ON ASSOCIATED VARIETY FOR LIE SUPERALGEBRAS

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1. Introduction

In this paper we suggest a notion of associated variety for a module over Lie superalgebras. This a superanalogue of an associated variety for Harish-Chandra modules. Associated varieties have many interesting applications in classical representation theory (see, for example, [7, 11, 13]).

The associated variety for a Lie superalgebra is a subvariety of a cone $X \subset \mathfrak{g}_1$ of self-commuting odd elements. This cone X was studied by Caroline Gruson, see [14, 5, 15]. She used geometric properties of X to obtain important results about cohomology of Lie superalgebras.

While the associated variety in classical representation theory is trivial if a module is finite-dimensional, finite-dimensional modules over classical Lie superalgebras have interesting associated varieties. Since finite-dimensional representation theory of superalgebras still has many open problems, we hope that our associated variety will have some application in this theory. In particular, it should help to describe analytic properties of supercharacters and cohomolgy groups. Some simple applications are given in Sections 3 and 7.

Let us outline the results of this paper. In Section 2 we give a definition and formulate simple properties of associated variety. In Section 3 we construct a coherent sheaf on X associated with M and prove a criterion of projectivity for certain Lie superalgebras. In Section 4 we discuss geometry of X. Section 5 contains main theorems (Theorem 5.3 and Theorem 5.4) about the associated varieties for simple classical contragredient superalgebras. In Section 6 we prove Theorem 5.3. In Section 7 we give some applications of Theorem 5.3 to supercharacters. Sections 8,9 and 10 contain the proof of Theorem 5.4.

2. Definition and basic properties

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional complex Lie superalgebra, G_0 denote a simply-connected connected Lie group with Lie algebra $\mathfrak{g}_0$. Let

$$X = \{x \in \mathfrak{g}_1 \mid [x, x] = 0\}.$$

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It is clear that X is G_0-invariant Zariski closed cone in $\mathfrak{g}_1$. Let M be a $\mathfrak{g}$-module. For each $x \in X$ put $M_x = \operatorname{Ker} x / xM$ and define

$$X_M = \{x \in X \mid M_x \neq 0\}.$$

We call X_M the *associated variety* of M.

Lemma 2.1. *If M is a finite-dimensional $\mathfrak{g}$-module, then X_M is Zarisky closed G_0-invariant subvariety.*

Proof. Since M is finite-dimensional, M is a G_0-module. For each $g \in G_0$ and $x \in M$ one has

$$M_{\mathrm{Ad}_g(x)} = gM_x,$$

that implies Lemma. □

Lemma 2.2. (1) *If $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M_0$ for some $\mathfrak{g}_0$-module M_0, then $X_M = \{0\}$;*
(2) *If $M = \mathbb{C}$ is trivial, then $X_M = X$;*
(3) *For any $\mathfrak{g}$-modules M and N, one has $X_{M \oplus N} = X_M \cup X_N$;*
(4) *For any $\mathfrak{g}$-modules M and N, one has $X_{M \otimes N} = X_M \cap X_N$;*
(5) *For any finite-dimensional $\mathfrak{g}$-module M, $X_{M^*} = X_M$;*
(6) *For any finite-dimensional $\mathfrak{g}$-module M and any $x \in X$, sdim M =sdim M_x.*

Proof. Properties 2,3,5 follow directly from definition. To prove 1, let $x \in X$ and $x \neq 0$. Let $\{v_j\}_{j \in J}$ be a basis of M_0 and $x_1, \dots, x_m$ be a basis of $\mathfrak{g}_1$ such that $x = x_1$. Then by PBW for Lie superalgebras $x_{i_1} x_{i_2} \dots x_{i_k} \otimes v_j$ for all $1 \leq i_1 < i_2 < \dots < i_k \leq m$, $j \in J$ form a basis of M. The action of $x = x_1$ in this basis is easy to write and it is clear that $\operatorname{Ker} x = xM$ is spanned by the vectors $x_1 x_{i_2} \dots x_{i_k} \otimes v_j$.

