Flow under Curvature: Singularity Formation, Minimal Surfaces, and Geodesics

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Abstract

We study hypersurfaces moving under flow that depends on the mean curvature. The approach is based on a numerical technique that embeds the evolving hypersurface as the zero level set of a family of evolving surfaces. In this setting, the resulting partial differential equation for the motion of the level set function ϕ may be solved by using numerical techniques borrowed from hyperbolic conservation laws. This technique is used to analyze a collection of problems. First we analyze the singularity produced by a dumbbell collapsing under its mean curvature and show that a multi-armed dumbbell develops a separate, residual closed interface at the center after the singularity forms. The level set approach is then used to generate a minimal surface attached to a one-dimensional wire frame in three space dimensions. The minimal surface technique is extended to construct a surface of any prescribed function of the curvature attached to a given bounding frame. Finally, the level set idea is used to study the flow of curves on 2-manifolds under geodesic curvature dependent speed.

1 Introduction

In this paper, the motion of hypersurfaces under flow that depends on the mean curvature is studied. The main tool is a numerical technique, introduced in [20], that accurately follows the evolving hypersurface by embedding it as the zero level set in a family of hypersurfaces. The resulting partial differential equations for the motion of the level set function may be solved by using numerical techniques borrowed from hyperbolic conservation laws. The advantage to this approach is that sharp corners and cusps are accurately tracked, and topological changes in the evolving hypersurface are handled naturally with no special attention.

Starting from the fundamental perspective of this "level set approach" to propagating interfaces, this paper extends the technology in several directions. First, the collapse of a hypersurface under motion by mean curvature is studied. In [23], numerical experiments were performed of the collapse of a dumbbell, and showed that the handle pinches off and splits the single dumbbell into two separate hypersurfaces, each of which collapses to a point. In this paper, we show that an extension of this problem produces an interesting result: a multi-armed dumbbell leaves a separate, residual closed object at the center after the singularity forms. This result is verified by studying a series of similar numerical problems, each showing this detached hypersurface. At the end of this section, hypersurfaces propagating under Gaussian curvature are briefly considered.

Next, the level set approach is used to generate minimal surfaces attached to a one-dimensional wire frame in three space dimensions. Given a wire frame, we construct a surface passing through that 1-D curve and view it as the zero level set of a higher dimensional function. The mean curvature equation for this function is then evolved in time, producing a minimal surface as the final limiting state. Using this technique, the minimal surface spanning two parallel rings is studied. As a test, the exact catenoid shape is compared to computed values. The rings are then pulled apart and the evolution of the spanning minimal surface is computed as it shrinks, breaks, and changes topology, resulting in the final shape of two disks. Minimal surfaces spanning a collection of other frames are also given.

Next, we compute hypersurfaces of constant non-zero mean curvature by adding a hyperbolic component to the flow partial differential equation. As examples, catenoid-like surfaces of a variety of non-zero curvatures are computed. The extension of the level set formulation to the computation of surfaces of any prescribed function of the curvature is given.

Finally, the curvature flow algorithm is generalized to apply to curves on 2-manifolds in R^3. In this context, the curves flow with speed dependent on the geodesic curvature of the curve. Examples of curves on a cube, sphere, and torus are given. The techniques used for computing minimal surfaces are then applied to this setting creating an algorithm for computing the geodesics of a manifold.

In summary, using the basic level set approach, this paper introduces and applies extensions to complex surfaces, flows under Gaussian curvature, computation of surfaces of non-constant curvature, and geodesics on manifolds. We hope that some of the complex and subtle phenomena exposed in this paper may lead to further conjectures and understanding of curvature-driven flow. Finally, as a point of reference, this report first appeared as a technical report of the Center for Pure and Applied Mathematics at Berkeley; a few examples from that work contributed to a overview report which appeared in the Computational Crystal Growers Workshop /citechopp-sethian4.

2 The Level Set Formulation

2.1 Equations of Motion

Consider a closed curve $\gamma(t)$ where t is time, $t \in [0, \infty)$, moving with speed F normal to itself. The speed F may depend on local properties of the curve such as the curvature or normal vector. The origin of the work to follow propagating interfaces began in [21, 22], where the role of curvature in the speed function F for the propagating front $\gamma(t)$ was shown to be analogous to the role of viscosity in the corresponding hyperbolic conservation law for the evolving slope of $\gamma(t)$. This led to the level set formulation of the propagating interface introduced in [20]. In general terms, let $\gamma(0)$ be a closed, non-intersecting, $(N-1)$ dimensional hypersurface and construct a function $\phi(\bar{x},t)$ defined from R^N to R such that the level set $\{\phi = 0\}$ is the front $\gamma(t)$, that is

$$\gamma(t) = \{\bar{x} : \phi(\bar{x},t) = 0\} \bar{x} \in R^N \tag{1}$$

In order to construct such a function $\phi(\bar{x},t)$, appropriate initial conditions $\phi(\bar{x},0)$ and associated partial differential equation for the time evolution of $\phi(\bar{x},t)$ must be supplied. We initialize ϕ by

$$\phi(\bar{x},0) = \pm d(\bar{x}) \tag{2}$$

where $d(\bar{x})$ is the signed distance from $\bar{x}$ to the initial front $\gamma(t=0)$. In order to derive the partial differential equation for the time evolution of ϕ, consider the motion of some level set $\{\phi(\bar{x},t) = C\}$. Let $\bar{x}(t)$ be the trajectory of some particle located on this level set, so that, (see [18]),

$$\phi(\bar{x}(t),t) = C \tag{3}$$

The particle velocity $\frac{\partial \bar{x}}{\partial t}$ in the direction $\bar{n}$ normal to the level set C is given by

