ON REPRESENTATIONS OF THE AFFINE SUPERALGEBRA $\mathfrak{q}(n)^{(2)}$

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Abstract. In this paper we study highest weight representations of the affine Lie superalgebra $\mathfrak{q}(n)^{(2)}$. We prove that any Verma module over this algebra is reducible and calculate the character of an irreducible $\mathfrak{q}(n)^{(2)}$- module with a generic highest weight. This formula is analogous to the Kac-Kazhdan formula for generic irreducible modules over affine Lie algebras at the critical level.

1. Introduction

1.1. Due to the existence of the Casimir operator a Verma module over a Kac-Moody superalgebra with a symmetrizable Cartan matrix is irreducible if its highest weight is generic.

In this paper we study the structure of Verma modules over the twisted affine superalgebra $\mathfrak{q}(n)^{(2)}, n \geq 3$. This superalgebra has a non-symmetrizable Cartan matrix. We will see that any Verma module over it is reducible.

We prove that a simple highest weight $\mathfrak{q}(n)^{(2)}$-module with a "generic" highest weight λ has the following character formula:

$$\operatorname{ch} L(\lambda) = e^{\lambda} \prod_{\alpha \in \hat{\Delta}^+_{re,\overline{0}}} (1 - e^{-\alpha})^{-1} \prod_{\alpha \in \hat{\Delta}^+_{\overline{1}}} (1 + e^{-\alpha}). \tag{1}$$

Here genericity means that λ lie in the compliment to countably many hypersurfaces in $\hat{\mathfrak{h}}^*$. This formula is analogous to the Kac-Kazhdan character formula, see below.

1.1.1. Let $\hat{\mathfrak{g}}$ be a complex affine Lie algebra. A Verma module with a generic weight at non-critical level is irreducible. A Verma module at the critical level is always reducible; if λ is a generic weight at the critical level then the simple module $L(\lambda)$ has the following character: $\operatorname{ch} L(\lambda) = e^{\lambda} \prod_{\alpha \in \hat{\Delta}^+_{re}} (1 - e^{-\alpha})^{-1}$. This formula was conjectured by V. G. Kac and D. A. Kazhdan in [KK] and proven by [Ku],[FF],[Sz]. Moreover, as it is shown in [Ku], there is a bijection between submodules of a Verma module with a "generic" highest weight at the critical level and graded ideals of a polynomial algebra in countable number of variables, in fact, the Jantzen filtration of a Verma module corresponds to the adic filtration of a polynomial algebra.

It turns out that the adic filtration comes to the picture as follows. Let $\mathcal{H}$ (resp., $\mathcal{H}_-$) be the sum of positive (resp., negative) imaginary root spaces of $\hat{\mathfrak{g}}$. Then $\mathcal{H}_- \oplus \mathbb{C}K \oplus \mathcal{H}$ is a countably dimensional Heisenberg algebra. A vacuum module of the Heisenberg algebra is irreducible if the central charge is non-zero. Let V^0 be the vacuum module with zero central charge. Identify V^0 with the universal enveloping algebra $\mathcal{U}(\mathcal{H}_-)$, which is a polynomial algebra in a countable number of variables. Recall that $\hat{\mathfrak{g}}$ is $\mathbb{Z}$-graded (by the eigenvalues of D) and view V^0 as a graded module via this idenitification. $\mathcal{H}_-$ inherits the $\mathbb{Z}$-grading. It is easy to see that the Jantzen filtration on V^0 corresponds to the adic filtration on $\mathcal{U}(\mathcal{H}_-)$. In this interpretation a Verma module $M(\lambda)$ with a generic highest weight at the critical level looks like V^0: there exists an isomorphism HC_+ from the space of singular vectors $M(\lambda)^{\hat{\mathfrak{n}}}$ to V^0 which induces a bijection between the submodules of $M(\lambda)$ and the homogeneous submodules of V^0; moreover, this bijection is compatible with the Jantzen filtrations.

A similar fact holds for affine Lie superalgebras with symmetrizable Cartan matrices if $\mathcal{H}_- \oplus \mathbb{C}K \oplus \mathcal{H}_+$ is a Heisenberg algebra (i.e., except the case $A(2k, 2l)^{(4)}$), see [G].

1.1.2. Let us go back to $\mathfrak{q}(n)^{(2)}$ for $n \geq 3$. In this case the subalgebra $\tilde{\mathcal{H}}$ generated by imaginary root spaces is not isomorphic to a Heisenberg algebra. It turns out that the even part of $\tilde{\mathcal{H}}$ is a centre. As a result, a Verma module over $\tilde{\mathcal{H}}$ is always reducible. We show that a Verma $\mathfrak{q}(n)^{(2)}$-module $M(\lambda)$ with a generic highest weight λ looks like a Verma $\tilde{\mathcal{H}}$-module $\mathcal{M}(\lambda)$: there exists an epimorphism $HC_+ : M(\lambda)^{\hat{\mathfrak{n}}} \to \mathcal{M}(\lambda)$ which extends to a bijection between the submodules of $M(\lambda)$ and the homogeneous submodules of $\mathcal{M}(\lambda)$. This implies the character formula (1).

It is possible to show (as it is done in [G]) that for a generic λ the Jantzen filtration on $M(\lambda)$ looks as follows: $M(\lambda)^0 = M(\lambda)$, $M(\lambda)^k = M'(\lambda)$ for any $k > 0$, where $M'(\lambda)$ is the maximal proper submodule of $M(\lambda)$. The structure of the Jantzen filtration on $\mathcal{M}(\lambda)$ is similar, and therefore the bijection between submodules in $M(\lambda)$ and in $\mathcal{M}(\lambda)$ is compatible with the Jantzen filtrations.

1.1.3. Recall that for affine Lie superalgebras with symmetrizable Cartan matrices all Shapovalov determinants are non-zero polynomials which admit a linear factorization (i.e. all their irreducible factors are linear).

For $\mathfrak{q}(n)^{(2)}$ this does not hold. The fact that any Verma module is not simple means that there exists a Shapovalov determinant $\det S_\nu$ which is identically equal to zero. Let δ be a minimal imaginary root. It turns out that $\det S_\nu = 0$ iff $\nu \geq 2\delta$. Another interesting feature of $\mathfrak{q}(n)^{(2)}$ is that $\det S_\delta$ has an irreducible factor of degree $n-1$: this follows from the fact that the leading term of $\det S_\delta$ is divisible by the irreducible polynomial $h_1 \dots h_n(\frac{1}{h_1} + \dots \frac{1}{h_n})$ which has degree $n-1$.

1.2. Main result. Let $\hat{\mathfrak{g}} = q\,(n)^{(2)}$, i.e.

$$\hat{\mathfrak{g}} = \mathfrak{sl}\,(n) \otimes \mathbb{C}\left[t^{\pm 2}\right] \oplus \mathfrak{sl}\,(n) \otimes t\mathbb{C}\left[t^{\pm 2}\right] \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

Fix triangular decompositions $\mathfrak{sl}(n) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}$, where

$$\hat{\mathfrak{n}}_- = \mathfrak{n}_- \oplus \mathfrak{sl}(n) \otimes t^{-1}\mathbb{C}[t^{-1}], \quad \hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D, \quad \hat{\mathfrak{n}} = \mathfrak{n} \oplus \mathfrak{sl}(n) \otimes t\mathbb{C}[t].$$

1.2.1. Set

$$\tilde{\mathcal{N}}^+ := \sum_{r\in\mathbb{Z}} t^r \otimes \mathfrak{n}, \quad \tilde{\mathcal{N}}^- := \sum_{r\in\mathbb{Z}} t^r \otimes \mathfrak{n}_-, \quad \tilde{\mathcal{H}} := \sum_{r\in\mathbb{Z}} t^r \otimes \mathfrak{h} \oplus \mathbb{C}K.$$

Notice that $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] = \tilde{\mathcal{N}}^+ \oplus \tilde{\mathcal{H}} \oplus \tilde{\mathcal{N}}^-$ is a triangular decomposition. Let $\mathrm{HC}_+ : \mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]) \to \mathcal{U}(\tilde{\mathcal{H}})$ be the projection along the kernel $\mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}])\tilde{\mathcal{N}}^- + \tilde{\mathcal{N}}^+\mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}])$. Set

$$\mathcal{H}_- := \tilde{\mathcal{H}} \cap \hat{\mathfrak{n}}_- = \sum_{r<0} t^r \otimes \mathfrak{h}, \qquad \mathcal{S} := \mathcal{U}(\mathcal{H}_{-,\bar{0}}).$$

Notice that $\mathcal{H}_{-,\bar{0}} = \sum_{r<0} t^{2r} \otimes \mathfrak{h}$ coincides with the centre of $\mathcal{H}_-$; in particular, $\mathcal{S}$ is a polynomial algebra in countably many variables.

Define $\mathrm{HC}_+ : M(\lambda) \to \mathcal{U}(\mathcal{H}_-)$ via the natural identification of $M(\lambda)$ with $\mathcal{U}(\hat{\mathfrak{n}}_-)$.

We call $v \in M(\lambda)$ *singular* if v is a weight vector and $v \in M(\lambda)^{\hat{\mathfrak{n}}}$.

1.2.2. Theorem. *Let λ be a generic weight.*

(i) *The restriction of* HC_+ *to* $M(\lambda)^{\hat{\mathfrak{n}}}$ *provides a bijection* $\mathrm{HC}_+ : M(\lambda)^{\hat{\mathfrak{n}}} \to \mathcal{S}$.
(ii) *One has* $[M(\lambda) : L(\lambda-\nu)] = \dim M(\lambda)^{\hat{\mathfrak{n}}}_{\lambda-\nu}$ *so any submodule of* $M(\lambda)$ *is generated by singular vectors.*
(iii) *A submodule generated by a singular vector is isomorphic to a Verma module* $M(\lambda - s\delta)$ *for some* $s \geq 0$.