Now let us show (4). We will prove that $M_x = 0$ implies $(M \otimes N)_x = 0$. Indeed, $M_x = 0$ implies that M is a free $\mathbb{C}[x]$-module. Tensor product of a free $\mathbb{C}[x]$-module with any $\mathbb{C}[x]$-module is free. Therefore $M \otimes N$ is free over $\mathbb{C}[x]$ and $(M \otimes N)_x = 0$.

Finally we will prove (6). Let $\Pi(N)$ stand for the superspace isomorphic to N with switched parity. Since $M/\operatorname{Ker} x$ is isomorphic to $\Pi(xM)$, then

$$\text{sdim } M = \text{sdim } \operatorname{Ker} x + \text{sdim } \Pi(xM) = \text{sdim } \operatorname{Ker} x - \text{sdim } xM = \text{sdim}(\operatorname{Ker} x / xM) = \text{sdim } M_x.$$

□

3. Localization and projective modules

Let $\mathcal{O}_X$ denote the structure sheaf of X. Then $\mathcal{O}_X \otimes M$ is the sheaf of sections of a trivial vector bundle with fiber isomorphic to M. Let $\partial : \mathcal{O}_X \otimes M \to \mathcal{O}_X \otimes M$ be the map defined by

$$\partial\varphi(x) = x\varphi(x)$$

for any $x \in X$, $\varphi \in \mathcal{O}_X \otimes M$. Clearly $\partial^2 = 0$ and the cohomology $\mathcal{M}$ of ∂ is a quasi-coherent sheaf on X. If M is finite-dimensional, then $\mathcal{M}$ is coherent.

For any $x \in X$ denote by $\mathcal{O}_x$ the local ring of x, by $\mathcal{I}_x$ the maximal ideal. Then the fiber $\mathcal{M}_x$ is the the cohomology of $\partial : \mathcal{O}_x \otimes M \to \mathcal{O}_x \otimes M$. The evaluation map $j_x : \mathcal{O}_x \otimes M \to M$ satisfies $j_x \circ \partial = x \circ j_x$. Hence we have the maps

$$j_x : \operatorname{Ker} \partial \to \operatorname{Ker} x,\ j_x : \operatorname{Im} \partial \to xM.$$

One can easily check that the latter map is surjective. Therefore j_x induces the map $\bar{j}_x \colon \mathcal{M}_x \to M_x$, and $\operatorname{Im} \bar{j}_x \cong \mathcal{M}_x / \mathcal{I}_x \mathcal{M}_x$.

Lemma 3.1. *Let M be a finite-dimensional $\mathfrak{g}$-module. The support of $\mathcal{M}$ is contained in X_M. The map $\bar{j}_x$ is surjective for a generic point $x \in X$. In particular, if $X_M = X$, then* $\operatorname{supp} \mathcal{M} = X$.

Proof. First, we will show that for any $x \in X \backslash X_M$ there exists a neighborhood U of x such that $\mathcal{M}(U) = 0$. Indeed, there exists a map $i_x : M \to M$ such that $x \circ i_x = id$. Therefore in some neighborhood U of x there exists a map $i : \mathcal{O}(U) \otimes M \to \mathcal{O}(U) \otimes M$ such that $\partial \circ i = \mathrm{id}$ and $i(x) = i_x$, hence $\mathcal{M}(U) = 0$. Thus, x does not belong to the support of $\mathcal{M}$ and we have obtained that $\operatorname{supp} \mathcal{M} \subset X_M$.

To prove the second statement let $x \in X$ be such that $\dim xM$ is maximal possible. Let $m \in \operatorname{Ker} x$. Then there exists some neighborhood U of x and $\varphi \in \mathcal{O}(U) \otimes M$ such that $\partial \varphi = 0$ and $\varphi(x) = m$. By definition $\varphi \in \mathcal{M}_x$ and $\bar{j}_x(\varphi) = m$. □

Corollary 3.2. *Let $x \in X$ be a generic point, then in some neighborhood U of x, the sheaf $\mathcal{M}_U$ coincides with the sheaf of section of a vector bundle with fiber M_x.*

Let $X_M \neq X$. Then $\mathcal{M}$ is the extension by zero of the sheaf $\mathcal{M}_{X_M}$. If we denote by $\mathcal{M}(x)$ the image of $\bar{j}_x$, then $\mathcal{M}_{X_M}$ locally is the sheaf of sections of the vector bundle with fiber $\mathcal{M}(x)$ for a generic $x \in X_M$. Note that $\mathcal{M}(x) \subset M_x$, but usually $\mathcal{M}(x) \neq M_x$, as one can see from the following example.

