HYPERSURFACES MOVING WITH CURVATURE-DEPENDENT SPEED:

HAMILTON-JACOBI EQUATIONS, CONSERVATION LAWS AND NUMERICAL ALGORITHMS

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In many physical problems, interfaces move with a speed that depends on the local curvature. Some common examples are flame propagation, crystal growth, and oil-water boundaries. We model the front as a closed, non-intersecting, initial hypersurface flowing along its gradient field with a speed that depends on the curvature. Because explicit solutions seldom exist, numerical approximations are often used. The goal of this paper is to show that algorithms based on direct parameterizations of the moving front face considerable difficulties. This is because such algorithms adhere to local properties of the solution, rather than the global structure. Conversely, the global properties of the motion can be captured by imbedding the surface in a higher-dimensional function. In this setting, the equations of motion can be solved using numerical techniques borrowed from hyperbolic conservation laws. We use these schemes to follow a variety of propagation problems, illustrating corner formation, breaking and merging.

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In many physical problems, interfaces move with speed that depends on the local curvature. Explicit solutions seldom exist. Thus, there is great interest in numerical algorithms that approximate the position of the moving front. The goal of this paper is to show that algorithms based on direct parameterizations of the moving front face considerable difficulties. This is because such algorithms adhere to local properties of the solution, rather than the global structure. Conversely, the global properties of the motion are preserved by imbedding the surface in a higher-dimensional function. In this setting, the resulting equations of motion can be accurately approximated using numerical techniques borrowed from hyperbolic conservation laws.

We now state the problem. Suppose we are given a closed (or periodic), non-intersecting, initial hypersurface and a function $F(K)$. Here, K is the curvature of the hypersurface. We wish to approximate the position of the surface as it flows along its gradient field with speed $F(K)$.

This paper is not written for experts in numerical analysis, Hamilton-Jacobi equations, or hyperbolic conservation laws. Here, we will show why some algorithms work and others fail, and avoid technical discussions of higher order extensions and error analysis of convergence rates. The details of how to design and apply numerical algorithms based on this higher-dimensional Hamilton-Jacobi formulation to propagating fronts were presented in [23], and we refer the interested reader there.

The outline of this paper is as follows. In Section 1, we describe two physical problems in which curvature plays a role in interface motion. In Section 2, we write down the equations of motion based on a parameterization of the moving front, and argue that numerical approximations to

these equations face considerable difficulties. We show that a simple approximation to a moving initial cosine curve goes unstable with a very fine discretization of the parameterization, even though the exact solution stays smooth.

In Section 3, we take a global approach and view the $(N-1)$ dimensional moving surface as a level set of a time-dependent function of N dimensions. The equation of motion for this function resembles an initial value Hamilton-Jacobi equation with parabolic right-hand-side and is closely related to a viscous hyperbolic conservation law. We then show that numerical schemes designed to approximate hyperbolic conservation laws may be used to approximate the motion of the propagating surface. Finally, in Section 4, we apply our new schemes to a variety of propagation problems, demonstrating cusp formation, breaking and merging.

1. MOTIVATION

The first mathematical model of combustion was formulated by Landau [16] in 1944. He idealized a flame as an infinitely thin boundary separating regions of constant steady-state velocity, density, and temperature. In addition, he postulated that flames burn at a constant speed V normal to themselves. Under these assumptions, linear stability analysis showed that any deviation from an absolutely flat flame will become unbounded as time progresses. Numerous experiments indicate otherwise. In 1962, Markstein [20] argued that cool convex fingers reaching out into unburnt gas must propagate slower than hot concave regions surrounding an unburnt pocket. Thus, he proposed a curvature-dependent flame speed of the form $V(1-\varepsilon K)$, where ε is a constant and K is the curvature (see Fig.1b). Here, $K>0$ for convex and $K<0$ for concave fingers. Sketches of these ideas are shown in Figure 1. The addition of curvature stabilizes perturbations and helped explain the formation of honey-combed "cellular flames" observed experimently. More complicated flame models are discussed in [7,21,23,31,33].

In crystal growth, the dendritic spikes of a snowflake result, in part, from surface freezing rates which depend on curvature (see [17]). Consider a solid ice pellet growing in a supercooled liquid. The rate of growth at any point on the boundary depends on the curvature through the Gibbs-Thomson relation $T(x)=T_M(1-\varepsilon K(x))$, where T_M is a constant melting temperature. Once again, $K>0$ for convex and $K<0$ for concave fingers. In this case, the lower temperature of the extended fingers implies that they grow faster. Thus, the boundary breaks into separate dendrites which grow rapidly out from the seed and spawn further dendrites, see [2,17,18,24,32]. A qualitative picture is shown in Figure 2.