$$\frac{\partial \bar{x}}{\partial t} \cdot \bar{n} = F \tag{4}$$

where the normal vector $\bar{n}$ is given by $\bar{n} = \nabla\phi / \|\nabla\phi\|$. By the chain rule,

$$\phi_t + \frac{\partial \bar{x}}{\partial t} \cdot \nabla\phi = 0 \tag{5}$$

and substitution yields

$$\phi_t + F\|\nabla\phi\| = 0 \tag{6}$$

$$\phi(\bar{x}, t=0) = \text{given}$$

Eqn. (6) yields the motion of the interface $\gamma(t)$ as the level set $\phi = 0$, thus

$$\gamma(t) = \{x | \phi(\bar{x},t) = 0\} \tag{7}$$

Eqn. (6) is referred to as the *level set formulation.* For certain speed functions F, it reduces to some familiar equations. For example, for $F = 1$, the equation becomes the eikonal equation for a front moving with constant speed. For $F = 1 - \epsilon\kappa$, where κ is the curvature of the front, Eqn. (6) becomes a Hamilton-Jacobi equation with parabolic right-hand-side, similar to those discussed in [6]. For $F = \kappa$, Eqn. (6) reduces to the equation for mean curvature flow. When required, the curvature κ may be determined from the level set function ϕ. For example, in three space dimensions the mean curvature is given by

$$\kappa = \frac{\begin{array}{r}(\phi_{xx})(\phi_y^2 + \phi_z^2) + (\phi_{yy})(\phi_x^2 + \phi_z^2) + (\phi_{zz})(\phi_x^2 + \phi_y^2) \\ -2(\phi_x\phi_y\phi_{xy} + \phi_y\phi_z\phi_{yz} + \phi_x\phi_z\phi_{xz})\end{array}}{2(\phi_x^2 + \phi_y^2 + \phi_z^2)^{3/2}} \tag{8}$$

Eqn. (6) is an Eulerian formulation for the hypersurface propagation problem, because it is written in terms of a fixed coordinate system in the physical domain. This is in contrast to a more geometry-based Lagrangian approach, in which the motion of the hypersurface is written in terms of a parameterization in $(N - 1)$-dimensional space. There are several advantages of the Eulerian approach given in Eqn. (6). First, the fixed coordinate system avoids the numerical stability problems that plague approximation techniques based on a parameterized approach. Second, topological changes are handled naturally, since the level surface $\phi = 0$ need not be simply connected. Third, the formulation clearly applies in any number of space dimensions.

As illustration, in Figure 1 the motion of circle in the xy-plane propagating outwards with constant speed is shown. Fig. 1a shows the initial circle, while Fig. 1b shows the same circle as the level set $\phi = 0$ of the initial surface $\phi(x, y, t = 0) = (x^2 + y^2)^{1/2} - 1$. The one-parameter family of moving curves $\gamma(t)$ is then matched with the one-parameter family of moving surfaces in Figs. 1c and 1d.

This level set approach to front propagation has been employed in a varifigures/eulerformula.eps

Figure 1: Eulerian formulation of equations of motion

ety of investigations. In numerical settings, it has been used to study flame propagation [26] and crystal growth and dendrite simulation [24]. The theoretical underpinnings of this approach have been examined in detail by Evans and Spruck [7, 8]; for further theoretical work, see also [3, **?**, 9, 13].

2.2 Numerical Approximation

A successful numerical scheme to approximate Eqn. (6) hinges on the link with hyperbolic conservation laws. As motivation, consider the simple case of a moving front in two space dimensions that remains a graph as it evolves, and consider the initial front given by the graph of $f(x)$ with f, f', periodic on $[0,1]$. Let $y(x,t)$ be the height of the propagating function at time t, thus $y(x,0) = f(x)$. The normal at (x,y) is $(-y_x, 1)$, and the equation of motion becomes $y_t = F(\kappa)(1+y_x^2)^{1/2}$. Using the speed function $F(\kappa) = 1 - \epsilon\kappa$, where the curvature $\kappa = y_{xx}/(1+y_x^2)^{3/2}$, we get

$$y_t - (1+y_x^2)^{1/2} = \epsilon \frac{y_{xx}}{(1+y_x^2)} \tag{9}$$

To construct an evolution equation for the slope $u = dy/dx$, we differentiate both sides of the above with respect to x and substitute to obtain

$$u_t + \left[-(1+u^2)^{1/2}\right]_x = \epsilon \left[\frac{u_x}{(1+u^2)}\right]_x \tag{10}$$

Thus, the derivative of the Hamilton-Jacobi equation with curvature-dependent right-hand-side for the changing height $y(x,t)$ is a viscous hyperbolic conservation law for the propagating slope u. With this hyperbolic conservation law, an associated entropy condition must be invoked to produce the correct weak solution beyond the development of a singularity in the evolving curvature. Complete details may be found in [23].

Consequently, considerable care must be taken in devising numerical schemes to approximate the level set Eqn. (6). Because a central difference approximation to the gradient produces the wrong weak solution, we instead exploit the technology of hyperbolic conservation laws in devising schemes which maintain sharp corners in the evolving hypersurface and choose the correct, entropy-satisfying weak solution. One of the easiest such schemes is a variation of the Engquist-Osher scheme presented in [20]. This scheme is upwind in order to follow the characteristics at boundaries of the computational domain. The scheme is as follows. Decompose the speed function F into $F = F_A + F_B$, where F_A is treated as the hyperbolic component which must be handled through upwind differencing, and the remainder F_B which is to approximated through central differencing. Let ϕ^n_{ijk} be the numerical approximation to the solution ϕ at the point $i\Delta x$, $j\Delta y$, $k\Delta z$, and at time $n\Delta t$, where Δx, Δy, Δz is the grid spacing and Δt is the time step. We can then advance from one time step to the next by means of the numerical scheme

$$\begin{aligned}
\phi^{n+1}_{ijk} \quad = \quad & \phi^n_{ijk} + F_A \Delta t \cdot \\
& \big((\min(D^-_x \phi_{ijk}, 0))^2 + (\max(D^+_x \phi_{ijk}, 0))^2 + (\min(D^-_y \phi_{ijk}, 0))^2 \\
& \quad + (\max(D^+_y \phi_{ijk}, 0))^2 + (\min(D^-_z \phi_{ijk}, 0))^2 + (\max(D^+_z \phi_{ijk}, 0))^2 \big)^{1/2} \\
& + \Delta t F_B \|\nabla \phi\| \qquad (11)
\end{aligned}$$