Example 3.3. Let $\mathfrak{g} = \mathfrak{sl}(1|n)$. Then $\mathfrak{g}_1 = \mathfrak{g}(-1) \oplus \mathfrak{g}(1)$, where $\mathfrak{g}(-1)$ and $\mathfrak{g}(1)$ are abelian superalgebras. Assume that M is a typical irreducible $\mathfrak{g}$-module. Then $X_M = \{0\}$, $M_0 = M$ and $\mathcal{M}(0) = M^{\mathfrak{g}(1)} \oplus M^{\mathfrak{g}(-1)}$.

Let $\mathcal{F}$ be the category of finite-dimensional $\mathfrak{g}$-modules semisimple over $\mathfrak{g}_0$. The latter condition is automatic if $\mathfrak{g}_0$ is semisimple.

Theorem 3.4. *Assume that $\mathfrak{g}_0$ is a reductive Lie algebra and elements of X span $\mathfrak{g}_1$. Then $M \in \mathcal{F}$ is projective iff $X_M = \{0\}$.*

Proof. Let M be projective. Since M is a quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M$, then M is a direct summand of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M$. By Lemma 2.2 (1) and (3) $X_M = \{0\}$.

To prove the assertion in opposite direction we need the following lemma. Let $H^{\cdot}_{\mathrm{red}}(\mathfrak{g}, M)$ denote the cohomology of $\mathfrak{g}$, induced by cocycles trivial on the center of $\mathfrak{g}_0$.

Lemma 3.5. *Let $\mathfrak{g}$ satisfy the condition of Theorem, $M \in \mathcal{F}$ and $X_M = \{0\}$. Then $H^1_{red}(\mathfrak{g}, M) = \{0\}$.*

Proof. Let $\varphi \in \mathfrak{g}^* \otimes M$ be a 1-cocycle. Then for any $x \in X$ we have $x\varphi(x) = 0$. Thus, φ induces a global section of $\mathcal{M}$. Since $\varphi(0) = 0$ and the $\operatorname{supp}\mathcal{M} = \{0\}$ by Lemma 3.1, this global section must be zero. Therefore there exists $\psi(x)$ such that $x\psi(x) = \varphi(x)$ for all $x \in X$. But φ is a linear function, therefore ψ is constant. If d is the differential in the cohomology complex, $\eta = \varphi - d\psi$ is a 1-cocycle homologically equivalent to φ. On the other hand, $\eta(x) = 0$ for any $x \in X$, and since elements of X span $\mathfrak{g}_1$, we have $\eta(\mathfrak{g}_1) = 0$. The restriction η on $\mathfrak{g}_0$ is a 1-cocycle for a Lie algebra $\mathfrak{g}_0$. But $\mathfrak{g}_0$ is reductive, hence $H^1_{\text{red}}(\mathfrak{g}_0, M) = 0$ and therefore $\eta = d\nu$. We have shown that φ induces the trivial cohomology class. Thus, $H^1_{\text{red}}(\mathfrak{g}, M) = \{0\}$. $\square$

Now, assume that $X_M = \{0\}$. We have to show that $\operatorname{Ext}^1(M, N) = \{0\}$, the latter is equivalent to $H^1_{\text{red}}(\mathfrak{g}, M^* \otimes N) = \{0\}$. By Lemma 2.2 $(4), (5)$ we have $X_{M^*\otimes N} = \{0\}$. Therefore $H^1_{\text{red}}(\mathfrak{g}, M^* \otimes N) = \{0\}$ and M is projective. $\square$

Remark 3.6. Note that the conditions of Theorem 3.4 hold for any simple classical superalgebra except $\mathfrak{osp}(1|2n)$. In case $\mathfrak{g} = \mathfrak{osp}(1|2n)$, $X = \{0\}$ and $\mathcal{F}$ is semisimple, hence every finite-dimensional module is projective. In general, however, Theorem 3.4 is not true if we drop the assumption that X spans $\mathfrak{g}_1$. Indeed, let $\mathfrak{g} = \mathfrak{q}(1)$, in other words $\mathfrak{g}$ has a basis of an even element C and an odd element T such that $[T, T] = C$. Then $X = \{0\}$, but not every module in $\mathcal{F}$ is projective. For example, the trivial one-dimensional $\mathfrak{g}$-module is not projective.