2. LAGRANGIAN FORMULATION: EQUATIONS AND APPROXIMATION

In this section, we derive equations of motion in terms of a parameterization of the moving front. For simplicity, we consider a closed curve moving in the plane. After some analysis, we study numerical schemes based on discrete approximations to these equations. A simple test problem shows the limitations of such schemes.

EQUATIONS OF MOTION

Let $\gamma(0)$ be a smooth, closed initial curve in R^2. Let $\gamma(t)$ be the one-parameter family of curves generated by moving $\gamma(0)$ along its normal vector field with speed $F(K)$. Here, $F(K)$ is a given scalar function of the curvature K. The natural approach is to parameterize the moving curve. Let $\vec{x}(s,t)$ be the position vector which parameterizes γ at time t. Here, $0 \le s \le S$, and we prescribe periodic boundary conditions $\vec{x}(0,t)=\vec{x}(S,t)$. The curve is parameterized so that the interior is on the left in the direction of increasing s. Let $\vec{n}(s,t)$ be the outward normal and $K(s,t)$ be the curvature. Then $\vec{n} \cdot \vec{x}_t = F(K)$. In terms of individual components $\vec{x}=(x,y)$, we have

$$x_t = F[\frac{y_{ss}x_s - x_{ss}y_s}{(x_s^2+y_s^2)^{3/2}}]\left[\frac{y_s}{(x_s^2+y_s^2)^{1/2}}\right] \qquad y_t = F[\frac{y_{ss}x_s - x_{ss}y_s}{(x_s^2+y_s^2)^{3/2}}]\left[\frac{-y_s}{(x_s^2+y_s^2)^{1/2}}\right]. \tag{1}$$

We call this a "Lagrangian" representation because the coordinate system moves with the front.

The special case of a closed curve shrinking under its own curvature has received considerable attention, see [6,8,9,10,13,25]. Recently, Grayson [13] has shown that any smooth initial curve must collapse smoothly to a point if $F(K)=-K$. This result will be demonstrated for some complicated initial curves using the algorithms described in Section 3.

GROWTH AND DECAY OF TOTAL VARIATION

What happens to oscillations in the initial curve as it moves? The main proposition of this section says that the growth or decay of oscillations depends only on the sign of F' at $K=0$. Let $g(s,t)=(x_s^2+y_s^2)^{1/2}$. Differentiating both sides of Eqn. (1) with respect to s we obtain the evolution equations for the metric and curvature, namely

$$K_t = -g^{-1}(F_s\, g^{-1})_s - K^2 F \tag{2}$$

$$g_t = g\, K\, F \tag{3}$$

Denote the total oscillation in the propagating curve at time t by $Var(t) = \int_0^S |K(s,t)|\, g(s,t)\, ds$.

The following proposition says that the total variation of the moving front decreases if $F'(0)<0$. Precise statement and proof are in [27].

Proposition: Consider a non-convex initial curve moving with speed $F(K)$. Suppose the moving curve stays smooth for $0<t<T$. Then

$$\text{1) If } F'(0)\le 0, \text{ then } \frac{d}{dt}Var(t)\le 0 . \qquad \text{2) If } F'(0)\ge 0, \text{ then } \frac{d}{dt}Var(t)\ge 0 .$$

The above inequalities can be made strict if $K_s(s,t)\neq 0$ whenever $K=0$.

A SIMPLE TEST PROBLEM

Consider the speed function $F(K)=1-\varepsilon K$, where ε is a non-negative constant. The curvature evolution equation becomes

$$K_t = \varepsilon K_{\alpha\alpha} + \varepsilon K^3 - K^2, \tag{4}$$

where we have changed variables and taken the derivative with respect to arclength α. Eqn. (4) is a reaction-diffusion equation. The drive toward singularities due to the reaction term ($\varepsilon K^3 - K^2$) is balanced by the smoothing effect of the diffusion term ($\varepsilon K_{\alpha\alpha}$). Indeed, with $\varepsilon=0$, we have a pure reaction equation $K_t=-K^2$. In this case, the solution is $K(s,t) = K(s,0)/(1+tK(s,0))$, which is singular at finite t if the initial curvature is anywhere negative. Thus, corners can form in the moving curve when $\varepsilon=0$.

An example will serve as a test problem for our numerical schemes. Consider the periodic initial cosine curve

$$\gamma(0) = (-s, [1+\cos 2\pi s]/2), \quad -\infty<s<\infty \tag{5}$$

and let it propagate upward with speed $F(K)=1-\varepsilon K$, $\varepsilon>0$. The troughs at $s=n+1/2, n=0,\pm 1,\pm 2,....$ are sharpened by the negative reaction term and smoothed by the positive diffusion term because $K<0$ at such points.