1.2.3. *Remark.* Theorem 1.2.2 holds for a generic weight at any level $k \in \mathbb{C}$, see Remark 7.3.

In Theorem 1.2.2 we may replace the projection HC_+ by the projection $\mathrm{HC}_- : \mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]) \to \mathcal{U}(\tilde{\mathcal{H}})$ along the kernel $\mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}])\tilde{\mathcal{N}}^+ + \tilde{\mathcal{N}}^-\mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}])$.

1.2.4. *Character formula (1).* The algebra $\mathcal{S}$ inherits the grading from $\mathcal{U}(\hat{\mathfrak{g}})$; if $I \subset \mathcal{S}$ is homogeneous, we denote by $\mathrm{ch}\, I$ its character with respect to this grading.

For a submodule N of $M(\lambda)$ set

$$H(N) := \mathrm{HC}_+(N^{\hat{\mathfrak{n}}}) \subset \mathcal{S}.$$

From Theorem 1.2.2 we see that H provides a one-to-one correspondence between the submodules of $M(\lambda)$ and the homogeneous ideals of $\mathcal{S}$.

The characters of N and of $H(N)$ are connected by the following formula:

$$\mathrm{ch}\, N = \mathrm{ch}\, L(\lambda) \cdot \mathrm{ch}(H(N)). \tag{2}$$

Applying this formula to $N = M(\lambda)$ we get

$$\operatorname{ch} M(\lambda) = \operatorname{ch} L(\lambda) \cdot \operatorname{ch} \mathcal{S}$$

which is equivalent to the character formula (1).

1.3. Outline of the proof of Theorem 1.2.2. Set $\mathfrak{q} := \mathfrak{n} + \sum_{r>0} t^r \otimes (\mathfrak{h} + \mathfrak{n})$. Using the Shapovalov form we prove that for generic λ any singular vector of $M(\lambda)$ has weight $\lambda - \alpha$ for some imaginary root α. Then it is easy to show that $v \in M(\lambda)_{\lambda-\alpha}$ is singular if $\mathfrak{q}v = 0$.

Let m be an even positive number and h be an element of $\mathfrak{h}$. Set

$$N := \mathbb{C}h + \sum_{s\geq 0} \tilde{\mathcal{N}}_s^+, \quad V := \mathbb{C}h + \sum_{0\leq s<m} \tilde{\mathcal{N}}_s^+,$$

and observe that N is a $\mathfrak{q}$-submodule of $\hat{\mathfrak{g}}$ and V is an $\mathfrak{n}^+$-submodule of N. Let $h^* \in V^*$ be the weight element dual to h. A cohomological lemma 8.1 implies that for generic λ there exists a unique $\mathfrak{q}$-homomorphism $\psi : V^* \to M(\lambda)$ such that $\psi(h^*)$ is the highest vector v_λ, see Lemma 6.3.3.

Let $T_m : \tilde{\mathcal{N}}^+ + \mathfrak{h} \to \tilde{\mathcal{N}}^+ + \mathfrak{h}$ be the linear map given by $T_m(a \otimes t^s) = a \otimes t^{s-m}$. Let $\gamma : \hat{\mathfrak{g}} \otimes M(\lambda) \to M(\lambda)$ be the natural map $\gamma(u \otimes v) = uv$, and let $\mathrm{id}' \in V \otimes N^*$ corresponds to the identity map $V \to V$. In Proposition 6.3.5, we prove that the vector

$$v(h, m) = \gamma\big((T_m \otimes \psi)\, \mathrm{id}'\big)$$

satisfies $\mathrm{HC}_+(v(h,m)) = h(-m)$ and is singular. As it was mentioned above, the singularity follows from $\mathfrak{q}v(h,m) = 0$ which is a consequence of the $\mathfrak{q}$-invariance of ψ.

1.4. Index of notations. Symbols used frequently are given below under the section number where they are first defined.

1.2.1	$\tilde{\mathcal{N}}^\pm, \tilde{\mathcal{H}}, \mathcal{H}, \mathcal{S}, \mathrm{HC}_+$
1.3	$\mathfrak{q}, N, V, T_m$
2.2	$\hat{\Delta}^+, \hat{Q}^+, \delta$
6.1	Λ
2.5.4	σ
6.2	$\mathcal{B}$

2. Preliminaries and notation

Our base field is $\mathbb{C}$. For a homogeneous element of a superspace we denote by $p(u)$ its $\mathbb{Z}_2$-degree. For a Lie superalgebra $\mathfrak{g}$ we denote by $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra and by $\mathcal{S}(\mathfrak{g})$ its symmetric algebra.

2.1. Description of $\mathfrak{q}(n)^{(2)}$. The Lie superalgebra $\mathfrak{q}(n)^{(2)}$ can be constructed as follows. Recall that $\mathfrak{sq}(n) \subset \mathfrak{gl}(n|n)$ consists of the matrices with the block form

$$X_{A,B} := \begin{pmatrix} A & | & B \\ -- & - & -- \\ B & | & A \end{pmatrix}$$

where A is an arbitrary $n \times n$ matrix, B is a traceless $n \times n$ matrix.

Let $\mathcal{L}(\mathfrak{sq}(n)) = \mathfrak{sq}(n) \otimes \mathbb{C}[t^{\pm 1}]$ be the corresponding loop superalgebra. Then $\mathfrak{q}(n)^{(1)} = \mathcal{L}(\mathfrak{sq}(n)) \oplus \mathbb{C}D$, where D acts on $\mathcal{L}(\mathfrak{sq}(n))$ by $[D, x \otimes t^k] = kx \otimes t^k$. Note that $\mathfrak{sq}(n),\ \mathfrak{q}(n)^{(1)}$ are not Kac-Moody superalgebra since their Cartan subalgebras contain odd elements.

Let ε be an automorphism of $\mathfrak{sq}(n)$ which acts by id on $\mathfrak{sq}(n)_{\bar{0}} = \mathfrak{gl}(n)$ and by $-\mathrm{id}$ on $\mathfrak{sq}(n)_{\bar{1}}$. Extend ε to $\mathfrak{q}(n)^{(1)}$ by $\varepsilon(t) = -t$, $\varepsilon(D) = D$. One can define $\mathfrak{q}(n)^{(2)}$ as the quotient of the subalgebra

$$(\mathfrak{q}(n)^{(1)})^{\varepsilon} = \big(\mathcal{L}(\mathfrak{sq}(n))\big)^{\varepsilon} \oplus \mathbb{C}D$$

by the abelian ideal $\sum_{i\neq 0} \mathbb{C} \otimes t^{2i}$. It is a Kac-Moody superalgebra, see for example [HS]. Its Cartan matrix is not symmetrizable. The algebras $\mathfrak{sq}(n),\ \mathfrak{q}(n)^{(1)}, \mathfrak{q}(n)^{(2)}$ do not have even non-degenerate invariant bilinear forms, but have the odd ones.

2.1.1. Using the above definition one can identify $\mathfrak{q}(n)^{(2)}$ with the vector space

$$\mathfrak{sl}(n) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$$

and define the commutator as

$$[x \otimes t^k, y \otimes t^m] = (xy - yx) \otimes t^{k+m}$$

if km is even,

$$[x \otimes t^k, y \otimes t^m] = (xy + yx - 2\,\mathrm{tr}(xy)) \otimes t^{k+m} + 2tr(xy)\delta_{-k,m}K$$

if km is odd.

Notice that the central element K does not lie in $[\mathfrak{q}(n)^{(2)}_{\bar{0}}, \mathfrak{q}(n)^{(2)}]$, but lies in $[\mathfrak{q}(n)^{(2)}_{\bar{1}}, \mathfrak{q}(n)^{(2)}_{\bar{1}}]$.

Let $\mathfrak{h}$ denote the diagonal subalgebra in $\mathfrak{sl}(n)$. The Cartan subalgebra of $\hat{\mathfrak{g}}$ is

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes 1 \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

It is convenient to identify $\mathfrak{h} \otimes 1 \oplus \mathbb{C}K$ with the diagonal subalgebra of $\mathfrak{gl}(n)$, by $h_1, ..., h_n$ we denote the standard basis there.

The superalgebra $\tilde{\mathcal{H}}$ generated by the imaginary root spaces has the following structure:

$$\tilde{\mathcal{H}}_{\bar{0}} = \sum_{r\in\mathbb{Z}} \mathfrak{h} \otimes t^{2r} \oplus \mathbb{C}K, \quad \tilde{\mathcal{H}}_{\bar{1}} = \sum_{r\in\mathbb{Z}} \mathfrak{h} \otimes t^{2r+1}.$$

The centre of $\tilde{\mathcal{H}}$ coincides with $\tilde{\mathcal{H}}_{\bar{0}}$; the other relations are

$$[x \otimes t^{2r+1}, y \otimes t^{2s+1}] = (xy + yx - 2\,\mathrm{tr}(xy)) \otimes t^{2(r+s+1)} + 2\,\mathrm{tr}(xy)\delta_{2r+1,-2s-1}K.$$

2.2. We denote by $\hat{\Delta}^+$ the set of positive roots of $\hat{\mathfrak{g}}$. Set

$$\hat{Q}^+ := \sum_{\alpha\in\hat{\Delta}^+} \mathbb{Z}_{\geq 0}\alpha.$$

Define a partial ordering on $\hat{\mathfrak{h}}^*$ by setting $\mu > \mu'$ if $\mu - \mu' \in \hat{Q}^+$.