Here, the difference operators D^-_x refers to the backward difference in the x

direction. The other difference operators are defined similarly.

2.3 Examples

In Figure 5, the motion of a closed two-dimensional spiral collapsing under its own curvature is given, that is, with $F(\kappa) = -\kappa$. Grayson [14] has shown that any non-intersecting closed curve must collapse smoothly to a circle; see also [10, 11, 12]. Consider the wound spiral traced out by

$$\gamma(0) = (0.1e^{(-10y(s))} - (0.1 - x(s))/20)(\cos(a(s)), \sin(a(s))) \qquad s \in [0,1]. \quad (12)$$

where

$$a(s) = 25 \tan^{-1}(10y(s))$$

$$(x(s), y(s)) = ((0.1)\cos(2\pi s) + 0.1, (.05)\sin(2\pi s) + 0.1).$$

The mesh is a 200 by 200 grid. In Figure 5, the unwrapping of the spiral and its eventual disappearance is depicted. Note that the calculation follows a family of spirals lying on the higher dimensional surface. The particular front corresponding to the propagating curve vanishes when the evolving surface moves entirely above the xy-plane, that is, when $\phi_{ij}^n > 0$.

As a different example, let the wound spiral in the previous example represent the boundary of a flame burning with speed $F(\kappa) = 1 - \epsilon\kappa$, $\epsilon = 0.1$. Here, the entropy condition is needed to account for the change in topology as the front burns together. In Fig. 6a, the initial spiral as the boundary of the shaded region is given. In Fig. 6b, the spiral expands, and pinches off due to the outward normal burning and separates into two flames, one propagating outwards and one burning in. In Figure 6c, the front is the boundary of the shaded region. The outer front expands and the inner front collapses and disappears. In Fig. 6d all that remains is the outer front which asymptotically approaches a circle.

3 Singularity Formation in Curvature Flow

3.1 Collapsing Dumbbells under Mean Curvature Flow

In this section, singularity formation of hypersurfaces in three space dimensions propagating under mean curvature is studied. Theoretical discussion of such flows have been made in [2, 15, 17]. Numerical calculations based on a marker Langrangian approach have been made in [1].

A well-known example is the collapse of a dumbbell, studied in [23]. In Fig. 7, the cross-section of the evolution of a dumbbell on a 214 by 72 by 72 grid collapsing under its mean curvaturel ($F(\kappa) = -\kappa$) is given. In Figure 7a, various time snapshots of the collapsing dumbbell are shown. As can be seen from the evolving slope, the center handle of the dumbbell pinches off, separating the collapsing hypersurface into two pieces.

An extension of this problem can be seen in Figure 8, where a periodic link of dumbbells is considered. As can be seen from the figures, each handle pinches off and breaks, leaving a collection of separate periodic closed hypersurfaces which each collapse into a sphere.

However, a different picture emerges if we consider multiple-armed dumbbells. In Figure 9, a three-armed dumbbell is shown. As this surface collapses under its mean curvature, the three handles pinch off, leaving a separate closed surface in the center. This "pillow" occurs because the mean curvature of each handle is larger than the saddle joints in the webbing between the spikes. Once this pillow separates off, it quickly collapses to a point.

A more dramatic and pronounced version is shown in Figure 10, which shows the collapse of a four-armed dumbbell. Once again, a residual pillow separates off in the center and collapses smoothly through a spherical shape to a point. The separated pillow is larger because the "webbing" between the arms collapses

GridSize	Diagonal	Volume
$30 \times 30 \times 10$	.35012	.00863
$46 \times 46 \times 16$	.34105	.00964
$61 \times 61 \times 60$	.35181	.01058
$61 \times 61 \times 61$	.35214	.01063

Table 1: Mesh Refinement of Detached Surface, four-armed case

slower as the number of arms increases. In Figure 11, the evolution of a six-armed dumbbell is calculated, showing the appearance of the isolated pillow. In this case, the pillow is almost the same size as the collapsing end balls.

To verify that these results are indeed real and not numerical artifacts, results of a quantitative study of the collapse of the four-armed dumbbell are given in Table 1. At the time when the dumbbell develops, the diagonal span of the pillow from one web to another is measured, as well as the total volume. Results obtained under considerable mesh refinement show that the results are independent of the chosen mesh size.

As a final demonstration of this process, Figures 12 and 13 show the collapse of a diagonal lattice of tubes. The lattice shown in Figure 12a (Time = 0.0) has periodic boundary conditions; thus, the figure represents one section of an infinite lattice. As the hypersurface collapses (Figure 12b, Time = 0.385), the pillow emerges at the intersection of the tubes. The bodies of the tubes collapse, leaving only the residual pillows which begin as pointed shapes (Figure 12c, Time = 0.405) and quickly evolve towards spherical shapes which collapse to points (Figure 12d, Time = 0.415). For comparison, in Figure 12e (Time = 0.385), the single hypersurface before breakage is shown from a slightly different angle.