Do corners develop in the moving front? For small initial times, the reaction term is stronger than the diffusion term, and the troughs begin to sharpen and close. For $\varepsilon=0$, we have shown that a corner must form. On the other hand, for $\varepsilon>0$, it can be shown (see [23,27] and Section 3), that the moving front stays C^∞. Thus, we can use an obvious extension of the proposition to periodic initial curves and the fact that $F'(K)=-\varepsilon<0$ to show that $dVar(t)/dt<0$. Hence, peaks and valleys in the

moving curve must decay.

The propagating curve is shown at various times in Fig. 3a. The front becomes flat as t increases. In Fig. 3b, we superimpose lines of constant s. The fact that s does not correspond to arclength is evident in the narrowing of constant parameter lines at the trough. In the next section, we show that this poses considerable difficulty for numerical approximation schemes.

NUMERICAL APPROXIMATIONS TO LAGRANGIAN FORMULATION

The key issues for any numerical scheme are accuracy and stability. The *accuracy* of a scheme determines how well the discrete formulation approximates the exact equation. *Stability* measures how sensitive the approximation is to small deviations. In this section, we argue that the stability requirement for Lagrangian schemes forces an unreasonably small time step.

We construct the simplest possible difference approximation to the Lagrangian equations of motion. Divide the parameterization interval $[0,S]$ into M equal intervals of size Δs, yielding $M+1$ mesh points $s_i = i\,\Delta s$, $i=0,...,M$. Divide time into equal intervals of length Δt. The image of each mesh point $i\,\Delta s$ at each time step $n\,\Delta t$ is a marker point (x_i^n, y_i^n) on the moving front. Our goal is a *numerical algorithm* which produces new values (x_i^{n+1}, y_i^{n+1}) from the previous positions. First, we approximate parameter derivatives at each marker point by using neighboring mesh points. The *central difference approximations* based on Taylor series are given by

$$(x_i^n)_s \approx \frac{x_{i+1}^n - x_{i-1}^n}{2\Delta s} \qquad (y_i^n)_s \approx \frac{y_{i+1}^n - y_{i-1}^n}{2\Delta s} \tag{6}$$

$$(x_i^n)_{ss} \approx \frac{x_{i+1}^n - 2x_i^n + x_{i-1}^n}{\Delta s^2} \qquad (y_i^n)_{ss} \approx \frac{y_{i+1}^n - 2y_i^n + y_{i-1}^n}{\Delta s^2}. \tag{7}$$

Similarly, time derivatives may be replaced by *forward difference approximations*

$$\frac{dx_i^n}{dt} \approx \frac{x_i^{n+1}-x_i^n}{\Delta t} \qquad \frac{dy_i^n}{dt} \approx \frac{y_i^{n+1}-y_i^n}{\Delta t}. \tag{8}$$

Substituting these approximations into the equations of motion (Eqn. (1)), we get

$$\left\lceil x_i^{n+1}, y_i^{n+1} \right\rceil = \left\lceil x_i^n, y_i^n \right\rceil + \Delta t \ \frac{F(K_i^n)}{((x_{i+1}^n - x_{i-1}^n)^2 + (y_{i+1}^n - y_{i-1}^n)^2)^{1/2}} \ \left\lceil y_{i+1}^n - y_{i-1}^n, -(x_{i+1}^n - x_{i-1}^n) \right\rceil \tag{9}$$

$$K_i^n = 4\,\frac{(y_{i+1}^n - 2y_i + y_{i-1}^n)(x_{i+1}^n - x_{i-1}^n) - (x_{i+1}^n - 2x_i + x_{i-1}^n)(y_{i+1}^n - y_{i-1}^n)}{((x_{i+1}^n - x_{i-1}^n)^2 + (y_{i+1}^n - y_{i-1}^n)^2)^{3/2}} \tag{10}$$

Using the periodicity of the curve, this is a complete recipe for updating the positions of the particles from one time step to the next.

We observe that the fixed discretization interval Δs has dropped out of the above expression. Consequently, as marker particles come together, quotients in the right-hand-side of Eqn. (9) approach zero over zero. This is a very sensitive calculation. The computed curvature can change drastically from one particle to the next because of small and unavoidable errors in the positions.

We can demonstrate this unstable growth of small errors by using our scheme to follow the initial cosine curve. Since $\varepsilon>0$, the exact solution is always smooth. We use 50 marker points and time step $\Delta t=0.01$. Although the propagating front begins to sharpen as expected (see Fig. 4a), oscillations soon develop which grow uncontrollably. These oscillations result from a feedback cycle: (1) small errors in approximate marker positions produce (2) local variations in the computed derivatives leading to (3) variation in the the computed particle velocities causing (4) uneven advancement of markers which yields (5) larger errors in approximate marker positions. Within a few time steps, the small oscillations in the curvature have grown wildly and the computed solution becomes unbounded. In Fig. 4a, we show the calculation until the computer program stops running.