Denote by δ the minimal imaginary root.

2.2.1. For $\alpha \in \hat{\Delta}^+$ let D_α be a matrix of the pairing $\hat{\mathfrak{g}}_\alpha \times \hat{\mathfrak{g}}_{-\alpha} \to \mathfrak{h}$.

Note that $D_\alpha = (h_\alpha)$ if α is real, and that $D_{m\delta}$ is an $(n-1)\times(n-1)$ matrix. By 2.1.1, $D_{2m\delta}$ is the zero matrix and $D_{(2m+1)\delta} = D_\delta$ for $m > 0$. A straightforward calculation shows that up to a multiplication on a non-zero scalar, one has

$$\det D_\delta = h_1 \dots h_n (\frac{1}{h_1} + \dots + \frac{1}{h_n}).$$

2.3. Set $\hat{\mathfrak{b}} := \hat{\mathfrak{h}} + \hat{\mathfrak{n}}$. For each $\lambda \in \hat{\mathfrak{h}}^*$ let $M(\lambda)$ be the Verma module of the highest weight λ, let v_λ be the canonical generator of $M(\lambda)$ and let $M'(\lambda)$ be the maximal proper submodule of $M(\lambda)$. The module $L(\lambda) := M(\lambda)/M'(\lambda)$ is simple.

2.3.1. Verma modules do not admit Jordan-Hölder series, since some Verma modules have an infinite length. However, so-called *local series* introduced in [DGK] are nice substitution for Jordan-Hölder ones. A series of weight modules $N = N_0 \supset N_1 \supset \dots \supset N_m = 0$ is called *local at* $\nu \in \hat{\mathfrak{h}}^*$ if either $N_i/N_{i+1} \cong L(\lambda_i)$ for some $\lambda_i \geq \nu$ or $(N_i/N_{i+1})_\mu = 0$ for all $\mu \geq \nu$. This allows to define the *multiplicity* $[N : L(\lambda)]$ as the number of i such that $N_i/N_{i+1} \cong L(\lambda)$ for a series local at some $\nu \leq \lambda$.

2.4. Projections HC **and** HC_+**.** Denote by HC the Harish-Chandra projection HC : $\mathcal{U}(\hat{\mathfrak{g}}) \to \mathcal{U}(\hat{\mathfrak{h}}) = \mathcal{S}(\hat{\mathfrak{h}})$ along the decomposition $\mathcal{U}(\hat{\mathfrak{g}}) = \mathcal{U}(\hat{\mathfrak{h}}) \oplus (\mathcal{U}(\hat{\mathfrak{g}})\hat{\mathfrak{n}}^+ + \hat{\mathfrak{n}}^-\mathcal{U}(\hat{\mathfrak{g}}))$. Recall that $\mathrm{HC}_+ : \mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]) \to \mathcal{U}(\mathcal{H})$ is the projection along the kernel $\mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}])\mathcal{N}^- + \mathcal{N}^+\mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}])$.

The restriction of HC to $\mathcal{U}(\hat{\mathfrak{g}})^{\hat{\mathfrak{h}}}$ is an algebra homomorphism. Similarly the restriction of HC_+ to $\mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}])^{\mathfrak{h}}$ is an algebra homomorphism.

2.5. Characters. We say that a module M admits a character if M is a diagonalizable $\mathfrak{h}$-module and all its weight spaces are finite dimensional; we write

$$\mathrm{ch}\, M = \sum_\mu \dim M_\mu e^\mu.$$

2.5.1. For each λ let C_λ be the collection of elements of the form $\sum_{\mu<\lambda} c_\mu e^\mu$ where $c_\lambda \in \mathbb{Z}_{\geq 0}$. Set $C := \{\sum_{i=1}^k x_i |\ x_i \in C_{\lambda_i}\}$. Note that $x, y \in C$ implies $xy \in C$. For $x, y \in C$ write $x \geq y$ if $x - y \in C$. In all our examples, $\mathrm{ch}\, M$ belongs to C.

2.5.2. For a diagonalizable $\mathfrak{h}$-module M we denote by $\Omega(M)$ the set of weights of M and by M_μ the weight space of weight μ.

2.5.3. Recall that $\hat{\mathfrak{g}} = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] \oplus \mathbb{C}D$. Therefore $M(\lambda)$ and $M(\lambda - r\delta)$ are isomorphic as $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$-modules, and hence $M'(\lambda) \cong M'(\lambda - r\delta)$. As a consequence, $\operatorname{ch} L(\lambda - r\delta) = e^{-r\delta} \operatorname{ch} L(\lambda)$.

2.5.4. *Choice of antiautomorphism.* Call a linear endomorphism σ of a superalgebra a "naive" antiautomorphism if σ is invertible and $\sigma([xy]) = [\sigma(y), \sigma(x)]$. Every Kac-Moody Lie superalgebra has a naive involutive antiautomorphism σ such that $\sigma|_{\hat{\mathfrak{h}}} = \mathrm{id}$, and $\sigma(\hat{\mathfrak{n}}) = \hat{\mathfrak{n}}_-$. Define a naive anti-involution of $\hat{\mathfrak{g}}$ by

$$D \mapsto D, \quad t^r \otimes a \mapsto t^{-r} \otimes a'$$

where a' is the transpose of a matrix $a \in \mathfrak{gl}(n)$.

If M is a $\hat{\mathfrak{g}}$-module with finite-dimensional weight spaces, then the restricted dual space M^* of M has the structure of $\hat{\mathfrak{g}}$-module defined by $Xf(m) = f(\sigma(X)m)$ for any $f \in M^*, m \in M, X \in \hat{\mathfrak{g}}$. The corresponding module will be denoted by M^σ. It is easy to see that σ is a contravariant functor on the category of weight modules with well defined characters, $\operatorname{ch} M = \operatorname{ch} M^\sigma$ and therefore $L(\lambda)^\sigma \cong L(\lambda)$. It is also clear that $[M : L(\lambda)] = [M^\sigma : L(\lambda)]$.

3. The Lie superalgebra $\tilde{\mathcal{H}}$

In this section we study Verma modules over $\tilde{\mathcal{H}}$.

3.1. Write $\tilde{\mathcal{H}} = \mathcal{H}_- \oplus \mathbb{C}K \oplus \mathfrak{h} \oplus \mathcal{H}$, where $\mathcal{H} = \tilde{\mathcal{H}} \cap \hat{\mathfrak{n}} = \sum_{r>0} \mathfrak{h} \otimes t^r$. For $\lambda \in \hat{\mathfrak{h}}$ let $\mathcal{M}(\lambda)$ be a Verma $\tilde{\mathcal{H}}$-module

$$\mathcal{M}(\lambda) = \mathcal{U}(\tilde{\mathcal{H}}) \otimes_{\mathcal{U}(\mathfrak{h}+\mathcal{H})} \mathbb{C}v_\lambda, \quad \mathcal{H}v_\lambda = 0, \ hv_\lambda = \lambda(h)v_\lambda \text{ for any } h \in \mathfrak{h}.$$

3.1.1. Note that $\hat{\mathfrak{h}}$ acts on $\tilde{\mathcal{H}}$; define the characters of $\tilde{\mathcal{H}}$-modules via this action. Then

$$\operatorname{ch} \mathcal{M}(\lambda) = e^\lambda \prod_{\alpha \in \Delta_{im,\bar{0}}} (1 - e^{-\alpha})^{-1} \prod_{\alpha \in \Delta_{im,\bar{1}}} (1 + e^{-\alpha}).$$

The subalgebra $\mathcal{S} = \mathcal{U}(\mathcal{H}_{-,\bar{0}})$ lies in the centre of $U(\tilde{\mathcal{H}})$ and acts freely on $\mathcal{M}(\lambda)$. Every ideal J of $\mathcal{S}$ defines the submodule $J\mathcal{M}(\lambda)$ in $\mathcal{M}(\lambda)$. On the other hand, any submodule $N \subset \mathcal{M}(\lambda)$ defines the ideal $J_N \subset \mathcal{S}$ such that $N \cap \mathcal{S} = J_N v_\lambda$.

Proposition. *Asume that* $\det D_\delta$ *is not zero when evaluated at* λ. *Then the map* $N \mapsto J_N$ *defines a bijection between submodules in* $\mathcal{M}(\lambda)$ *and ideals in* $\mathcal{S}$.

Proof. It suffices to check that $N = J_N\mathcal{M}(\lambda)$ and it is clear that $J_N\mathcal{M}(\lambda) \subset N$. Let $V = \mathcal{M}(\lambda)/J_N\mathcal{M}(\lambda)$ and W be the image of N in V under the natural projection. We have to show that $W = 0$. Set $R = \mathcal{S}/J_N$. It is easy to see that V is a free R-module and that $W \cap Rv_\lambda = 0$.