4. The structure of X for contragredient simple Lie superalgebras

Let $\mathfrak{g}$ be a contragredient finite-dimensional Lie superalgebra with indecomposable Cartan matrix, i.e. $\mathfrak{g}$ is isomorphic to one from the following list: $\mathfrak{sl}(m|n)$ if $m \neq n$, $\mathfrak{gl}(n|n)$, $\mathfrak{osp}(m|2n)$, $D(\alpha)$, F_4 or G_3 (for definitions see [1]).

Remark 4.1. The Lie superalgebras we consider are simple except one case. For a simple Lie superalgebra $\mathfrak{psl}(n|n)$ the Cartan matrix is degenerate and we consider the corresponding Kac-Moody Lie superalgebra which is isomorphic to $\mathfrak{gl}(n|n)$. Later we will do the proofs for $\mathfrak{gl}(m|n)$ even if $m \neq n$, in this case $\mathfrak{gl}(m|n) \cong \mathfrak{sl}(m|n) \oplus \mathbb{C}$.

We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. In this case the Cartan subalgebra of $\mathfrak{g}$ coincides with a Cartan subalgebra of $\mathfrak{g}_0$ and $\mathfrak{g}$ has a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha,$$

each root space $\mathfrak{g}_\alpha$ is one dimensional. The parity of $\alpha \in \Delta$ by definition is equal to the parity of the root space $\mathfrak{g}_\alpha$. The invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ is not positive definite and some of odd roots are isotropic. For a non-isotropic β we denote by $\beta^\vee$ the element of $\mathfrak{h}$ such that $\alpha(\beta^\vee) = \frac{2(\alpha,\beta)}{(\beta,\beta)}$. Let S denote the set of subsets of mutually orthogonal linearly independent isotropic roots of Δ_1, i.e. an element of S is $\mathrm{A} = \{\alpha_1, \dots, \alpha_k \mid (\alpha_i, \alpha_j) = 0\}$. The Weyl group W of $\mathfrak{g}_0$ acts on S in the obvious way. Put $S_k = \{\mathrm{A} \in S \mid |\mathrm{A}| = k\}$, here $S_0 = \{\varnothing\}$.

Theorem 4.2. *There are finitely many G_0-orbits on X. These orbits are in one-to one correspondence with W-orbits in S.*

Proof. We define the map $\Phi : S \to X/G_0$ in the following way. Let $\mathrm{A} = \{\alpha_1, \dots, \alpha_k\} \in S$, choose a non-zero $x_i \in \mathfrak{g}_{\alpha_i}$ and put $x = x_1 + \dots + x_k \in X$. By definition $\Phi(\mathrm{A}) = G_0 x$. To see that $\Phi(\mathrm{A})$ does not depend on a choice of x_i note that since $\alpha_1, \dots, \alpha_k$ are linearly independent, for any other choice

$$x' = \Sigma x'_i = \Sigma c_i x_i$$

there is $h \in \mathfrak{h}$ such that $c_i = e^{\alpha_i(h)}$ and therefore

$$x' = \exp(\mathrm{ad}(h))(x).$$

If $\mathrm{B} = w(\mathrm{A})$ for some $w \in W$, then clearly $\Phi(\mathrm{B})$ and $\Phi(\mathrm{A})$ belong to the same orbit. Therefore Φ induces the map $\bar{\Phi} : S/W \to X/G_0$. We check case by case that $\bar{\Phi}$ is injective and surjective.