A wholly different result is shown in Figure 13, where the same tube lattice is shown, only this time with thicker tubes (Figure 13a). In this case, the separate pillows appear in the *holes* of the lattice, as the evolving surface collapses around them (Figures 13b, 13c, 13d).

Next, a three-dimensional version of the spiral collapsing under mean curvature is computed. The three-dimensional spiral hypersurface shown in Figure 14 is actually hollow on the inside; the opening on the right end extends all the way through the object to the leftmost tip. The inner boundary of the spiral hypersurface is only a short thickness away from the outer boundary. As the hypersurface collapses under its mean curvature, the inner sleeve shrinks faster than the outer sleeve, and withdraws to the rightmost edge before the outer sleeve collapses around it.

3.2 Collapsing Surfaces under Gaussian Curvature Flow

A variation on the above study can performed using the Gaussian curvature instead of the mean curvature. Starting with the definition of Gaussian curvature $\kappa_{\mathrm{Gaussian}}$ for a surface (see [16]), an expression for $\kappa_{\mathrm{Gaussian}}$ in terms of the level set function ϕ can be obtained, namely

$$\kappa_{\mathrm{Gaussian}} = \frac{\begin{array}{c}\phi_x^2(\phi_{yy}\phi_{zz} - \phi_{yz}^2) + \phi_y^2(\phi_{xx}\phi_{zz} - \phi_{xz}^2) + \phi_z^2(\phi_{xx}\phi_{yy} - \phi_{xy}^2) \\ + 2[\phi_x\phi_y(\phi_{xz}\phi_{yz} - \phi_{xy}\phi_{zz}) + \phi_y\phi_z(\phi_{xy}\phi_{xz} - \phi_{yz}\phi_{xx}) \\ + \phi_x\phi_z(\phi_{xy}\phi_{yz} - \phi_{xz}\phi_{yy})]\end{array}}{(\phi_z^2 + \phi_y^2 + \phi_x^2)^2} \tag{13}$$

Suppose we consider flow of surfaces under Gaussian curvature. If the closed hypersurface is convex, the Gaussian curvature will not change sign, and the surface should collapse as it flows, see [19]. In Figure 15, the motion of a flat disk-like surface collapsing under its Gaussian curvature is shown. The sharply curved regions move in quickly, since they are regions of high Gaussian curvature, and the hypersurface moves towards a spheroidal shape.

In the case of non-convex closed hypersurfaces, the situation is delicate, due to the fact that Gaussian curvature is the *product* of the two principle curvatures. In general, the problem acts like the backwards heat equation, and goes unstable in most cases. We illustrate with two examples. In Figure 16, a very slightly depressed dumbbell is shown. The balls have radius .5, while the inner handle has radius .45. The distance between the centers of the two end balls is 2.0. Because the variation away from a cylindrical shape is small, the strong positive Gaussian curvature on the ends pulls the surface inwards, and it seems that the calculation does not go unstable and the surface collapses. In contrast, in Figure 17 the evolution of two spheres glued together is shown. The ring connecting the two spheres has a fairly narrow radius, and thus the Gaussian curvature along the edges of the ring is initially large and negative. This carries the indentation area outwards, and wild oscillations develop, showing the instability.

4 Construction of Minimal Surfaces

In this section, the level set perspective is used to construct minimal surfaces. Consider a closed curve $\Gamma(s)$ in R^3; $\Gamma : [0,1] \to R^3$. The goal is to construct a membrane with boundary Γ and mean curvature zero.

Given the bounding wire frame Γ, consider some initial surface $S(t=0)$ whose boundary is Γ. Let $S(t)$ be the family of surfaces parameterized by t obtained by allowing the initial surface $S(t=0)$ to evolve under mean curvature, with boundary given by Γ. Defining the surface S by $S = \lim_{t\to\infty} S(t)$, one expects that the surface S will be a minimal surface for the boundary Γ. Thus, given an initial surface $S(0)$ passing through Γ, construct a family of neighboring surfaces by viewing $S(0)$ as the zero level set of some function ϕ over all of R^3. Using the level set Eqn. (6), evolve ϕ according to the speed law $F(\kappa) = -\kappa$.

figures/twopoints.eps

Figure 2: Grid points around the boundary

Then the minimal surface S will be given by

$$S = \lim_{t \to \infty} \{x | \phi(x,t) = 0\} \tag{14}$$

The difficult challenge with the above approach is to ensure that the evolving zero level set always remains attached to the boundary Γ. This is accomplished by creating a set of boundary conditions on those grid points closest to the wire frame and link together the neighboring values of ϕ to force the level set $\phi = 0$ through Γ. The underlying idea is most easily explained through a one-dimensional example. Here, we follow the discussion in Chopp [5].

Consider the simple problem of finding the shortest distance between two points A and B in the plane. The goal is to give conditions on a function ϕ defined in R^2 so that $\phi(A,t) = \phi(B,t) = 0$ for all time. In Figure 2, an initial curve is shown which is the level set $\phi(x,t) = 0$ and the boundary points A and B. Suppose that the point A is located in the middle between points g_i and g_d (Figure 3). In order to have that $\phi(A,t) = 0$ for all time, we require that $\phi(g_i,t) = -\phi(g_d,t)$. Label the subscripts d and i for dependent and independent, and set the dependent point in terms of the independent point. This binds the dependent points to the independent points in a way that forces the zero level set through the points A and B. In general, the boundary conditions will be represented as a vector equation of the form

$$v_{\text{dep}} = A v_{\text{ind}} \tag{15}$$

figures/pointa.eps

Figure 3: Grid points around the boundary

where

$$v_{\text{dep}} = \left\{ \begin{array}{c} \phi(g_{d,1}) \\ \phi(g_{d,2}) \\ \vdots \\ \phi(g_{d,m}) \end{array} \right\} \qquad v_{\text{ind}} = \left\{ \begin{array}{c} \phi(g_{i,1}) \\ \phi(g_{i,2}) \\ \vdots \\ \phi(g_{i,n}) \end{array} \right\} \tag{16}$$

and A is an $m \times n$ matrix. The matrix A is determined from the chosen mesh and wire frame, and hence both the classification of dependent and independent points and the matrix A need only be computed once at the beginning of the calculation. This links the set of all dependent points in terms of the set of all independent points, in such a way that the level set $\phi = 0$ is forced to pass through the wire frame. Complete details of the automatic technique for generating this list of boundary conditions may be found in [5].