Let $u_1, ..., u_{n-1}$ be a basis in $\mathfrak{h}$, $X_{i,j} = u_i \otimes t^{-2j-1}$, $Y_{i,j} = u_i \otimes t^{2j+1}$. Recall that $[X_{i,j}, Y_{k,j}] = [X_{i,0}, Y_{k,0}]$. Due to the assumption on λ one can choose $u_1, ..., u_{n-1}$ so that $\lambda([X_{i,j}, Y_{k,j}]) = \delta_{i,k}$. Clearly $X_{i,j}$ for $i = 1, ..., n-1$, $j \geq 0$ form a basis in $\tilde{\mathcal{H}}_{-,\bar{1}}$. Introduce the order on $X_{i,j}$ by setting $X_{i,j} \geq X_{k,l}$ if $j \geq l$ or $j = l$ and $i \geq k$. Then $X_{i_1,j_1}...X_{i_k,j_k}v_\lambda$ for all $X_{i_1,j_1} \geq ... \geq X_{i_k,j_k}$ form a basis of V over R. Now assume that $W \neq 0$. Since $W \cap Rv_\lambda = 0$, one can pick up a non-zero $v \in W$ such that the maximal $X_{i,j}$ which appears in the decomposition of v is minimal among all non-zero vectors in W. One can write in the unique way $v = X_{i,j}w + u$ for some non-zero w and u such that in the decomposition of u, w only $X_{k,l} < X_{i,j}$ appear. Then $Y_{i,j}u = Y_{i,j}w = 0$ and $Y_{i,j}v = w$. Then $w \in W$, $w \neq 0$ and all $X_{k,l}$ in the decomposition of w are less than $X_{i,j}$ Contradiction. $\square$

3.2. Now consider $\mathcal{M}(\lambda)$ as a module over $\mathbb{C}D + \tilde{\mathcal{H}}$. Then $\mathcal{M}$ has a unique maximal submodule $\mathcal{M}'(\lambda)$ and a unique simple quotient $\mathcal{L}(\lambda) = \mathcal{M}(\lambda)/\mathcal{M}'(\lambda)$. Moreover, there is a bijection between submodules of $\mathcal{M}(\lambda)$ and graded ideals of $\mathcal{S}$, where the grading on $\mathcal{S}$ is defined by D. The above proposition implies the following

Corollary. *For a generic weight $\lambda \in \mathfrak{h}^*$ one has*

(i) $\mathcal{M}(\lambda)^{\mathcal{H}} = \mathcal{S}v_\lambda$;
(ii) $[\mathcal{M}(\lambda) : \mathcal{L}(\lambda - s\delta)] = \mathcal{M}(\lambda)^{\mathcal{H}}_{\lambda - s\delta}$ *for all s;*
(iii) *any submodule N of $\mathcal{M}(\lambda)$ is generated by $\mathcal{M}(\lambda)^{\mathcal{H}}$;*
(iv) $\operatorname{ch}\mathcal{L}(\lambda) = e^\lambda \prod_{\alpha \in \Delta_{im,\bar{1}}}(1 + e^{-\alpha})$.

4. On the reducibility of a Verma module over a Kac-Moody superalgebra

Let $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}^- + \hat{\mathfrak{h}} + \hat{\mathfrak{n}}^+$ be an arbitrary Kac-Moody superalgebra. For each positive root α denote by D_α the pairing $\hat{\mathfrak{g}}_\alpha \times \hat{\mathfrak{g}}_{-\alpha} \to \hat{\mathfrak{h}}$. Let $\Delta' \subset \Delta^+$ consists of the roots where D_α is degenerate (i.e., $[e, \hat{\mathfrak{g}}_{-\alpha}] = 0$ for some non-zero $e \in \hat{\mathfrak{g}}_\alpha$). Set

$$Q' := \sum_{\alpha \in \Delta'} \mathbb{Z}_{\geq 0}\alpha.$$

4.1. Theorem. *If $\mu \notin Q'$ then the weight space $M(\lambda)_{\lambda-\nu}$ does not contain a singular vector for a generic $\lambda \in \hat{\mathfrak{h}}^*$.*

Remark. "Genericity" here means that the property holds on the compliment to a hypersurface.

Proof. Assume that $M(\lambda)^{\hat{\mathfrak{n}}^+}_{\lambda-\mu} \neq 0$ for all $\lambda \in \mathfrak{h}^*$. This means that a certain system of homogeneous linear equations with coefficients in $\mathcal{S}(\mathfrak{h})$ has a non-zero solution for each $\lambda \in \mathfrak{h}^*$. Then we can write a "generic" solution: a non-zero element $u \in \mathcal{U}(\hat{\mathfrak{n}}^-)_{-\mu}\mathcal{S}(\mathfrak{h})$ such that for any $\lambda \in \mathfrak{h}^*$ the vector $u(\lambda)v_\lambda \in M(\lambda)_{\lambda-\mu}$ is singular. Note that $(\operatorname{ad}\hat{\mathfrak{n}}^+)\mathcal{U}(\hat{\mathfrak{n}}^-) \subset \mathcal{U}(\hat{\mathfrak{n}}^-)\mathcal{S}(\mathfrak{h})$ and define the action of $\hat{\mathfrak{n}}^+$ on $\mathcal{U}(\hat{\mathfrak{n}}^-)\mathcal{S}(\mathfrak{h})$ by

$$e.(xs) := (\operatorname{ad} e)(x)s \quad \text{for } e \in \hat{\mathfrak{n}}^+, x \in \mathcal{U}(\hat{\mathfrak{n}}^-), s \in \mathcal{S}(\mathfrak{h}).$$

Then $u(\lambda)v_\lambda \in M(\lambda)^{\hat{\mathfrak{n}}^+}_{\lambda-\mu}$ means that

$$\hat{\mathfrak{n}}^+.u = 0.$$

Let F be the canonical increasing filtration on $\mathcal{U}(\mathfrak{h}) = \mathcal{S}(\mathfrak{h})$ (by the total degree); extend F to $\mathcal{U}(\hat{\mathfrak{n}}^-)\mathcal{S}(\mathfrak{h})$ by setting $F^r(\mathcal{U}(\hat{\mathfrak{n}}^-)\mathcal{S}(\mathfrak{h})) = \mathcal{U}(\hat{\mathfrak{n}}^-)F^r(\mathcal{S}(\mathfrak{h}))$. Note that

$$\hat{\mathfrak{n}}^+.F^p(\mathcal{U}(\hat{\mathfrak{n}}^-)\mathcal{S}(\mathfrak{h})) \subset F^{p+1}(\mathcal{U}(\hat{\mathfrak{n}}^-)\mathcal{S}(\mathfrak{h}))$$

and for a fixed $e \in \hat{\mathfrak{n}}^+$ denote by $\tau'_e : \mathcal{U}(\hat{\mathfrak{n}}^-)\mathcal{S}(\mathfrak{h}) \to \operatorname{gr}_F \mathcal{U}(\hat{\mathfrak{n}}^-)\mathcal{S}(\mathfrak{h})$ the map

$$x \mapsto \operatorname{gr}_{\deg_F(x)+1}(e.x)$$

where $\deg_F(x)$ stands for the degree of x with respect to the filtration F. Now let F_- be the canonical filtration on $\mathcal{U}(\hat{\mathfrak{n}}^-)$; extend F_- to $\operatorname{gr}_F \mathcal{U}(\hat{\mathfrak{n}}^-)\mathcal{S}(\mathfrak{h}) \cong \mathcal{U}(\hat{\mathfrak{n}}^-)\mathcal{S}(\mathfrak{h})$ by setting $F^k_-(\mathcal{U}(\hat{\mathfrak{n}}^-)\mathcal{S}(\mathfrak{h})) = F^k_-(\mathcal{U}(\hat{\mathfrak{n}}^-))\mathcal{S}(\mathfrak{h})$. Clearly, $\operatorname{gr}_-(\mathcal{U}(\hat{\mathfrak{n}}^-)\mathcal{S}(\mathfrak{h})) = \mathcal{S}(\hat{\mathfrak{b}}^-)$ where $\hat{\mathfrak{b}}^- := \mathfrak{h}+\hat{\mathfrak{n}}^-$. Consider the map $\tau_e := \operatorname{gr}_- \circ \tau'_e$ which is the map $\mathcal{U}(\hat{\mathfrak{n}}^-)\mathcal{S}(\mathfrak{h}) \to \mathcal{S}(\hat{\mathfrak{b}}^-)$ given by

$$\tau_e(x) = \operatorname{gr}_- \operatorname{gr}_{\deg_F(x)+1}(e.x).$$

Define the action of $\hat{\mathfrak{n}}^+$ on $\hat{\mathfrak{b}}^-$ by

$$e * y = [e, y]\delta_{\beta,\gamma} \text{ if } e \in \hat{\mathfrak{n}}^+_\beta, \quad y \in \hat{\mathfrak{g}}_{-\gamma}$$

and extend this action to a derivation of $\mathcal{S}(\hat{\mathfrak{b}}^-)$. One can easily check that

$$\tau_e(x) = e * \operatorname{gr}_- \operatorname{gr} x.$$

Set

$$\begin{aligned} \Delta'' : &= \{\beta \in \Delta^+ | \beta \leq \mu \ \&\ \beta \notin \Delta'\}, \\ V_1 : &= \textstyle\sum_{\beta\in\Delta''} \hat{\mathfrak{n}}^-_{-\beta}, \\ V_2 : &= \textstyle\sum_{\beta\in\Delta''} \hat{\mathfrak{n}}^+_{\beta}, \\ W : &= \mathfrak{h} + \textstyle\sum_{\beta\in\Delta'} \hat{\mathfrak{n}}^-_{-\beta}. \end{aligned}$$

Note that $\dim V_1 = \dim V_2 < \infty$. Define a map $D : V_2 \otimes \mathcal{S}(V_1 \oplus W) \to \mathcal{S}(V_1 \oplus W)$ by

$$D(e, z) := e * z.$$

Note that $\operatorname{gr}_- \operatorname{gr} u$ lies in $\mathcal{S}(V_1 \oplus W)$. The equality $\hat{\mathfrak{n}}^+.u = 0$ forces $\tau'_e(u) = 0$ and thus $\tau_e(u) = 0$ for any $e \in \hat{\mathfrak{n}}^+$. Hence

$$D(e, \operatorname{gr}_- \operatorname{gr} u) = e * \operatorname{gr}_- \operatorname{gr} u = 0 \quad \text{for any } e \in \hat{\mathfrak{n}}^+.$$