Suppose we try to increase accuracy by using a smaller time step. In Fig.4b and Fig.4c, we show calculations with $\Delta t=.001$ and $\Delta t=.0001$, respectively. Once again, the solution becomes unstable, and the smooth decay of the trough is not seen. It is important to point out that for any $\varepsilon>0$, there is a bound for the minimum distance between particles. Thus, a small enough time step does exist to insure stability. The issue here is *practicality*. Such a time step may be so

unreasonably small that the calculation simply takes too long. If we choose a bigger ε, we can choose a bigger time step without violating stability.

Two remedies are often proposed to circumvent these problems. One option is to "smooth" the speed function $F(K)$ so that constant parameter curves stay far enough apart. Another option is to redistribute marker particles according to arclength every few time steps. Both options are designed to produce a practical time step that maintains stability.

Ultimately, serious difficulties remain. With both remedies, the equations have been altered in non-obvious ways. Significant amounts of smoothing may be required to insure a practical time step. Thus, one has chosen to sacrifice the most interesting propagation characteristics, such as front sharpening and curvature singularities, in order to keep the calculation alive. Similarly, calculation of arclength adds an additional smoothing term to the speed function and is difficult to analyze. Due to these two effects, the computed solution may be far from the desired one. In the worst case, time and effort are spent solving an unrelated problem.

Topological changes in the moving front are also problematic for Lagrangian approximation schemes. Consider two separate regions of growing substance in a plane, each surrounded by a closed curve. Suppose these patches merge and the boundary becomes a single curve. It is difficult to produce a systematic way of removing those markers that no longer sit on the actual boundary. The bookkeeping of removing, redistributing, and connecting markers becomes even more complicated for higher dimensional interface problems.

To summarize, Lagrangian approximations suffer from instability and topological limitations because they follow a local representation of the front. In the next section, we take a global approach.

3. EULERIAN FORMULATION: EQUATIONS AND APPROXIMATIONS

In this section, we reformulate the problem and show a link between the resulting Hamilton-Jacobi equation and a hyperbolic conservation law. We then show that the central difference approximation fails, because it ignores a global "entropy condition" which applies when sharp corners develop. Finally, we compute the motion of the propagating cosine curve using the simplest possible entropy-satisfying algorithm.

EQUATIONS OF MOTION

We will motivate our approach with a simple example. Suppose the initial front γ at $t=0$ is a circle in the (x,y) plane (Fig. 5a). We imagine that the circle is the level set $\psi=0$ of an initial surface $z=\psi(x,y,t=0)$ in R^3 (see Fig. 5b). We can then match the one-parameter family of moving curves $\gamma(t)$ with a one-parameter family of moving surfaces in such a way that the level set $\psi=0$ always yields the moving front (Fig. 5c, 5d). All that remains is to find an equation of motion for the evolving surface.

In the general case, let $\gamma(0)$ be a closed, non-intersecting, $(N-1)$ dimensional hypersurface. Let $\psi(\bar{x},\bar{t})$, $\bar{x} \in R^n$, be a scalar function such that $\psi(\bar{x},0)=\pm d(\bar{x})$, where $d(\bar{x})$ is the signed distance from $\bar{x}$ to the hypersurface $\gamma(0)$. We use the plus sign if $\bar{x}$ is outside $\gamma(0)$ and the minus sign if $\bar{x}$ is inside. Each level set of ψ flows along its gradient field with speed $F(K)$. The gradient $\nabla\psi(\bar{x},\bar{t})$ is normal to the $(N-1)$ dimensional level set passing through $\bar{x}$. Let $K(\bar{x},\bar{t})$ be the curvature of that level set at $\bar{x}$. We may then express K in terms of ψ. For example, if $\bar{x} \in R^2$, then $K \equiv (\psi_{yy}\psi_x^2 - 2\psi_x\psi_y\psi_{xy} + \psi_{xx}\psi_y^2)/(\psi_x^2+\psi_y^2)^{3/2}$. In higher dimensions, appropriate expressions may be obtained for the mean curvature or for the Gaussian curvature. Thus, the motion of each level set is given by

$$\psi_t + F(K)\,|\nabla\psi| = 0 \tag{11}$$

At any time, the moving front $\gamma(t)$ is just the level set $\psi=0$.