If $\mathfrak{g}$ is $\mathfrak{sl}(m|n)$ or $\mathfrak{gl}(n|n)$, $\mathfrak{g}$ has a natural $\mathbb{Z}$ grading $\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ such that $\mathfrak{g}_0 = \mathfrak{g}(0)$, $\mathfrak{g}_1 = \mathfrak{g}(1) \oplus \mathfrak{g}(-1)$. The orbits of W on S are enumerated by the pairs of numbers (p, q), where $p = |\mathrm{A} \cap \Delta(\mathfrak{g}(1))|$, $q = |\mathrm{A} \cap \Delta(\mathfrak{g}(-1))|$. The orbits of G_0 on X are enumerated by the same pairs of numbers (p, q) in the following way. If $x = x^+ + x^-$, where $x^\pm \in \mathfrak{g}(\pm 1)$, then $p = \mathrm{rank}(x^+)$, $q = \mathrm{rank}(x^-)$. We can see by the construction of $\bar{\Phi}$, that $\bar{\Phi}$ maps (p, q)-orbit on S to the (p, q)-orbit on X.

Let $\mathfrak{g} = \mathfrak{osp}(m|2n)$. If $m = 2l + 1$ or $m = 2l$ with $l > n$, then the W-orbits on S are in one-to-one correspondence with $\{0, 1, 2, \dots, \min(l, n)\}$. Namely, A and B are on the same orbit if they have the same number of elements. As it was shown in [14], X can be identified with the set of all linear maps $x : \mathbb{C}^m \to \mathbb{C}^{2n}$, such that $\mathrm{Im}\, x$ is an isotropic subspace in $\mathbb{C}^{2n}$ and $\mathrm{Im}\, x^*$ is an isotropic subspace in $\mathbb{C}^m$. Furthermore, $x, y \in X$ belong to the same G_0-orbit iff $\mathrm{rank}(x) = \mathrm{rank}(y)$. One can see that rank $\Phi(\mathrm{A}) = |\mathrm{A}|$.

Now let $\mathfrak{g} = \mathfrak{osp}(2l|2n)$ where $l \leq n$. If $\mathrm{A}, \mathrm{B} \in S$ and $|\mathrm{A}| = |\mathrm{B}| < l$, then A and B are on the same W-orbit. In the same way if $\mathrm{rank}(x) = \mathrm{rank}(y) < l$, then x and y are on the same G_0-orbit. However, the set of all $x \in \mathfrak{g}_1$ of maximal rank splits in two orbits, since the Grassmannian of maximal isotropic subspaces in $\mathbb{C}^{2l}$ has two connected components. In the same way S_l splits in two W-orbits. Hence in this case again $\bar{\Phi}$ is a bijection.

If $\mathfrak{g}$ is one of exceptional Lie superalgebras $D(\alpha)$, G_3 or F_4, then the direct calculation shows that X has two G_0-orbits: $\{0\}$ and the orbit of a highest vector in $\mathfrak{g}_1$. The set S also consists of two W-orbits: $\varnothing$ and the set of all isotropic roots in Δ. $\square$

Remark 4.3. Note that in our situation the representation of G_0 in $\mathfrak{g}_1$ is symplectic and multiplicity free (see [8]). The cone X is the preimage of 0 under the moment map $\mathfrak{g}_1 \to \mathfrak{g}_0^*$.

We use the notation $\Phi : S \to X/G_0$ introduced in the proof of Theorem 4.2. Using the explicit description of G_0-orbits on X and the description of roots systems, which can be found in [1], one can check the following statements case by case. We omit this checking here.

Lemma 4.4. *Let $A, B \in S$.*

1. *If $\alpha \in \Delta$ is a linear combination of roots from A, then $\alpha \in A \cup -A$;*
2. *If $|A| \leq |B|$, then there exists $w \in W$ such that $w(A) \subset B \cup -B$;*
3. *$\Phi(A)$ lies in the closure of $\Phi(B)$ iff $w(A) \subset B$ for some $w \in W$.*

By $\mathrm{A}^{\perp}$ we denote the set of all weights orthogonal to A with respect to the standard form on $\mathfrak{h}^*$.

Theorem 4.5. *Let $A \in S$. Then* $\dim \Phi(A) = \frac{|\Delta_1 \backslash A^{\perp}|}{2} + |A|$.