By the construction, the map $z \mapsto D(e,z)$ is a derivation for each $e \in V_2$. Observe that $D(V_2 \otimes V_1) \subset \mathfrak{h} \subset W$. By the definition of Δ'' one has

$$\{v \in V_1 |\ \forall f \in V_2\ D(f,v) = 0\} = 0.$$

Using Lemma 4.2 we obtain $\mathrm{gr}_{-}\,\mathrm{gr}\,u \in \mathcal{S}(W)$. However, $\Omega(\mathcal{S}(W)) = -Q'$ so $\mathcal{S}(W)_\mu = 0$. Hence $u = 0$. □

4.2. Lemma. *Let V_1, V_2 be finite dimensional vector spaces of the same dimension and let W be an even vector space. Let $d : V_2 \otimes V_1 \to W$ be a non-degenerate pairing i.e. $d(V_2, v) \neq 0$ for a non-zero $v \in V_1$. Define $D : V_2 \otimes \mathcal{S}(V_1 \oplus W) \to \mathcal{S}(V_1 \oplus W)$ by the following properties*

(i) *$D(f,v) = d(f,v)$ for $v \in V_1$ and $D(f,w) = 0$ for $w \in W$.*
(ii) *For each $f \in V^*$ the map $x \mapsto D(f,x)$ is a derivation of the algebra $\mathcal{S}(V_1 \oplus W)$.*

Then

$$\{x \in \mathcal{S}(V_1 \oplus W) |\ \forall f \in V_2\ D(f,x) = 0\} = \mathcal{S}(W).$$

Proof. Let F be the field of fractions of $\mathcal{S}(W)$ and let A be the localization of $\mathcal{S}(V_1 \oplus W)$ by the non-zero elements of W that is

$$A := \mathcal{S}(V_1 \oplus W) \otimes_{\mathcal{S}(W)} F.$$

Extend D to a map $F \otimes V_2 \otimes A \to A$ by setting $D(s, f, xw^{-1}) := sD(f,x)w^{-1}$ for $s \in F, x \in \mathcal{S}(V_1 \oplus W), w \in \mathcal{S}(W)$. Then for each pair (s,f) the map $a \mapsto D(s,f,a)$ is a derivation of the algebra A. Identify $F \otimes V_1$ with the F-subspace $FV_1 \subset A$ spanned by V_1. Extend d to a map $F \otimes V_2 \otimes FV_1 \to F$ by setting $d(s,f,s'v) := ss'd(f,v)$ for $s, s' \in F$. We have

$$D(s,f,v) = d(s,f,v) \quad \text{for } v \in FV_1.$$

View $F \otimes V_2,\ FV_1$ as vector spaces over F and note that d is a bilinear map. Choose F-bases $\{f_i\}$ in $F \otimes V_2$ and $\{v_j\}$ in FV_1 satisfying $d(f_i, v_j) = \delta_{ij}$. View $a \in A$ as a rational function in $\{v_j\}$ and a basis of W; note that the denominator is a polynomial in $\mathcal{S}(W)$. We have

$$D(f_i, a) = \frac{\partial a}{\partial v_i}.$$

Therefore,

$$\{a \in A |\ \forall i\ D(f_i, a) = 0\} = F.$$

Clearly,

$$\{x \in \mathcal{S}(V \oplus W) |\ \forall f \in V_2\ D(f,x) = 0\} = \mathcal{S}(V \oplus W) \cap \{a \in A |\ \forall i\ D(f_i, a) = 0\}$$

and this gives the assertion. □

4.3. Proposition. *If $[M(\lambda):L(\lambda')]>0$ then either $M(\lambda)$ has a singular vector of weight λ' or $[M(\lambda):L(\lambda'')],\ [M(\lambda''):L(\lambda')]>0$ for some λ'' such that $\lambda'<\lambda''<\lambda$.*

Proof. Let us assume that $M(\lambda)$ has no singular vectors of weight λ'. Then $\lambda\neq\lambda'$, hence one can find a proper submodule $N\subset M(\lambda)$ such that $[N:L(\lambda')]>0$. By our assumption $N^{\hat{\mathfrak{n}}}_{\lambda'}=0$. By duality $N^{\sigma}_{\lambda'}\subset\hat{\mathfrak{n}}^-N^{\sigma}$.

Define the increasing flag $F_0\subset F_1\subset F_2\subset ...$ of $\hat{\mathfrak{g}}$-submodules of N^{σ} such that $F_0=0$ and F_i/F_{i-1} is generated by all singular vectors in N^{σ}/F_{i-1} of weights greater than λ'. Since eventually all weights $\mu\geq\lambda'$ of singular vectors will be exausted, $N^{\sigma}_{\lambda'}\subset\hat{\mathfrak{n}}^-N^{\sigma}$ implies that $(F_i)_{\lambda'}=N^{\sigma}_{\lambda'}$. Therefore $[F_i:L(\lambda')]>0$ for some $i>0$. Choose the minimal i such that $[F_i:L(\lambda')]>0$. Then $[(F_i/F_{i-1}):L(\lambda')]>0$. By definition F_i/F_{i-1} is a quotient of a direct sum of Verma modules. Hence $[M(\lambda''):L(\lambda')]>0$ at least for one of these Verma modules $M(\lambda)$. On the other hand, $[N^{\sigma}:L(\lambda'')]>0$, hence by duality $[N:L(\lambda'')]>0$. Thus, we have $[M(\lambda''):L(\lambda')]>0$ and $[M(\lambda):L(\lambda'')]>0$ as required. □

4.3.1. Corollary. *If $\hat{\mathfrak{g}}$ is a Lie algebra then*

$$[M(\lambda):L(\lambda')]>0\Longrightarrow\ M(\lambda)^{\hat{\mathfrak{n}}}_{\lambda'}\neq 0.$$

Proof. If $\hat{\mathfrak{g}}$ is a Lie algebra then $M(\nu)^{\hat{\mathfrak{n}}}_{\nu'}\neq 0$ implies that $M(\nu')$ is a submodule of $M(\nu)$; the assertion follows from Proposition 4.3 by induction on $\lambda-\lambda'$. □

4.4. We will use the following lemma.

Lemma. *Assume that λ is such that $[M(\lambda):L(\mu)]=M(\lambda)^{\hat{\mathfrak{n}}}_{\mu}$ for all $\mu\in\hat{\mathfrak{h}}^*$. Then any submodule M of $M(\lambda)$ is generated by its singular vectors.*

Proof. Let $\mathcal{Y}$ be the set of subquotients of $M(\lambda)$ and $\mathcal{Y}':=\{M\in\mathcal{Y}|\ \forall\mu\ [M:L(\mu)]=\dim M^{\hat{\mathfrak{n}}}_{\mu}\}$. Let us show that $\mathcal{Y}'=\mathcal{Y}$ and that for any $M,N\in\mathcal{Y}$ such that N is a submodule of M one has

$$\dim M^{\hat{\mathfrak{n}}}_{\mu}=\dim N^{\hat{\mathfrak{n}}}_{\mu}+\dim(M/N)^{\hat{\mathfrak{n}}}_{\mu}. \tag{3}$$

Take $M\in\mathcal{Y}'$ and let N be a submodule of M. One has

$$[M:L(\mu)]=[N:L(\mu)]+[M/N:L(\mu)]\geq\dim N^{\hat{\mathfrak{n}}}_{\mu}+\dim(M/N)^{\hat{\mathfrak{n}}}_{\mu}\geq\dim M^{\hat{\mathfrak{n}}}_{\mu}.$$

This gives $N,\ M/N\in\mathcal{Y}'$ and implies (3). Since $M(\lambda)\in Y'$ we obtain $\mathcal{Y}'=\mathcal{Y}$.

Now let M be any submodule of $M(\lambda)$ and let N be the submodule generated by $M^{\hat{\mathfrak{n}}}$. By (3), $\dim(M/N)^{\hat{\mathfrak{n}}}_{\mu}=0$ for any μ. Hence $M/N=0$ as required. □

5. Towards the proof of Theorem 1.2.2 (i), (ii)

5.1. In this section we reduce Theorem 1.2.2 (i), (ii) to the following assertions:

(A) $\operatorname{ch} L(\lambda) \geq e^{\lambda} \prod_{\alpha \in \hat{\Delta}^+_{re;0}} (1-e^{-\alpha})^{-1} \prod_{\alpha \in \hat{\Delta}^+_1} (1+e^{-\alpha})$.

(B) $\mathrm{HC}_+(M(\lambda)^{\hat{\mathfrak{n}}})$ contains $\mathcal{H}_{-,\overline{0}}$.

5.2. Algebra structure on $\sum_s M(\lambda)^{\hat{\mathfrak{n}}}_{s\delta}$. Identify $\operatorname{End}_{[\hat{\mathfrak{g}},\hat{\mathfrak{g}}]}(M(\lambda))$ with $\sum_s M(\lambda)^{\hat{\mathfrak{n}}}_{s\delta}$ via the map $\phi \mapsto \phi(v_\lambda)$. Endow $\sum_s M(\lambda)^{\hat{\mathfrak{n}}}_{s\delta}$ with the algebra structure via this bijection.

5.2.1. Lemma. *The restriction* $\mathrm{HC}_+ : \sum_s M(\lambda)^{\hat{\mathfrak{n}}}_{s\delta} \to \mathcal{U}(\mathcal{H}_-)$ *is an algebra homomorphism.*

Proof. Let ι stands for the natural identification $M(\lambda)$ with $\mathcal{U}(\hat{\mathfrak{n}}_-)$. Clearly, the algebra structure on $\sum_s M(\lambda)^{\hat{\mathfrak{n}}}_{s\delta}$ is compatible with ι i.e., $\iota(x)\iota(y) = \iota(xy)$.