There is one final issue that comes into play in the evolution of the level set function ϕ towards a minimal surface. By the above set of boundary conditions, only the zero level set $\phi = 0$ is constrained. Thus the other level surfaces are free to move at will, which means on one side of the level set $\phi = 0$ the surfaces will crowd together, while on the other side they will pull away from the zero level set. This causes numerical difficulties in the evaluation of derivatives over such a steep gradient. A reinitialization procedure is used to remedy this; after a given number of time steps, the level set $\phi = 0$ is computed, and the function ϕ is reinitialized using the signed distance function as given in Section 2. This uniformly redistributes the level sets so that the calculation can proceed.

As a test example, the minimal surface spanning two rings has an exact solution given by the catenoid

$$r(x) = a \cosh(x/a) \tag{17}$$

where $r(x)$ is the radius of the catenoid at a point x along the x axis, and a is the radius of the catenoid at the center point $x = 0$. Suppose that the boundary consists of two rings of radius R located at $\pm b$ on the x-axis. Then the parameter a is determined from the expression

$$R = a \cosh(b/a) \tag{18}$$

If there is no real value of a which solves this expression, then a catenoid solution between the rings does not exist. Thus, for a given R, if the rings are closer than some minimal distance $2b_{\max}$ apart, there are two distinct catenoid solutions, one of which is stable and the other is not. For rings exactly $b_{\max}$ apart, there is only one solution. For rings more than $b_{\max}$ apart, there is no catenoid solution.

In Figure 18, the minimal surface spanning two rings each of radius 0.5 and at positions $x = \pm.277259$ is computed. A cylinder spanning the two rings is taken as the initial level set $\phi = 0$. A $27 \times 47 \times 47$ mesh with space step 0.025 is used. The final shape is shown in from several different angles in Figure 18. Next, in Figure 19, this same problem is computed, but the rings are placed far enough apart so that a catenoid solution cannot exist. Starting with a cylinder as the initial surface, the evolution of this cylinder is computed as it collapses under mean curvature while remaining attached to the two wire frames. As the surface evolves, the middle pinches off and the surface splits into two surfaces, each of which quickly collapses into a disk. The final shape of a disk spanning each ring is indeed a minimal surface for this problem. This example illustrates one of the virtues of the level set approach. No special cutting or *ad hoc* decisions are employed to decide when to break the surface. Instead the characterization of the zero level set as but one member of a family of flowing surfaces allows this smooth transition.

In Figure 20, six squares which are initially spanned by the union of three cylinders with square cross-section are shown. Each square has side of length 1/2. On the x-axis, the squares are located at ± 0.375, on the y-axis at ± 0.775, and on the z-axis at ± 1.275. The different distances cause the surface to break at three different times. More complex examples of minimal surfaces are given in [5].

5 Extensions to Surfaces of Prescribed Curvature

5.1 Surfaces of Constant Mean Curvature

The above technique can be extended to produce surfaces of constant but non-zero mean curvature. To do so requires further inspection of the suggestive example of a front propagating with speed $F(\kappa) = 1 - \epsilon\kappa$. Suppose that $\epsilon = 1$, and consider the evolution of the partial differential equation

$$\phi_t = (1 - \kappa)\|\nabla\phi\| \tag{19}$$

where again the mean curvature κ is given by Eqn. 8. Furthermore, consider initial data given by

$$\phi(x, y, z, t = 0) = (x^2 + y^2 + z^2)^{1/2} - 1 \tag{20}$$

The zero level set is the sphere of radius one, which remains fixed under the motion $F(\kappa) = 1 - \kappa$. All level surfaces inside the unit sphere have mean curvature greater than one, and hence propagate inwards, while all level surfaces outside the unit sphere have mean curvature less than one, and hence propagate outwards. Thus, the level sets on either side of the zero level set unit sphere pull apart. If one were to apply the level set algorithm in free space, the gradient $\|\nabla\phi\|$ would smooth out to zero along the unit sphere surface, and this eventually causes numerical difficulties. However, the reinitialization process describe earlier periodically rescales the labeling of the level sets, and thus $\|\nabla\phi\|$ is renormalized.

Thus, in order to construct a surface of constant curvature κ_0, start with any initial surface passing through the initial wire frame and allow it to propagate with speed

$$F(\kappa) = \kappa_0 - \kappa \tag{21}$$

Here, as before, the "constant advection term" κ_0 is taken as the hyperbolic component F_A in Eqn. 11, and treated using the entropy-satisfying upwind difference solver, while the parabolic term κ is taken as F_B, and is approximated using central differences.