Proof. Let $\mathrm{A} = \{\alpha_1, \dots, \alpha_k\}$, $x = x_1 + \dots + x_k$ for some choice of $x_i \in \mathfrak{g}_{\alpha_i}$, $y = y_1 + \dots + y_k$ for some $y_i \in \mathfrak{g}_{-\alpha_i}$. Let $h = [x, y]$, $h_i = [x_i, y_i]$. Clearly, $h = h_1 + \dots + h_k$ and h, x, y generate the $\mathfrak{sl}(1|1)$-subalgebra in $\mathfrak{g}$. With respect to this subalgebra $\mathfrak{g}$ has a decomposition

$$\mathfrak{g} = \oplus_{\mu} \mathfrak{g}^{\mu},$$

where

$$\mathfrak{g}^{\mu} = \{g \in \mathfrak{g} \mid [h, g] = \mu g\}.$$

Note that

$$\dim[\mathfrak{g}, x] = \sum_{\mu} \dim[\mathfrak{g}^{\mu}, x],$$

and from the description of irreducible $\mathfrak{sl}(1|1)$-modules for $\mu \neq 0$

$$\dim[\mathfrak{g}^{\mu}, x] = \frac{\dim \mathfrak{g}^{\mu}}{2}.$$

On the other hand, for $\mu \neq 0$ sdim $\mathfrak{g}^{\mu} = 0$, and therefore

$$\dim \mathfrak{g}^{\mu} = 2 \dim \mathfrak{g}_1^{\mu}.$$

Observe that for a generic choice of $x_i \in \mathfrak{g}_{\alpha_i}$, $\mathfrak{g}_{\beta} \subset \mathfrak{g}^0$ iff $(\beta, \alpha_i) = 0$ for all $i \leq k$. Indeed, for generic choice of x_i the condition $\beta(h) = 0$ implies $\beta(h_i) = 0$ for all i, and therefore $(\beta, \alpha_i) = 0$ for all i. Hence

$$\oplus_{\mu \neq 0} \mathfrak{g}_1^{\mu} = \oplus_{\alpha \in \Delta_1 \backslash \mathrm{A}^{\perp}} \mathfrak{g}_{\alpha}$$

and

$$\sum_{\mu \neq 0} \dim[\mathfrak{g}^{\mu}, x] = \sum_{\mu \neq 0} \dim \mathfrak{g}_1^{\mu} = |\Delta_1 \backslash \mathrm{A}^{\perp}|.$$

To calculate $\dim[\mathfrak{g}^0, x]$ note that

$$\mathfrak{g}^0 = \mathfrak{h} \oplus \oplus_{\beta \in \Delta \cap \mathrm{A}^{\perp}} \mathfrak{g}_{\beta}.$$

We claim that

$$[\mathfrak{g}^0, x] = \oplus_{i=1}^{k} \mathbb{C} h_i \oplus \oplus_{i=1}^{k} \mathfrak{g}_{\alpha_i},$$

hence $\dim[\mathfrak{g}^0, x] = 2k$. Indeed, if $(\beta, \alpha_i) = 0, \beta \neq \pm\alpha_i$ then $\beta \pm \alpha_i \notin \Delta$. Therefore $[x, \mathfrak{g}_\beta] = 0$ for any $\beta \in \Delta \cap \mathrm{A}^\perp, \beta \neq -\alpha_i$. Furthermore, $[x, \mathfrak{g}_{-\alpha_i}] = \mathbb{C}h_i$ and $[x, \mathfrak{h}] = \oplus_{i=1}^k \mathfrak{g}_{\alpha_i}$. Thus, we obtain

$$\dim[\mathfrak{g}, x] = |\Delta_1 \backslash \mathrm{A}^\perp| + 2k. \tag{4.1}$$

Now the statement will follow from the lemma.

Lemma 4.6. *sdim* $[\mathfrak{g}, x] = 0$.