Notice that ι maps $\sum_s M(\lambda)_{s\delta}$ to $\mathcal{U}(\hat{\mathfrak{n}}_-)^{\mathfrak{h}'}$, where $\mathfrak{h}' = \hat{\mathfrak{h}} \cap [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$. Now the assertion follows from the fact that the restriction of HC_+ to $\mathcal{U}(\hat{\mathfrak{n}}_-)^{\mathfrak{h}'}$ is an algebra homomorphism. $\square$

5.3. Proof of Theorem 1.2.2 (i),(ii). From Lemma 4.4 we see that the formula

$$[M(\lambda) : L(\lambda-\nu)] = \dim M(\lambda)^{\hat{\mathfrak{n}}}_{\lambda-\nu} \tag{4}$$

implies that any submodule of $M(\lambda)$ is generated by singular vectors.

Let us prove (4). From Lemma 5.2.1 and the assertion (B) it follows that $\mathrm{HC}_+(M(\lambda)^{\hat{\mathfrak{n}}})$ contains $\mathcal{S}$. Then

$$[M(\lambda) : L(\lambda-\nu)] \geq \dim M(\lambda)^{\hat{\mathfrak{n}}}_{\lambda-\nu} \geq \dim \mathcal{S}_{-\nu}. \tag{5}$$

Using 2.5.3, we get

$$\operatorname{ch} M(\lambda) \geq \sum_s [M(\lambda) : L(\lambda - s\delta)] e^{-s\delta} \operatorname{ch} L(\lambda) \geq \operatorname{ch} \mathcal{S} \cdot \operatorname{ch} L(\lambda), \tag{6}$$

where the first inequality is strict if $[M(\lambda) : L(\lambda-\nu)] \neq 0$ for some $\nu \notin \mathbb{Z}\delta$ and the second inequality is strict if at least one of the inequalities of (5) is strict. The inequality (A) can be rewritten as

$$\operatorname{ch} \mathcal{S} \cdot \operatorname{ch} L(\lambda) \geq \operatorname{ch} M(\lambda). \tag{7}$$

Comparing (6) with (7) we conclude that all inequalities in (6) are in fact equalities and thus all inequalities in (5) and the inequality (A) are equalities as well. In particular, this gives (4).

Since the inequalities (5) are equalities we have $\operatorname{ch} M(\lambda)^{\hat{\mathfrak{n}}} = e^{\lambda}\mathcal{S}$. By above, $\mathrm{HC}_+(M(\lambda)^{\hat{\mathfrak{n}}})$ contains $\mathcal{S}$. As a result, the restriction of HC_+ to $M(\lambda)^{\hat{\mathfrak{n}}}$ gives a bijection: $\mathrm{HC}_+ :$

$M(\lambda)^{\hat{\mathfrak{n}}} \xrightarrow{\sim} \mathcal{S}$; note that this is an algebra isomorphism. This proves Theorem 1.2.2 (i).

6. Explicit construction of singular vectors

In this section we prove (B) of 5.1 for $\lambda \in \Lambda$, where Λ is described in 6.1. More precisely, for $\lambda \in \Lambda$ in Proposition 6.3.5 we will explicitly construct a singular vector $v(h,m) \in M(\lambda)$ satisfying $\mathrm{HC}_+(v(h,m)) = t^{-m} \otimes h$ where h is any element of $\mathfrak{h}$ and m is any even positive number.

Note that Λ is dense in $\hat{\mathfrak{h}}^*$ so (B) holds for any $\lambda \in \hat{\mathfrak{h}}$.

6.1. The set Λ. Let Λ be the set of λs such that any singular vector of $M(\lambda)$ has weight of the form $\lambda - 2s\delta$ for some $s \geq 0$:

$$\Lambda := \bigcap_{\mu \in \hat{Q}^+, \mu \neq 2s\delta} \Lambda_\mu, \quad \text{where} \quad \Lambda_\mu := \{\lambda \in \mathfrak{h}^* |\ M(\lambda)^{\hat{\mathfrak{n}}^+}_{\lambda-\mu} = 0\}.$$

Recall that $M(\lambda) \cong M(\lambda - 2\delta)$ as $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$-modules. As a consequence, Λ is invariant under the shift by 2δ: if $\lambda \in \Lambda$ then $\lambda - 2\delta \in \Lambda$.

6.1.1. Lemma. *For any $\lambda \in \Lambda$ all simple subquotients of $M(\lambda)$ are of the form $L(\lambda - 2s\delta)$.*

Proof. Let $\nu \in \mathbb{Q}^+ \setminus \mathbb{N}(2\delta)$ be minimal such that $[M(\lambda) : L(\lambda - \nu)] \neq 0$ for some $\lambda \in \Lambda$. By Proposition 4.3 there exists ξ such that $0 < \xi < \nu$ and $[M(\lambda) : L(\lambda - \xi)]$, $[M(\lambda - \xi) : L(\lambda - \nu)] \neq 0$. Then $\xi \in \mathbb{N}(2\delta)$ so $\lambda - \xi \in \Lambda$. The minimality of ν gives $(\lambda - \xi) - (\lambda - \nu) \in \mathbb{N}(2\delta)$ so $\nu \in \mathbb{N}(2\delta)$. The assertion follows. $\square$

6.1.2. Lemma. *The set Λ is Zariski dense in $\hat{\mathfrak{h}}^*$.*

Proof. Retain notation of Sect. 4. By 2.2.1, $\Delta' = \{2s\delta\}_{s>0}$ and so $Q' = \{2s\delta\}_{s>0}$. By Theorem 4.1, Λ_μ is the compliment to a surface in $\hat{\mathfrak{h}}^*$ for $\mu \neq 2s\delta$. As a consequence, Λ is the compliment to a union of countably many hypersurfaces in $\hat{\mathfrak{h}}^*$. This implies the assertion. $\square$

6.2. The subalgebra $\mathfrak{q}$. Put $\mathcal{B} := \tilde{\mathcal{N}}^+ \oplus \mathcal{H}$; Recall that $\mathfrak{q} = \mathcal{B} \cap \hat{\mathfrak{n}}$.

6.2.1. Lemma. *Take $\lambda \in \Lambda$. A vector $v \in M(\lambda)_{\lambda-2s\delta}$ is singular iff $\mathfrak{q}v = 0$.*

Proof. Take $v \in M(\lambda)_{\lambda-s\delta}$ satisfying $\mathfrak{q}v = 0$. The subspace $\mathcal{U}(\hat{\mathfrak{n}})v$ contains a singular vector. Write $\hat{\mathfrak{n}} = \mathfrak{q} \oplus \mathfrak{s}$ where $\mathfrak{s} := \sum_{i>0} \mathfrak{n}_- \otimes t^i$. Notice that $\mathfrak{s}$ is a subalgebra of $\hat{\mathfrak{n}}$ and $\mathcal{U}(\hat{\mathfrak{n}})v = \mathcal{U}(\mathfrak{s})v$ because $\mathfrak{q}v = 0$. Weight vectors in $\mathcal{U}(\mathfrak{s})v$ which are not proportional to v have weights of the form $\lambda - \mu$ where $\mu \in \hat{Q}^+,\ \mu \not\in \mathbb{N}\delta$; such vectors are not singular. Hence v is singular. □

6.2.2. Set

$$\mathcal{N}^+ = \sum_{r\geq 0} \mathfrak{n} \otimes t^r = \tilde{\mathcal{N}}^+ \cap \hat{\mathfrak{n}}.$$

Lemma. *Take $\lambda \in \Lambda,\ \mu \in \Omega(\mathcal{N}^+)$. One has*

$$L(\lambda)_{\lambda-\mu} \subset \sigma(\mathfrak{q})L(\lambda), \quad M'(\lambda)_{\lambda-\mu} \subset \sigma(\mathfrak{q})M'(\lambda).$$

Proof. Writing $\hat{\mathfrak{n}}_- = \sigma(\mathfrak{q}) \oplus \sigma(\mathfrak{s})$ where $\mathfrak{s}$ is introduced in the proof of Lemma 6.2.1, we get $\mathcal{U}(\hat{\mathfrak{n}}_-) = \mathcal{U}(\sigma(\mathfrak{s})) + \sigma(\mathfrak{q})\mathcal{U}(\hat{\mathfrak{n}}_-)$.

One has $L(\lambda) = \mathcal{U}(\hat{\mathfrak{n}}_-)v_\lambda$. The condition $\lambda \in \Lambda$ ensures that $\mathcal{U}(\sigma(\mathfrak{s}))v_\lambda$ does not meet $L(\lambda)_{\lambda-\mu}$. Thus $L(\lambda)_{\lambda-\mu} \subset \sigma(\mathfrak{q})L(\lambda)$.