We call this an *Eulerian formulation* for front propagation, because it is written in terms of a fixed coordinate system in the physical domain. There are two advantages to this approach. First, since the underlying coordinate system is fixed, discrete mesh points do not move and the stability problems that plagued the Lagrangian approximations may be avoided. Second, topological changes are handled naturally, since the level surface $\psi=0$ need not be simply connected.

HAMILTON-JACOBI EQUATIONS AND HYPERBOLIC CONSERVATION LAWS

Consider the typical speed function $F(K)=1-\varepsilon K$. The equation of motion

$$\psi_t + |\nabla\psi| = \varepsilon K\,|\nabla\psi| \tag{12}$$

is a member of a broad class known as "Hamilton-Jacobi equations with viscosity" (see [3]). The left-hand-side is the Hamilton-Jacobi equation part, and the "viscosity" refers to the second order parabolic right-hand-side.

Assume the moving front is a curve in two space dimensions that remains a graph and consider the initial front given by the graph of $f(x)$, with f and f' periodic on [0,1]. Let ϕ be the height of the propagating function at time t, thus $\phi(x,0)=f(x)$. The normal at (x,ϕ) is $(1,\phi_x)$, and

the

equation of motion becomes $\phi_t = F(K)(1+\phi_x^2)^{1/2}$. Using the speed function $F(K)=1-\varepsilon K$ and the formula $K=-\phi_{xx}/(1+\phi_x^2)^{3/2}$, we get

$$\phi_t - (1+\phi_x^2)^{1/2} = \varepsilon \frac{\phi_{xx}}{1+\phi_x^2}. \tag{13}$$

Suppose we now try to construct an evolution equation for the slope $u=d\psi/dx$ of the propagating front. Differentiating both sides of the above with respect to x and substituting, we get

$$u_t + [-(1+u^2)^{1/2}]_x = \varepsilon[\frac{u_x}{1+u^2}]_x. \tag{14}$$

Thus, the derivative of the Hamilton-Jacobi equation with parabolic right-hand-side for the changing height ϕ is a viscous hyperbolic conservation law for the propagating slope u (see [29]).

Much is known about hyperbolic conservation laws, see [5,11,19,22]. For $\varepsilon>0$, the parabolic right-hand-side diffuses steep gradients and enforces smooth solutions (this is main fact underlying these statements in Section 2). However, for $\varepsilon=0$, discontinuous solutions can arise from smooth initial data. A variety of weak solutions which satisfy an integral version of Eqn. (14) are possible beyond the occurrence of the singularity. Of all such weak solutions, we are interested in the one that is the limit of smooth solutions as $\varepsilon\to 0$. This particular weak solution can be selected with the help of a so-called entropy condition.

We now illustrate these ideas by studying our propagating cosine curve. We have already seen that with $\varepsilon>0$, the exact solution develops a sharpening trough which ultimately stays smooth. On the other hand, for $\varepsilon=0$ a corner must develop, (see Fig. 6a). Thus, jump discontinuities in the slope arise from smooth initial data. How do we proceed once a corner develops? It is unclear how to construct the normal at the corner and continue the evolution, since the derivative is not defined there. One possibility is the "swallowtail" solution formed by letting the front pass through itself, (see Fig. 6b). However, geometrically it seems clear that the front at time t should only consist of the set of all points located a distance t from the initial curve. (This is known as the Huygens principle construction). Roughly speaking, we want to remove the "tail" from the "swallowtail". In Fig.

6c, we show this alternate weak solution. Another way to characterize this weak solution is through the following "entropy condition" (see [26]): If the front is viewed as a burning flame, *then once a particle is burnt it stays burnt.* Careful adherence to this stipulation produces the Huygen's principle construction. Furthermore, this physically reasonable weak solution has an equally appealing mathematical quality: It is the formal limit of the smooth solutions $\varepsilon>0$ as the curvature term vanishes, (see [23,27]).

A NUMERICAL ALGORITHM THAT IGNORES THE ENTROPY CONDITION

Not all numerical algorithms pick out the correct weak solution. In this section, we show that a straightforward central difference approximation to the equations of motion chooses the wrong limiting solution. Let $F(K)=1$ and consider the initial value problem

$$\phi_t = (1 + \phi_x^2)^{1/2} \tag{15}$$

$$\phi(x,0) = f(x) = \left\{ \begin{matrix} (1/2 - x) & x \le 1/2 \\ (x - 1/2) & x > 1/2 \end{matrix} \right\},$$

The initial front is a "V" formed by rays meeting at (1/2,0). By our entropy condition, the solution at any time t is the set of all point located a distance t from the initial "V", (see Fig. 7a). Divide the interval [0,1] into $2M-1$ points, and form the central difference approximation to the spatial derivative in Eqn. (15), namely

$$\phi_t = \left[1 + \left[\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x} \right]^2 \right]^{1/2}. \tag{16}$$