Using the two ring "catenoid" problem as a guide, in Figure 21 this technique is used to compute the surface of constant curvature spanning the two rings. In each case, the initial shape is the cylinder spanned by the rings. The final computed shapes for a variety of different mean curvatures are shown. In Figure 21a, a surface of mean curvature 2.50 spanning the rings is given: the rings are located a distance .61 apart and have diameter 1.0. The resulting surface bulges out to fit against the two rings. In Figure 21b, a surface of mean curvature 1.0 is found, which corresponds to the initial surface. The slight bowing is due to the relatively coarse $40 \times 40 \times 40$ mesh. In Figure 21c, the catenoid surface of mean curvature 0.0 is given. We isolated the value of -0.33 as a value close to the breaking point (Figure 21d). In Figure 21e, a mean curvature value of -0.35 is prescribed, causing the initial bounding cylinder to collapse onto the two rings and bulge out slightly. Finally, in Figure 21f, bowing out disks corresponding to surfaces of mean curvature -1.00 are shown.

5.2 Surfaces of Non-Constant Mean Curvature

To complete the construction, the technique is extended to allow the calculation of surfaces of a prescribed function of the curvature. Suppose we wish to find a surface of curvature $A(\bar{x})$ passing through a given wire frame, where A is some given function of a point $\bar{x}$ in three-dimensional space. Using the above approach, the initial bounded zero level surface is evolved with speed

$$F(\kappa) = A(\bar{x}) - \kappa \tag{22}$$

As a simple example, the surface spanning two rings with curvature at any point x along the x-axis given by $10\cos(10x)$ is constructed. The obtained wavy surface with prescribed curvature is shown in Figure 22. The grid size is $40 \times 75 \times 75$, with uniform mesh size of $\Delta x = 0.02$. The rings are located at ± 0.305, with radius .5. After 100 time steps, the change in ϕ is less than 10^{-5} per time step of size 10^{-4}, indicating the computed value has converged to the solution.

6 Geodesic Curvature Flow

The curvature flow algorithm can be generalized to more complicated two-dimensional spaces. For example, we may let the level set function ϕ be defined on a differentiable 2-manifold in R^3 with speed depending on geodesic curvature. By restricting the level set function ϕ to coordinate patches, it is possible to study single curves on non-simply connected manifolds , e.g. a torus. The fixed boundary condition techniques for minimal surfaces can also be applied here. In this case, a curve with fixed endpoints should flow towards a geodesic of the manifold, i.e. a curve with constant geodesic curvature zero.

6.1 Equations of Motion

Consider a 2-dimensional manifold $M \subset R^3$. Let $\gamma(t) \subset M$, for $t \in [0, \infty)$, be a family of closed curves moving with speed $F(\kappa_g)$ in the direction normal to itself on M. Here, κ_g is the geodesic curvature of $\gamma(t)$ on M. Let $g_t(s)$ be the parameterization of $\gamma(t)$ by arc length.

First assume that M is orientable. In this case, the unit normal map N is continuous on M. At every point $g_t(s)$, a natural coordinate system for T_M is given by the vectors $g_t'(s)$, $N \times g_t'(s)$. Thus, for any point $x(t) \in \gamma(t)$, the velocity under this flow is given by

$$\dot{x} \cdot (N \times g_t') = F(\kappa_g).$$

The expression for geodesic curvature is given by

$$\kappa_g = (N \times g_t') \cdot g_t''. \tag{23}$$

Note that a change in sign of the unit normal N results in a corresponding change in sign of κ_g. If F is an odd function, then $\dot{x}$ is independent of the choice of N. However, if F is not an odd function, then the choice of the normal changes the flow. Therefore, if M is not orientable, then only odd speed functions F are allowed. The algorithm presented here also requires that F be an odd function when M is not simply connected.

Assume the manifold M is given by $M = f^{-1}(0)$, where $f : R^3 \to R$. We break the manifold into a collection of coordinate maps, $\{(U_i, \pi_i)\}$ such that $M = \cup U_i$, each set U_i is simply connected, and $\pi_i : U_i \to V_i \subset R^2$ is a bijection. The computing is done on the collection of sets $V_i = \pi_i(U_i)$. We define the function $\Phi_i : V_i \to R$ by $\Phi_i(x, t) = \phi(\pi_i^{-1}(x), t)$, so that $\phi(x, t)|_{U_i} = \Phi_i(\pi_i(x), t)$.

In order to write the equations of motion in the level set representation, we must compute a velocity field on the entire manifold M. We compute the velocity field on each coordinate patch and then see that it is consistent in regions of overlap. For this section, assume $\phi = \phi|_{U_i}$. At any point $x \in U_i$, the velocity vector will be normal to the level set of ϕ containing x, towards the center of curvature in the tangent space $T_M(x)$, and have length $F(\kappa_g)$.

The geodesic curvature of the curve $\phi^{-1}(C)$ in terms of ϕ is given by

$$\kappa_g = \frac{[N \times \tau] \cdot n}{1 - (n \cdot N)^2} \left\{ \tau \cdot \left[\left(\frac{n \cdot N}{\|\nabla f\|} \nabla^2 f - \frac{1}{\|\nabla \phi\|} \nabla^2 \phi \right) \cdot \tau \right] \right\}$$

where

$$n = \frac{\nabla \phi}{\|\nabla \phi\|} \quad N = \frac{\nabla f}{\|\nabla f\|} \quad \text{and} \quad \tau = \frac{\nabla f \times \nabla \phi}{\|\nabla f \times \nabla \phi\|}.$$

