{i-1})^2)^{3/2}} \right]$$

$$\bar{x_i} = x_i^n + \Delta t\,(u_i^n) \qquad \bar{y_i} = y_i^n + \Delta t\,(v_i^n)$$

and $\bar{u_i}$, $\bar{v_i}$, and $\bar{K_i}$ are computed using $\bar{x_i}$ and $\bar{y_i}$.

This scheme is unstable. As the curvature builds, the marker particles come close together, so that small errors in their positions produce wild variations in the curvature. Even with double precision arithmetic and very small time steps, it is very hard to get the calculation, for small values of ε, to proceed past $t=1.5$ without oscillations and instability.

This problem was eliminated through a regridding technique designed to keep the horizontal distance between mesh points fixed. We let $s_1, \cdots s_i, \cdots s_N$ be N evenly spaced points in the interval $[-\pi,\pi]$, $s_1=-\pi$, $s_N=\pi$, and let $x_i^0=\alpha(s_i)=s_i$, $y_i^0=\beta(s_i)$ Using the above difference scheme, we then advance the points (x_i^n,y_i^n) ahead one time step to produce new values (x_i^*,y_i^*). These new markers yield the positions of the updated curve. Let $x_i^{n+1}=s_i$. To find y_i^{n+1}, we use points on the curve and linearly interpolate to find the intersection between the vertical line $x=s_i$ and the curve determined by the updated positions. This provides an updated set of marker particles (x_i^{n+1},y_i^{n+1}) such that $(x_i^{n+1}-x_{i-1}^{n+1})$ is constant. We varied N from 100 to 400 points and Δt from 10^{-3} to 10^{-5} to ensure that results did not depend on grid size or time step, comparing position and energy of the curve.

In Figure 1, we plot $Var(t)$ as a function of time for various values of ε, as well as $Var_{Ent}(t)$ for the entropy-satisfying solution ($\varepsilon=0$). The dashed line represents the initial value $Var(0)=\pi$. It is easy to show analytically that the energy for $\varepsilon=0$ is constant until t^*, where $t^* = |\pi/4|\,(1+\sec^2(\pi/4))^{1/2}$; at this time the inflection point of the curve hits the y axis. As $\varepsilon\to 0$, the graphs approach that of the entropy-constructed solution, (labeled "cusp"), and in particular converge to the constant value $Var(t)=\pi$ for $t<t^*$. In Figure 2, we plot the positions of the front for

various values of ε. The front locations tend towards the cusp solution; in addition, the decay in total variation can easily be seen.

It is clear that our numerical method contains some artificial smoothing in both the finite difference approximations and in the linear interpolation used in the regridding technique. However, since the technique show the solution approaching the cusp solution as $\varepsilon \to 0$, this gives strong indication of the convergence.

B) Discrete Parameterization vs. Volume of Fluid Techniques

In general, numerical methods based on discrete parameterizations of the front face some serious obstacles. First of all, as pointed our earlier, marker points come close together in areas where the curvature builds, even with the influence of the diffusion-like term in the reaction diffusion equation for curvature (Eqn. (2.16)). Since curvature is a second derivative, small errors in the particle positions cause large errors in the computed curvature. In those regions where marker particles move far apart, one may be faced with a very poor approximation to the curvature, even though in those regions the time step may satisfy the stability requirement. In addition, the numerical algorithm contains some stability requirement depending on the ratio of Δt to $\Delta\alpha$, where $\Delta\alpha$ is the arclength spacing between neighboring marker points; indications of this come from the simplistic argument that the equations of motion at least partially look like the heat equation. If the marker points come close together, $\Delta\alpha$ decreases and the time step must be correspondingly decreased to ensure stability, making the calculation prohibitively expensive.

A natural way around these problems is to reparameterize the front and redistribute marker points so that they stay far enough apart to allow reasonable time steps. This reparameterization is, unfortunately, a smoothing process; one uses information from the front to calculate various curvature and arclength approximations to determine the new set of marker points.

A separate obstacle for discrete parameterization techniques is the question of merging. If two burning patches burn into each other, it is difficult to decide which marker points to throw away so

that those remaining constitute the boundary of the new combined region.

As evidence of the above, it is important to point out how much the numerical calculation of an advancing cosine wave is rigged. First of all, the front can always be written as a single-valued function over a fixed interval, since the solution is periodic and nowhere has derivative greater than one in magnitude (since the $\varepsilon=0$ solution is known and can be seen to bound the $\varepsilon>0$ solution). It is not a trivial issue to extend our technique to a curve that cannot be represented as a function. Second, the solution is symmetric around $x=0$, thus we spread points from $-\pi$ to 0 and use symmetric boundary conditions at $x=0$. This insured that we always had a regridding line and hence a marker point at the point of maximum curvature. Thus, this point was never readjusted. If this was not done and the calculation was performed over the entire interval, then round-off error caused the center point to drift off the line $x=0$ and the interpolating technique used to create a new marker on that line severely smoothed the curvature.

As an alternative, numerical methods based on "volume of fluid" constructions do not rely on discrete parameterizations of the moving front and can easily handle topological issues such as merging. Such techniques can be used in conjunction with Huyghens principle so that the entropy condition comes about in a natural way. For details, see [15], [3], [17]. Here, we shall briefly describe one such method for the case $\varepsilon=0$, and demonstrate its use on a simple problem involving corner development.