Since $x_M = 1/2$, by symmetry, $\phi_{M+1} = \phi_{M-1}$, thus $\phi_t(1/2,0)=1$. However, for all $x \neq 1/2$, ϕ_t is correctly calculated to be $\sqrt{2}$, since the graph is linear on either side of the corner and thus the central difference approximation is exact. Note that this has nothing to do with the size of the space step Δx or

the time step Δt. *No matter how small we take the numerical parameters, the approximation to* ϕ_t *at* $x=1/2$ *gets no better*. It is simply due to the way in which the derivative ϕ_x is approximated. In Figs. 7b and 7c, we show results using this scheme. In contrast with the correct solution (shown in Fig. 4a), here we clearly see that the trough is "too deep", because of the error in the approximation to ψ_x at $x=1/2$. As the calculation progresses, things get worse and worse.

It is easy to see what has gone wrong. In the exact solution, $\psi_t = \sqrt{2}$ for all $x \neq 1/2$. This should also hold at $x=1/2$ where the slope is not defined; the Huygen's construction sets $\psi_t(x=1/2,t)$ equal to $\lim_{x \to 1/2} \psi_t$. Unfortunately, the central difference approximation chooses a different (and, for our purpose, wrong) limiting solution. It sets the undefined slope ψ_x equal to the average of the left and right slopes. What we want is a scheme chooses the appropriate limiting form at discontinuities. In the next section, we give the simplest such "entropy-satisfying" scheme.

ENTROPY-SATISFYING ALGORITHMS

It can be shown that the entropy condition for propagating fronts is identical to the one for hyperbolic conservation laws, where entropy-satisfying numerical algorithms have a rich history. The most effective of these schemes have resulted from a combination of partial differential equa-

tions theory, numerical analysis, physical intuition, computing experience and clever fine tuning (see [12,30]). In this section, we show that the simplest possible first order algorithms transform into effective algorithms for our moving surface problems.

A single, non-linear equation of the form

$$u_t + [H(u)]_x = 0 \tag{17}$$

is called a *hyperbolic conservation law*. Eqn. (14) for the propagating slope $u=\phi_x$ is a conservation law in the limiting case $\varepsilon\to 0$ with $H(u)=-(1+u^2)^{1/2}$. Since the propagating front can develop corners, we know that discontinuities may develop in the slope u from smooth initial data. Thus, we study an integral version of the conservation law which admits discontinuous solutions. Consider a closed interval $[a,b]$. We may integrate both sides of Eqn. (17) to produce

$$\frac{d}{dt}\int_a^b u(x,t)dx = H[u(a,t)]-H[u(b,t)]. \tag{18}$$

We say that u is a weak solution of the conservation law if it satisfies the above integral equation. Note that u need not be differentiable to satisfy the integral form of the conservation law. We sidestep the smoothness issue by devising numerical algorithms to approximate the integral rather than differential equation of motion.

Definition: Let u_i^n be the value of u at mesh point $i\Delta x$ at time $n\Delta t$. We say that a 3-point difference scheme is in *conservation form* if there exists a function $g(u_1,u_2)$ such that the scheme can be written in the form

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = -\frac{g(u_i^n,u_{i+1}^n)-g(u_{i-1}^n,u_i^n)}{\Delta x} \quad \text{where} \quad g(u,u)=H(u) \tag{19}$$

This definition is natural; the scheme must approximate the hyperbolic conservation law, subject to the consistency requirement $g(u,u)=H(u)$. Thus, any scheme that can be put into conservation form

gives a weak solution. But how do we guarantee that the scheme picks out the correct entropy-satisfying weak solution? We must restrict things further. We say that a 3-point finite difference scheme of the form $u_i^{n+1} = F(u_{i-1}^n, u_i^n, u_{i+1}^n)$ is *monotone* if F is an increasing function of all its arguments. We now can state the main fact: A conservative, monotone scheme produces a solution that satisfies the entropy condition. Thus, we need only check monotonicity and conservation form to verify that a scheme gives the correct entropy condition. One such simple scheme, called the Lax-Friedrichs method, is built from central difference approximations and given by

$$u_i^{n+1} = \frac{1}{2}\,(u_{i-1}^n + u_{i+1}^n) - \frac{\Delta t}{\Delta x}[H(u_{i+1}^n) - H(u_{i-1}^n)]. \tag{20}$$

The reader can easily verify that this is monotone and may be put in conservation form via the numerical flux function

$$g(u_1, u_2) = -\frac{\Delta t}{2\Delta x}(u_2 - u_1) + \frac{1}{2}[H(u_2) + H(u_1)]. \tag{21}$$