For the second inclusion, observe that $\lambda \in \Lambda$ forces $M'(\lambda) = \mathcal{U}(\hat{\mathfrak{n}}_-)M''$ where $M'' := \sum_{s>0} M'(\lambda)_{\lambda-2s\delta}$. The assumption $\mu \in \Omega(\mathcal{N}^+)$ ensures that $\mathcal{U}(\sigma(\mathfrak{s}))M''$ does not meet $M(\lambda)_{\lambda-\mu}$. Thus $M'(\lambda)_{\lambda-\mu} \subset \sigma(\mathfrak{q})M''$ and the second inclusion follows. □

6.2.3. Proposition. *For any $\lambda \in \Lambda,\ \mu \in \Omega(\mathcal{N}^+)$ one has*

$$H^r(\mathfrak{q}, L(\lambda))_{\lambda-\mu} = 0 \ \ \textit{and} \ \ H^r(\mathfrak{q}, M(\lambda))_{\lambda-\mu} = 0 \ \ \textit{for } r = 0, 1.$$

Proof. The first formula follows from Lemma 8.1 and Lemma 6.2.2. The second formula is an easy consequence of the first one. Indeed, $M(\lambda)$ has a local series at $\lambda - \mu$ with simple quotients $L(\lambda_i)$ where, by Lemma 6.2.1 (ii), $\lambda_i = \lambda - 2s_i\delta$ for some $s_i \geq 1$. By the long exact sequence of Lie algebra cohomology, it is enough to show that for $r = 0, 1$ one has $H^r(\mathfrak{q}, L(\lambda_i))_{\lambda-\mu} = 0$ for all indexes i. The last follows from the first formula and the fact that $\lambda_i \in \Lambda$ since Λ is τ_{2s}-stable. □

6.3. Let m be an even positive number. Fix $u = h(-m) \in \mathcal{H}_-$. In this subsection we construct for each $\lambda \in \Lambda$ a singular vector $v(h,m) \in M(\lambda)$ satisfying

$$\mathrm{HC}_+(v(h,m)) = h(-m).$$

6.3.1. Let $T_{2k}:\mathcal{N}^+ + \mathbb{C}h\to\mathcal{B}$ be a linear map given by

$$T_{2k}(t^s\otimes u)=t^{s-2k}\otimes u.$$

Observe that T_{2k} is $\mathcal{N}^+$-invariant..

6.3.2. *Notation.* View $\mathcal{B}=\mathcal{H}\oplus\mathcal{N}^+$ as a $\mathfrak{q}$-module via the adjoint action. Consider the natural grading on $\hat{\mathfrak{g}}$: $\hat{\mathfrak{g}}_0=\mathfrak{gl}(n), \hat{\mathfrak{g}}_s=t^s\otimes\mathfrak{sl}(n)$ for $s\neq 0$ and for a homogeneous subspace $X\subset\hat{\mathfrak{g}}$ set $X_s:=X\cap\hat{\mathfrak{g}}_s$. Set

$$N:=\mathbb{C}h+\sum_{s\geq 0}t^s\otimes\mathfrak{n}^+,\quad N':=\sum_{s\geq m}t^s\otimes\mathfrak{n}^+,\quad V:=\mathbb{C}h+\sum_{0\leq s<m}t^s\otimes\mathfrak{n}^+;$$

note that N,N' are $\mathfrak{q}$-submodule of $\mathcal{B}$ and V is an $\hat{\mathfrak{n}}_0$-submodule of N.

Let V^* be the orthogonal compliment of N' in N^* that is

$$V^*:=\{f\in\operatorname{Hom}(N,\mathbb{C})|\ f(N')=0\}.$$

Notice that V^* viewed as $\hat{\mathfrak{n}}_0$-module is dual to V.

Both N,N' are $(\mathfrak{q}+\hat{\mathfrak{h}})$-modules. View N^* and V^* as $(\mathfrak{q}+\hat{\mathfrak{h}})$-modules via the antiautomorphism $-\operatorname{id}$. Let $h^*\in V^*$ be the "dual to" h that is $h^*(h)=1, h^*(a)=0$ for all $a\in\sum_{s\geq 0}t^s\otimes\mathfrak{n}^+$.

6.3.3. Lemma. *For any $\lambda\in\Lambda$ there exists a unique $\mathfrak{q}$-homomorphism $\psi:V^*\to M(\lambda)$ such that $\psi(h^*)=v_\lambda$.*

Proof. Recall that $\Omega(\mathfrak{q})\subset\hat{Q}^+$ and so the action of $\mathfrak{q}$ raises the weight. As a result, V^* admits an increasing $(\hat{\mathfrak{h}}+\mathfrak{q})$-filtration $\{W^k\}_{k\geq 0}$ with one-dimensional factors and $W^0=\mathbb{C}h^*$.

Define a twisted $\hat{\mathfrak{h}}$-action on $M(\lambda)$ by $h.v=(h-(\lambda)(h))v$ for $h\in\hat{\mathfrak{h}}, v\in M(\lambda)$. The twisted action is compatible with the action of $\mathfrak{q}$; view $M(\lambda)$ as $(\hat{\mathfrak{h}}+\mathfrak{q})$-module with respect to this action and notice that ψ is a $(\hat{\mathfrak{h}}+\mathfrak{q})$-homomorphism . In the formulas below we use this twisted $(\hat{\mathfrak{h}}+\mathfrak{q})$-module structure on $M(\lambda)$; Hom stands for $\operatorname{Hom}_{(\hat{\mathfrak{h}}+\mathfrak{q})}$.

Let T^ν be a one-dimensional $(\hat{\mathfrak{h}}+\mathfrak{q})$-module of weight ν (i.e. $\mathfrak{q}T^\nu=0,\ h|_{T^\nu}=\nu(h)\operatorname{id}$). The short exact sequence

$$0\to W^{k-1}\to W^k\to T^\nu\to 0$$

gives

$$\begin{aligned}0\to\operatorname{Hom}(T^\nu,M(\lambda))=H^0(\mathfrak{q},M(\lambda))_{\nu'}\to\operatorname{Hom}(W^k,M(\lambda))\to\\ \to\operatorname{Hom}(W^{k-1},M(\lambda))\to\operatorname{Ext}^1(T^\nu,M(\lambda))=H^1(\mathfrak{q},M(\lambda))_{\nu'},\end{aligned}$$

where all Hom stands for $\operatorname{Hom}_{(\hat{\mathfrak{h}}+\mathfrak{q})}$ and $\nu':=\lambda-s\delta+\nu$. For $k>0$ one has $\nu-s\delta\in\hat{Q}^-_-$ and so

$$\operatorname{Hom}(T^\nu,M(\lambda))=\operatorname{Ext}^1(T^\nu,M(\lambda))=0,$$

by Proposition 6.2.3. Therefore $\mathrm{Hom}(W^k, M(\lambda)) = \mathrm{Hom}(W^{k-1}, M(\lambda))$. As a result, $\mathrm{Hom}(W^k, M(\lambda)) = \mathrm{Hom}(W^0, M(\lambda)) = \mathrm{Hom}(\mathbb{C}h^*, M(\lambda))$. The assertion follows. $\square$

6.3.4. Remark that ψ shifts weights by $\lambda + r\delta$ that is $\psi(V^*_{-r\delta-\mu}) \subset M(\lambda)_{\lambda-\mu}$.

6.3.5. Retain notation of 6.3.1, 6.3.2.

Proposition. *Let m be an even positive number. Fix $\lambda \in \Lambda$ and $h \in \mathfrak{h}$. Let $\psi : V^* \to M(\lambda)$ be a $\mathfrak{q}$-homomorphism constructed in Lemma 6.3.3, let $\gamma : \hat{\mathfrak{g}} \otimes M(\lambda) \to M(\lambda)$ be the natural map $\gamma(u \otimes v) = uv$, and let* $\mathrm{id}' \in V \otimes V^*$ *corresponds to the identity map $V \to V$. Then*

$$v(h,m) = \gamma\big((T_m \otimes \psi)\,\mathrm{id}'\big)$$

satisfies $\mathrm{HC}_+(v(h,m)) = h(-m)$ *and $v(h,m)$ is singular.*

Proof. Let B be a weight basis of $V \cap \mathcal{N}^+$; then

$$B' := \{h\} \cup B$$

is a weight basis of V. For $b \in B$ denote by b^* the element of the dual basis $\{h^*\} \cup B^*$ of V^*. One has

$$v(h,m) = h(-m)v_\lambda + \sum_{b \in B} T_m(b)\psi(b^*).$$

Since $\mathcal{N}^+$ is T_m-stable, $T_m(b) \in \mathcal{N}^+$ and so $\mathrm{HC}_+(v(h,m)) = h(-m)$.

Let us check that $v(h,m)$ is singular. In the light of Lemma 6.2.1 it is enough to verify that $\mathfrak{q}v(h,m) = 0$.

It is easy to see that

$$T_m([g,x]) = [g, T_m(x)] \text{ for all } g \in \mathfrak{q}, x \in V. \tag{8}$$

Take $u \in \mathfrak{q}$. Using (8) and the fact that ψ is a $\mathfrak{q}$-homomorphism we have

$$\begin{aligned} uv(h,m) &= \textstyle\sum_{b\in B'} uT_m(b)\psi(b^*) \\ &= \textstyle\sum_{b\in B'} [u, T_m(b)]\psi(b^*) + (-1)^{p(u)p(b)} T_m(b)u\psi(b^*) \\ &= \textstyle\sum_{b\in B'} T_m([u,b])\psi(b^*) + (-1)^{p(u)p(b)} T_m(b)u\psi(b^*). \end{aligned} \tag{9}$$

For $b_s \in B'$ write

$$[u, b_s] = \sum c_{sj}b_j + w_s \quad \text{where } w_s \in \sum_{t \geq -m} \mathcal{N}_t^+.$$

Then

$$ub_s^* = (-1)^{p(u)p(b_s^*)+1} \sum c_{js} b_j^*$$

and (9) gives

$$uv(h,m) = \sum_s T_m(w_s)\psi(b_s^*) = \psi\big(\sum_s T_m(w_s)b_s^*\big),$$

where the last equality follows from the fact that ψ is $\mathfrak{q}$-invariant and $T_m(\sum_{t\geq m} \mathcal{N}_t^+) \subset \mathfrak{q}$.