A square grid i,j of uniform mesh size is imposed on the domain, and a number f_{ij}, $0\le f_{ij}\le 1$ is assigned to each cell, corresponding to the fraction of material within the cell to be propagated. To move the material within a cell in the direction $\vec{u}=(u,v)$, the algorithm performs a fractional step in each direction. First, an interface is drawn in the cell representing the boundary of the material, where the orientation of the interface depends on the value of f_{ij} and the f_{ij}'s in the cell's neighbors. This interface is then transported in the x direction a distance $u\Delta t$ and the process is then repeated for the sweep in the y direction, providing a new f_{ij}.

This algorithm can be used to advance a front along its normal vector field using Huyghens principle. For the moment, assume that $F(K)=1$ and consider L angles, $\Theta_l=(2\pi(l-1)/L$, $l=1,...,L$. Given any cell with volume fraction f_{ij}^n, the material in that cell is moved a distance Δt in each of the l directions $(\cos\Theta_l,\sin\Theta_l)$. As $L\to\infty$ and the mesh size goes to zero, this corresponds to drawing a disk of unit radius around the center of the cell. The Huyghens principle construction says that the envelope formed by all such disks (that is, for all i,j) gives the front advanced a unit distance along its normal field. Thus, let $f_{ij}^n\Theta_l$ be the array of volume fractions obtained by moving the fractions in the direction Θ_l and let $f_{ij}^n\Theta_0=f_{ij}^n$. The new volume fractions approximating the front advanced one time step will be given by

$$f_{ij}^{n+1} = \max_{0\le l\le L} f_{ij}^n\Theta_l .$$

Of course, since many of the volume fractions are either one or zero, with careful programming one can limit all the computing effort to the boundary between the burnt and unburnt regions. In addition, we point out that one advantage to such a scheme is that it easily carries over to three dimensions.

As an example of this algorithm, we consider a smooth, non-convex curve that develops a corner as it moves. We used 8 angles and a 60 x 60 mesh of cells. Let $\gamma(s)=(\alpha(s),\beta(s))$, $s\in[0,6\pi]$, where α and β are defined as follows:

$$\alpha(s) = \left\{ \begin{array}{cc} -\cos(s) & 0\le s<\pi/2 \\ -\cos(s) & \pi/2\le s<3\pi/2 \\ 3\cos(s/3) & 3\pi/2\le s<9\pi/2 \\ -\cos(s) & 9\pi/2\le s<11\pi/2 \\ -\cos(s) & 11\pi/2\le s\le 6\pi \end{array} \right\} \qquad \beta(s) = \left\{ \begin{array}{cc} \sin(s) & 0\le s<\pi/2 \\ -\sin(s)+2 & \pi/2\le s<3\pi/2 \\ 3\sin(s/3) & 3\pi/2\le s<9\pi/2 \\ -\sin(s)-2 & 9\pi/2\le s<11\pi/2 \\ \sin(s) & 11\pi/2\le s\le 6\pi \end{array} \right\}$$

Let the front propagate with speed $=1/2$. A lengthy but straightforward calculation shows that the position of the front $(x(s,t),y(s,t))$ is given by

$$x(s,t) = \frac{1}{2}\,\beta_s t + \alpha \quad y(s,t) = -\frac{1}{2}\,\alpha_s t + \beta$$

$$0 \le s \le 6\pi$$

for $t \le 1$, and

$$x(s,t) = \frac{1}{2}\,\beta_s t + \alpha \quad y(s,t) = -\frac{1}{2}\,\alpha_s t + \beta$$

$$\left\lceil \cos^{-1}\left(\frac{2}{(1+t)}\right)+\frac{\pi}{2} \right\rceil \le s \le 6\pi - \left\lceil \cos^{-1}\left(\frac{2}{(1+t)}\right)+\frac{\pi}{2}\right)\ \Bigg\rceil$$

for $t > 1$.

The position of the front at various values of t is shown in Figure 3. As the front moves, a corners forms at $t=1$, and travels along the positive x axis, "swallowing up" sections of the front. In Figure 4, the results of the application of the numerical algorithm algorithm to this problem for various values of t are shown. The method does an excellent job of showing the formation and absorption of the corner.

The above method filters out high wavelengths in the solution by limiting the oscillations of the front to the order of one cell width. Thus, the solution is necessarily smoothed. In order to extend the method to a front moving with speed $F(K)$, the curvature must be determined from the volume fractions and it is clear that the maximum allowable curvature is limited by the one cell width resolution of the front. A method for determining the curvature based on fitting osculating circles to volume fractions has been developed [2]; there, the cell size determines the smallest possible osculating circle and hence bounds the curvature and smooths the solution.

In spite of this smoothing, such a curvature algorithm can provide a valuable tool for analyzing a moving front, since the entropy condition naturally generates weak solutions and the mesh size can be systematically refined to allow larger curvatures in a controlled way. Thus, if a mesh size h is used with maximum allowable curvature K_h, the algorithm produces a weak solution with this bound; by refining the mesh size, one can investigate both possible blow up in the curvature and nature of the solution beyond the singularity.

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