The direction of the velocity vector is the same as the orthogonal projection of $\nabla \phi$ onto T_M, so

$$v = \tau \times N = \frac{n - (n \cdot N)N}{\|n - (n \cdot N)N\|}$$

and the velocity at x on M is described by

$$\dot{x} \cdot v = F(\kappa_g).$$

The computing is done on the sets $\pi_i(U_i)$, so we want the equation of motion in terms of Φ_i. Therefore, for a point $x \in \pi_i(U_i)$, the equation of motion is described by

$$\dot{x} \cdot \eta = F(\kappa_g) D\pi_i(v) \cdot \eta$$

where $\eta = \nabla \Phi_i / \|\nabla \Phi_i\|$ is the unit normal to the level set containing $\Phi_i(x)$. Let

$$\tilde{F}(\kappa_g) = F(\kappa_g) D\pi_i(v) \cdot \nabla \Phi_i / \|\nabla \Phi_i\|, \tag{24}$$

then the equation of motion in terms of Φ is given by

$$0 = \Phi_{it} + \tilde{F}(\kappa_g) \|\nabla \Phi_i\|. \tag{25}$$

6.2 Geodesic Curvature Algorithm

Putting everything together, the general algorithm for curvature flow on a manifold can be stated as follows:

1. Choose coordinate patches and maps to represent the manifold M.

2. Initialize the functions Φ_i on each coordinate patch.

3. Compute the boundary values in each coordinate patch based upon overlap values with neighboring patches.

4. Advance each Φ_i in time according to the differential equation (25).

5. Go to step 3.

At any time t, the curve $\gamma(t)$ can be reconstructed from

$$\gamma(t) = \cup \pi_i^{-1}(\Phi_i^{-1}(0,t)) \tag{26}$$

We will now discuss the details of each of these steps.

Given a manifold M, it is important to choose simply connected coordinate patches $\{U_i, \pi_i\}$, so that any simple curve can be represented by a level set of a function ϕ on U_i. The equations given above are for the case when π_i maps onto a rectangular coordinate system in R^2. In the overlap sets, where $U_i \cap U_j \neq \emptyset$, there must be at least a three grid point overlap between the sets $V_i = \pi_i(U_i)$ and $V_j - \pi_j(U_j)$. Computing the boundary conditions for each V_i is made easier if the grid points and projection maps π_i are chosen so that if $x \in V_i$ is a grid point in V_i and $\pi_i^{-1}(x) \in U_i \cap U_j$, then $\pi_j(\pi_i^{-1}(x))$ is also a grid point in V_j.

For example, let M be a torus with large radius R and small radius r symmetric about the z-axis. One choice of coordinate patches is

$$U_1 \quad = \quad \left\{(x,y,z) : \sqrt{x^2+y^2} > R - \epsilon, x > -\epsilon\right\}$$

figures/toruspatch.eps

Figure 4: Example of coordinate patches on a torus.

$$\begin{aligned} U_2 &= \left\{(x,y,z) : \sqrt{x^2+y^2} < R+\epsilon, x > -\epsilon\right\} \\ U_3 &= \left\{(x,y,z) : \sqrt{x^2+y^2} < R-\epsilon, x < \epsilon\right\} \\ U_4 &= \left\{(x,y,z) : \sqrt{x^2+y^2} > R-\epsilon, x < \epsilon\right\} \end{aligned}$$

where each $\pi_i : U_i \to V_i = (-\pi/2-\epsilon, \pi/2+\epsilon) \times (-\pi/2-\epsilon, \pi/2+\epsilon)$ in the natural way. Then a uniform rectangular grid is placed on the closure of V_i. A diagram of this choice of coordinate maps is given in figure 4.

The objective when initializing the functions Φ_i is to satisfy Eqn. (26). We use the signed distance function where the distance is computed in the manifold space. The sign of Φ_i is assigned on each coordinate patch independently. This ensures that at each grid point $x \in U_i \cap U_j$,

$$|\Phi_i(\pi_i(x))| = |\Phi_j(\pi_j(x))|.$$

This is important for ensuring consistent motion in the overlap regions. If F is not an odd function, we must additionally require $\Phi_i(\pi_i(x)) = \Phi_j(\pi_j(x))$.

The evolution on each patch is computed on the interior grid points of each patch V_i. The values at the boundary are taken from neighboring patches. Let x be a grid point on the boundary of V_i. By construction, $\pi_i^{-1}(x)$ is in the interior of some other patch U_j. Therefore, we have $\Phi_i(x) = \pm\Phi_j(\pi_j(\pi_i^{-1}(x)))$, where the sign is determined by whether $\Phi_i \circ \pi_i$ and $\Phi_j \circ \pi_j$ are of equal (plus) or opposite (minus) sign in the interior of the region $U_i \cap U_j$.

Two coordinate patches may have two or more disconnected overlap regions on M. The sign convention must be determined individually for each distinct overlap. For example, the two outer patches on the torus example above overlap in two distinct locations. It is possible for both patches to share the same sign convention in one overlap while having the opposite convention in the other overlap. This property makes it possible to model a single curve on a torus as a level set of a function within each coordinate patch.

Following the argument in section 5.1, we break the function F into the constant and non-constant parts, $F(\kappa) = F_1 - F_2(\kappa)$. Eqn. (25) then becomes,

$$\Phi_{i t} + \tilde{F}_1 \|\nabla\Phi_i\| = \tilde{F}_2(\kappa_g)\|\nabla\Phi_i\| \tag{27}$$

As in section 2, upwind techniques from hyperbolic conservation laws are used to compute the left hand side and central differences are used on the right hand side.

6.3 Examples

We begin with flow on a sphere. The sphere is constructed with a single coordinate patch with the projection mapping the sphere onto its spherical coordinate system, the square $[-\pi, \pi] \times [-\pi, \pi]$. The gap in figures 23 and 24 shows the boundary of the coordinate patch. Figure 23 shows an initial circle just smaller than a great circle shrinking to a point at the top. Figure 24 shows a periodic curve symmetric with respect to the equator collapsing to the equator.

Next, we show flow on a torus. If the torus is constructed with a single coordinate patch, then it is not possible to model a single non-contractible curve using a level set approach. For curves which are not too complicated, it is possible to construct a second curve to allow for the level set formulation. In figure 25, a single coordinate patch is used and the flow of two non-intersecting curves is computed.