Denote by $\mathrm{wt}\, a$ the weight of a. One has $\mathrm{wt}\, w_s = \mathrm{wt}\, u + \mathrm{wt}\, b_s$ and $\mathrm{wt}\, b_s^* = -\mathrm{wt}\, b_s$ so $\mathrm{wt}\, T_m(w_s)b_s^* = m\delta + \mathrm{wt}\, u \in \Omega(\mathcal{B})$ because $\mathrm{wt}\, u \in \Omega(\mathcal{N}^+)$. However $T_m(w_s)b_s^* \in V^*$ and $\Omega(V^*) = -\Omega(V) \subset -\Omega(\mathcal{N}^+) \cup \{0\}$. Thus $T_m(w_s)b_s^* = 0$ or $T_m(w_s)b_s^*$ has zero weight. If $\mathrm{wt}\, u \neq m\delta$ we obtain $T_m(w_s)b_s^* = 0$ so $uv(h,m) = 0$ as required.

It remains to check the case when u has weight $m\delta$ that is $u = h'(m)$ for some $h' \in \mathfrak{h}$. Let us show that

$$\sum_s T_m(w_s)b_s^* = 0.$$

As we have shown the left-hand side has zero weight so is proportional to h^*. Thus we need to verify that $\sum_s (T_m(w_s)b_s^*)(h) = 0$. Observe that for any $x \in V$ one has $[h'(m), x] \in \sum_{t\geq m} \mathcal{N}_t^+$ and so $w_s = [h'(m), b_s]$. Therefore $T_m(w_s) = T_m([h'(m), b_s]) = h'(\mathrm{wt}\, b_s)b_s$ because $h'(m)$ is even. Then

$$\sum_s (T_m(w_s)b_s^*)(h) = \sum_s h'(\mathrm{wt}\, b_s)(b_s b_s^*)(h) = \sum_s (-1)^{p(b_s)+1} h'(\mathrm{wt}\, b_s)h(\mathrm{wt}\, b_s) = \mathrm{str}_V(\mathrm{ad}\, h)(\mathrm{ad}\, h').$$

Recall that $V = \mathbb{C}h + \sum_{j=0}^{m-1} \mathfrak{n}^+ \otimes t^j$. One has

$$\mathrm{str}_{\mathfrak{n}^+\otimes t^j}(\mathrm{ad}\, h)(\mathrm{ad}\, h') = (-1)^j\, \mathrm{tr}_{\mathfrak{n}^+}(\mathrm{ad}\, h)(\mathrm{ad}\, h');$$

since m is even we obtain $\mathrm{str}_V(\mathrm{ad}\, h)(\mathrm{ad}\, h') = 0$ and this completes the proof. □

7. Proof of 5.1 (A) and of Theorem 1.2.2 (iii)

7.1. Proof of 5.1 (A). Set

$$N_+ := \sum_{r>0} t^{-r} \otimes \mathfrak{n}^+,\ N_- := \sum_{r\geq 0} t^{-r} \otimes \mathfrak{n}^-,\ H_1 := \sum_{r>0} t^{-2r+1} \otimes \mathfrak{h} = \mathcal{H}_{-,\overline{1}}.$$

Notice that N_+, N_-, H_1 are Lie subalgebras of $\hat{\mathfrak{n}}^-$. In a Shapovalov matrix S_ν consider the minor corresponding to the space $\mathcal{U}(N_+)\mathcal{U}(N_-)\mathcal{U}(H_1)$. The leading term of this minor takes form

$$\prod_{\alpha\in\hat{\Delta}^+\setminus\hat{\Delta}^+_{\mathrm{Im},\overline{0}}} D_\alpha^{r_\alpha(\nu)},$$

for some $r_\alpha(\nu) \geq 0$, see [GS]. From 2.2.1 it follows that this minor is a non-zero polynomial. Therefore for a generic λ the space $\mathcal{U}(N_+)\mathcal{U}(N_-)\mathcal{U}(H_1)v_\lambda$ does not meet $M'(\lambda)$. This proves (A) of 5.1.

7.2. Proof of Theorem 1.2.2 (iii). We need to show that for a generic λ the submodule generated by a singular weight vector in $M(\lambda)$ is a Verma module.

Take λ such that for any $s \geq 0$ the assertion of Theorem 1.2.2 (i) holds for $\lambda' := \lambda - 2s\delta$. Let $v \in M(\lambda)_{\lambda-2s\delta}$ be a singular vector. The submodule M generated by v is a quotient of $M(\lambda-2s\delta)$. If $M \neq M(\lambda-2s\delta)$ then $M = M(\lambda-2s\delta)/M''$ where M'' contains a singular vector v'. Write $v = uv_\lambda, v' = u'v_{\lambda-2s\delta}$, $(u, u' \in \mathcal{U}(\hat{\mathfrak{n}}_-))$. Then $uu' = 0$. By Theorem 1.2.2

(i) $\mathrm{HC}_+(u), \mathrm{HC}_+(u') \neq 0$. Hence $\mathrm{HC}_+(uu') = \mathrm{HC}_+(u')\,\mathrm{HC}_+(u) \neq 0$. and this contradicts to $u'u = 0$. This proves Theorem 1.2.2 (iii).

7.3. *Remark.* Now we can precisely formulate the condition on λ in Theorem 1.2.2.

From 7.1 we see that Theorem 1.2.2 (i), (ii) hold if the minors corresponding to $\big(\mathcal{U}(N_+)\mathcal{U}(N_-)\mathcal{U}(H_1)\big)_\nu$ in the Shapovalov matrices S_ν are non-zero at λ for each $\nu \in \hat{Q}^+$. Since $\hat{Q}^+$ this condition excludes countably many hypersurfaces.

From 7.2 we see that for Theorem 1.2.2 (iii) it is enough if Theorem 1.2.2 (i), (ii) hold for $\lambda - 2s\delta$ for each $s \geq 0$. This again excludes countably many hypersurfaces.

Using a Shapovalov technique it is easy to see that the leading term of the above minors is not divisible by K; thus these minors are not identically equal to zero at any hyperplane $\{\lambda : K(\lambda) = k\}$ $(k \in \mathbb{C})$. Hence Theorem 1.2.2 holds for a generic weight at each level.

8. A vanishing lemma

If $\mathfrak{p}$ is a Lie algebra, N is a $\mathfrak{p}$-module and N' is a subspace of N, denote by $\mathfrak{p}N'$ the vector space spanned by xv where $x \in \mathfrak{p},\ v \in N'$.

8.1. Lemma. *Let $\mathfrak{m}$ be a subalgebra of $\hat{\mathfrak{n}}$. Assume that $\lambda, \mu \in \hat{\mathfrak{h}}^*$ are such that*

$$L(\lambda)_\mu \subset \sigma(\mathfrak{m})L(\lambda), \quad M'(\lambda)_\mu \subset \sigma(\mathfrak{m})M'(\lambda)$$

then

$$H^r(\mathfrak{m}, L(\lambda))_\mu = 0 \quad \textit{for } r = 0, 1.$$

Proof. Let $\mathfrak{l} = \mathfrak{m} + \hat{\mathfrak{h}}$. Obviously,

$$\mathrm{Ext}^r_{\mathfrak{l}}(V, W) = \mathrm{Ext}^r_{\sigma(\mathfrak{l})}(W^\sigma, V^\sigma).$$

In what follows we consider only extensions which are semi-simple over $\hat{\mathfrak{h}}$. If T^ν is the one-dimensional $\mathfrak{l}$ module of weight ν, we have to show that

$$\mathrm{Ext}^r_{\mathfrak{l}}(T^{\lambda-\nu}, L(\lambda)) = 0$$

for $r = 0, 1$, which is equivalent to proving

$$\mathrm{Ext}^r_{\sigma(\mathfrak{l})}(L(\lambda), T^{\lambda-\nu}) = 0,$$

since $(T^\mu)^\sigma \cong T^\mu$ and $L(\lambda)^\sigma \cong L(\lambda)$.

First we note that that $V_\mu \subset \sigma(\mathfrak{m})V$ implies $\mathrm{Hom}_{\sigma(\mathfrak{l})}(V, T^\mu) = 0$ for any $\sigma(\mathfrak{l})$-module V. Indeed, let $f \in \mathrm{Hom}_{\sigma(\mathfrak{l})}(V, T^\mu)$, then $f(v) \neq 0$ for some $v \in V_\mu$. But $v = Xw$ for some $X \in \sigma(\mathfrak{m}), w \in V$ and $f(w) = 0$ because the weight of w is greater than μ. Contradiction.

Hence $\mathrm{Hom}_{\sigma(\mathfrak{l})}(L(\lambda), T^{\lambda-\nu}) = 0$ and $\mathrm{Hom}_{\sigma(\mathfrak{l})}(M'(\lambda), T^{\lambda-\nu}) = 0$. Now we use the exact sequence

$$0 \to M'(\lambda) \to M(\lambda) \to L(\lambda) \to 0.$$

The Verma module $M(\lambda)$ is free over $\sigma(\mathfrak{m})$ and hence projective over $\sigma(\mathfrak{l})$. Therefore

$$\mathrm{Ext}^1_{\sigma(\mathfrak{l})}(M(\lambda), T^{\lambda-\nu}) = 0.$$

Now applying the long exact sequence for Ext and using

$$\mathrm{Hom}_{\sigma(\mathfrak{l})}(M'(\lambda), T^{\lambda-\nu}) = 0$$

one immediately obtains

$$\mathrm{Ext}^1_{\sigma(\mathfrak{l})}(L(\lambda), T^{\lambda-\nu}) = 0$$

as required. □

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