How do we go from a scheme for the slope u to a scheme for the front ϕ itself? One idea is to simply solve for u and integrate. However, there is another way. In the limiting case $\varepsilon \to 0$, we may rewrite the front propagation equation as $\phi_t + H(d\phi/dx) = 0$, where $H(u) = -(1+u^2)^{1/2}$. Using a forward difference scheme in time, we have $\phi_i^{n+1} = \phi_i^n - \Delta t H(u)$. Since our numerical flux function g approximates H, we can simply write

$$\phi_i^{n+1} = \phi_i^n - \Delta t\, g(\phi_i^n - \phi_{i-1}^n)/dx, \phi_{i+1}^n - \phi_i^n)/dx), \tag{22}$$

where g is defined above. This algorithm produces the correct entropy-satisfying weak solution, and will be used to propagate our initial cosine curve. Finally, if $\varepsilon \neq 0$, we may use a straightforward finite difference scheme to approximate spatial derivatives for the parabolic curvature-dependent right-hand-side.

These ideas may be extended to produce entropy-satisfying algorithms for the full Eulerian equations of motion given in Eqn. (11), (see [23]). One further point should be made. It is advantageous to use *upwind schemes* which calculate derivatives in the direction of the outward flowing normals. If such schemes are used, necessary numerical boundary conditions far from the region of interest do not flow backwards and create spurious solutions. One of the most straightforward

upwind schemes is the Engquist-Osher scheme [4,23]. In the next section, we show the application of this upwind scheme to a variety of hypersurface propagation problems.

4. EXAMPLES

In this section, we use our entropy-satisfying, upwind algorithm to compute the motion of a collection of test problems. In these examples, the input parameters are the initial position of the hypersurface, the time step Δt, the number of grid points in each coordinate direction N, and the particular speed function $F(K)$. Formation of cusps, generation of the entropy-satisfying weak solution, and changes in topology (merging and breaking) are handled automatically by the Hamilton-Jacobi formulation.

PROPAGATING INITIAL COSINE CURVE: $F(K)=1-\varepsilon K$.

We first demonstrate the diffusive effects of curvature on the formation of singularities in the propagating front. We consider an initial cosine curve $\gamma(0)=\cos(8\pi x)$, $0\le x\le 1$ propagating with speed $F(K)=1-\varepsilon K$. Periodic boundary conditions are employed at $x=0$ and $x=1$. We use 160 mesh points and time step $\Delta t=.001$. Because the moving front always remains the graph of a function, we solve the Hamilton-Jacobi with viscosity equation given in Eqn. (13). In Figure 8, we graph the front at various times. In the case $\varepsilon=0$ (Fig. 8a), corners form in the moving front, and these curvature singularities propagate upwards. In the case $\varepsilon=0.025$, (Fig. 8b), the front stays smooth due to the diffusive curvature term. In the case $\varepsilon=0.1$, (Fig. 8c), diffusion is so large that the peaks move down at first ($1-\varepsilon K<0$) before they flatten out enough to propagate upwards. These calculations were difficult to perform using the marker particle Lagrangian representation of Section II, even though there was an order of magnitude more smoothing ($\varepsilon=0.1$ compared with $\varepsilon=0.01$ here).

STAR-SHAPED FRONT BURNING OUT: $F(K)=1$.

We consider a seven-pointed star

$$\gamma(0) = (0.1+(0.065)\sin(7\cdot 2\pi s)\,\big[\cos(2\pi s),\sin(2\pi s)\big] \quad 0\le s\le 1$$

as the initial curve and solve the Hamilton-Jacobi level set formulation (Eqn. 11). We choose speed function $F(K)=1$ and view the boundary as a flame burning outwards. The computational domain is

a square centered at the origin of side length 1/2. We use 300 mesh points per side and a time step Δt=.0005. We follow an entire family of concentric star-shaped curves moving on the higher-dimensional surface $\psi(x,y,t)$. At any time $n\Delta t$, the front is plotted by passing the discrete grid function Ψ_{ij}^{n} to a standard contour plotter and asking for the contour Ψ=0. The initial curve corresponds to the boundary of the shaded region, and the position of the front at various times is shown in Fig. 9. The smooth initial curve develops sharp corners which then open up as the front burns, asymptotically approaching a circle.

STAR-SHAPED CURVE AND SPIRAL COLLAPSING UNDER CURVATURE: $F(K)=-K$.

With the same initial curve as above, we let $F(K)=-K$, corresponding to a front moving in with speed equal to its curvature. It has recently been shown [13] that any non-intersecting curve must collapse smoothly to a circle under this motion. With 300 mesh points and Δt=0.0005, we show the front at time t=0.0, 0.01, 0.02, 0.03, 0.04, 0.05 in Fig. 10. The plots show the relaxation of the peaks and troughs.