However, if multiple coordinate patches are used, then a required sign change can be handled by the communications between coordinate patches. Therefore, it is possible to model a single curve on a torus. Figure 26 shows a single curve flowing on a torus. The coordinate patches for the torus are described in section 6.2.

Subsets of manifolds can also be used. For example, we compute the flow of an oval and a periodic curve on a helicoid. The boundary conditions are periodic at the top and bottom, one sided derivatives on the sides of a single rectangular coordinate patch. Figure 27 shows the oval as it shrinks to a point, while figure 28 shows the periodic curve flowing towards the central axis of the helicoid.

Another example of flow on a submanifold is when the manifold is the graph of a function $f : R^2 \to R$. In this example, we use $f(x, y) = 2\cos(2\sqrt{x^2 + y^2})$. One-sided derivatives are used for all boundaries of a rectangular region of this graph. Figure 29 shows a straight line perturbed off-center flowing away from the center over a ridge.

Finally, we show several flows on a cube. The cube is constructed with six coordinate patches corresponding to the faces of the cube. The first experiment on the cube involves a comparing the flow on a cube with orthogonal edges to the flow on a cube with constructed round edges. Figures 30–33 show the orthogonal cube followed by three different rounded cubes with the same initial curve. The initial curve is flatter on the front faces than on the top. We see that the flow is similar in all cases with the curve collapsing to a point near the corner. Two other flows on a cube are shown in figures 34 and 35.

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figures/spiral1.ps figures/spiral2.ps

figures/spiral3.ps figures/spiral4.ps

Figure 5: Collapsing 2-dimensional spiral

figures/flame1.ps figures/flame2.ps

figures/flame3.ps figures/flame4.ps

figures/flame5.ps figures/flame6.ps

Figure 6: Burning spiral

Figure 7: Collapsing Dumbbell

figures/dstring1.ps figures/dstring2.ps

figures/dstring3.ps figures/dstring4.ps

Figure 8: Collapse of a dumbbell string

figures/threearm1.ps

figures/threearm2.ps figures/threearm3.ps

figures/threearm4.ps figures/threearm5.ps

Figure 9: Collapse of a three-armed dumbbell

figures/fourarm1.ps

figures/fourarm2.ps

figures/fourarm3.ps

figures/fourarm4.ps

figures/fourarm5.ps

figures/fourarm6.ps

Figure 10: Collapse of a four-armed dumbbell

figures/sixarm1.ps

figures/sixarm2.ps figures/sixarm3.ps

figures/sixarm4.ps figures/sixarm5.ps

Figure 11: Collapse of a six-armed dumbbell

figures/lattice1.ps

figures/lattice2.ps figures/lattice3.ps

figures/lattice4.ps figures/lattice5.ps

Figure 12: Collapse of a periodic lattice, small tubes

figures/biglattice1.ps figures/biglattice2.ps

figures/biglattice3.ps figures/biglattice4.ps

figures/biglattice5.ps figures/biglattice6.ps

Figure 13: Collapse of a periodic lattice, large tubes

figures/testtube1.ps

figures/testtube2.ps figures/testtube3.ps

figures/testtube4.ps figures/testtube5.ps

Figure 14: Collapse of a twisted testtube

figures/mint1.ps figures/mint2.ps

figures/mint3.ps figures/mint4.ps

figures/mint5.ps figures/mint6.ps

Figure 15: Collapse of a surface under Gaussian curvature

figures/shallow1.ps figures/shallow2.ps

figures/shallow3.ps figures/shallow4.ps

figures/shallow5.ps figures/shallow6.ps

Figure 16: Collapse of a slightly non-convex surface under Gaussian curvature

figures/balls1.ps figures/balls2.ps

figures/balls3.ps figures/balls4.ps

Figure 17: Collapse of a slightly non-convex surface under Gaussian curvature

figures/cat1.ps figures/cat2.ps

figures/cat3.ps figures/cat4.ps

figures/cat5.ps figures/cat6.ps

Figure 18: Euler's catenoid minimal surface

figures/catsplit1.ps figures/catsplit2.ps

figures/catsplit3.ps figures/catsplit4.ps

figures/catsplit5.ps figures/catsplit6.ps

Figure 19: Splitting catenoid evolution

figures/sixsplit1.ps figures/sixsplit2.ps

figures/sixsplit3.ps figures/sixsplit4.ps

figures/sixsplit5.ps figures/sixsplit6.ps

Figure 20: Splitting six-armed catenoid evolution

figures/curv1.ps figures/curv2.ps

figures/curv3.ps figures/curv4.ps

figures/curv5.ps figures/curv6.ps

Figure 21: Constant mean curvature surfaces with fixed boundary

figures/undulate1.ps figures/undulate2.ps

figures/undulate3.ps figures/undulate4.ps

Figure 22: Non-constant prescribed curvature surface

Figure 23: Circle shrinking on a sphere

Figure 24: Periodic curve shrinking to a great circle

Figure 25: Two curves flowing on a torus

Figure 26: A single curve flowing on a torus

figures/helicoid2.ps

Figure 27: A single loop shrinking on a helicoid

figures/helicoid1.ps

Figure 28: A periodic curve on a helicoid

figures/sombrero.ps

Figure 29: A curve flowing on the graph of $f(x, y) = 2\cos(2\sqrt{x^2 + y^2})$

Figure 30: A single loop flowing on a cube with orthogonal edges

Figure 31: A single loop flowing on a cube with large rounded edges

Figure 32: A single loop flowing on a cube with medium rounded edges

Figure 33: A single loop flowing on a cube with small rounded edges

Figure 34: A single loop pulled over two opposite corners on a face

Figure 35: A single loop pulled over alternating corners