In Figure 11, we show the results of the same motion applied to a different initial curve, namely the wound spiral traced out by

$$\gamma(0) = (0.1e^{(-10y(s))} - (0.1-x(s))/20)(\cos(a(s)), \sin(a(s))) \quad s \in [0,1]$$

where $a(s) = 25\tan^{-1}(10y(s))$ and $(x(s),y(s)) = ((0.1)\cos(2\pi s)+0.1, (.05)\sin(2\pi s)+0.1)$. With 200 mesh points and Δt=0.0001, Here, we have rescaled time by a factor of 100 because the real front moves so quickly. Figure 11 shows the unwrapping of the spiral from t=0 to t=0.65. In Figures 11a-d we show the collapse to a circle and eventual disappearance at t=.295. Again, we follow a family of nested initial spirals lying on the higher-dimensional surface. The particular front we are interested in vanishes when the evolving surface moves entirely above the xy-plane, that is, when $\Psi_{ij}^{n} > 0$ for all ij.

BURNING SPIRAL AND CHANGE OF TOPOLOGY: MERGING AND BREAKING

Suppose the wound spiral in the previous example represents the boundary of a flame burning with speed $F(K)=1-\varepsilon K$, $\varepsilon=.01$. We use 200 mesh points and $\Delta t=.0001$. Figure 12a shows the initial curve as the boundary of the shaded region. In Fig. 12b, the spiral expands and pinches off due to the outward normal burning, separating into two flame fronts, one propagating outwards and one burning in. In Fig. 12c, the front at $t=.04$ is the boundary of the shaded region. The outer front expands and the inner front collapses and disappears. In Fig. 12d, all that remains is the outer front which asymptotically approaches a circle.

BURNING TORUS: CHANGE OF TOPOLOGY, MERGING AND BREAKING, $F(K)=1-\varepsilon K$, $\varepsilon=.01$.

We evolve the toroidal initial surface, described by the set of all points (x,y,z) satisfying

$$z^2 = (0.5)^2-((x^2+y^2)^{1/2}-0.05)^2.$$

This is a torus with main radius .5 and smaller radius .05. The computational domain is a rectangular parallelpiped with lower left corner (–1,–1,–0.8) and upper right corner (1,1,0.8). We evolve the surface with $F(K) = 1-\varepsilon K$, $\varepsilon=.01$, $\Delta t=.01$, and 90 points per x and y side of the domain and 72 mesh points in the z direction. Physically, we might think of this problem as the boundary of a torus separating products on the inside from reactants outside, with the burning interface propagating outwards. We follow a nested set of concentric toroidal shapes, and look for the level surface $\psi(x,y,z,t)=0$. In Figure 13 (Figs. 13a and 13b), we plot this surface at various times. First, the torus burns smoothly (and reversibly) until the inner radius collapses to zero. At that time (T=0.3), the genus goes from 1 to 0, normals collide, and the entropy condition is automatically invoked. The surface then looks like a sphere with deep inward spikes at the top and bottom which open up as the surface asymptotically approaches a sphere. Note that in the final four figures, the boundary of the expanding torus intersects the edge of the computational domain. This is reflected in the slicing of the level surface $\psi=0$ by the sides of the box. This demonstrates the advantage of an upwind formulation, since information flows out of the computational box.

COLLAPSE OF TORUS UNDER ITS MEAN CURVATURE: $F(K) = -K$.

Finally, we show the compute the motion of a torus under its mean curvature. This problem has been studied in [1,14,15]. The inner radius is .25 and the outer radius is .5. We imbed the problem in a unit cube of side length 2., and use a fairly coarse mesh with 45 points per side.

First, we perform the calculation with time step $\Delta t = .05$. Again, time is rescaled by a factor of 100 because the flow proceeds so quickly. Soon after the front starts to collapse, the calculation goes unstable, (see Fig. 14). This is manifested by the contour plotter finding numerous small spheres of radius one cell size having value $\psi = 0$. As time progresses, the evolving surface degenerates into noise as the contour plotter desperately tries to find zero level surfaces of a wildly oscillating function. What has happened is that we have violated the stability criteria for our numerical algorithm.

However, because the grid size is fixed, we need only decrease the time step to satisfy stability. This one of the strongest arguments for an Eulerian formulation. In Figures 15a and 15b, we repeat the calculation with time step $\Delta t = .01$. The torus deflates smoothly and collapses to the ring shown at T=4.1 before it vanishes. The final shape shown is the smallest surface that can be resolved on the given mesh size.

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