Proof. Define the odd skew-symmetric form on $\mathfrak{g}$ by

$$\omega(y, z) = (x, [y, z]).$$

Obviously the kernel of ω coincides with centralizer $C_\mathfrak{g}(x)$. Thus, ω is non-degenerate odd skew-symmetric form on $C_\mathfrak{g}(x)$. Hence sdim $\mathfrak{g}/C_\mathfrak{g}(x) = 0$. But $[\mathfrak{g}, x] \cong \Pi(\mathfrak{g}/C_\mathfrak{g}(x))$, which implies the lemma. □

Lemma implies that $\dim[\mathfrak{g}_0, x] = 1/2 \dim[\mathfrak{g}, x]$. Since $\dim G_0 x = \dim[\mathfrak{g}_0, x]$, the theorem follows from (4.1).

□

Corollary 4.7. *If* $|A| = |B|$, *then* $\dim \Phi(A) = \dim \Phi(B)$.

Proof. Follows from Theorem 4.2 and Lemma 4.4 (2). □

The maximal number of isotropic mutually orthogonal linearly independent roots is called the *defect* of $\mathfrak{g}$. This notion was introduced in [16]. One can see that the defect of $\mathfrak{g}$ is equal to the dimension of maximal isotropic subspace in $\mathfrak{h}^*$. All exceptional Lie superalgebras has defect 1. The defect of $\mathfrak{sl}(m|n)$ is $\min(m, n)$, the defect $\mathfrak{osp}(2l+1|2n)$ and $\mathfrak{osp}(2l|2n)$ is $\min(l, n)$.

Corollary 4.8. *Let d be the defect of $\mathfrak{g}$. Then the irreducible components of X are in bijection with W-orbits on S_d. If all odd roots of $\mathfrak{g}$ are isotropic, then the dimension of each component equals* $\frac{\dim \mathfrak{g}_1}{2} = \frac{|\Delta_1|}{2}$.

Proof. As follows from Theorem 4.2 and Lemma 4.4 (3), each irreducible component is the closure of $\Phi(\mathrm{A})$ for a maximal $\mathrm{A} \in S$. By Lemma 4.4 (2) $|\mathrm{A}| = d$. Hence the first statement. Theorem 4.5 immediately implies the statement about dimension. □

Corollary 4.9. *If all odd roots of $\mathfrak{g}$ are isotropic, then the codimension of $\Phi(A)$ in X equals* $\frac{|\Delta_1 \cap A^\perp|}{2} - |A|$.

Proof. The codimension equals $\dim X - \dim \Phi(\mathrm{A})$. Using Theorem 4.5 and Corollary 4.8

$$\operatorname{codim} \Phi(\mathrm{A}) = \frac{|\Delta_1| - |\Delta_1 \backslash \mathrm{A}^\perp|}{2} - |\mathrm{A}| = \frac{|\Delta_1 \cap \mathrm{A}^\perp|}{2} - |\mathrm{A}|.$$

□

5. Central character and the main theorems

Let us fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ by choosing a decomposition $\Delta = \Delta^+ \cup \Delta^-$. Note that this choice is not unique but our consideration will not depend on it. Later we will use different Borel subalgebras in some proofs. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha - \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha,$$

and define the shifted action of W on $\mathfrak{h}^*$ by

$$\lambda^w = w(\lambda + \rho) - \rho.$$

By M_λ we denote the Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_\lambda$, and by L_λ we denote the unique irreducible quotient of M_λ. We say that $\lambda \in \mathfrak{h}^*$ is *integral dominant* if L_λ is finite-dimensional. We denote by Σ the set of all integral dominant weights.

Let Z denote the center of the universal enveloping algebra $U(\mathfrak{g})$. One can see that any $z \in Z$ acts as a scalar $\chi_\lambda(z)$ on M_λ and L_λ. Therefore $\lambda \in \mathfrak{h}^*$ defines a central character $\chi_\lambda : Z \to \mathbb{C}$. Let

$$\mathfrak{h}_\chi = \{\mu \in \mathfrak{h}^* \mid \chi_\mu = \chi\}.$$

Lemma 5.1. *Let $\chi = \chi_\lambda$, $A \in S$ be a maximal set of linearly independent mutually orthogonal isotropic roots orthogonal to $\lambda + \rho$ and $\mathfrak{t}_\lambda = \lambda + \oplus_{\alpha \in A} \mathbb{C}\alpha$. Then*

$$\mathfrak{h}_\chi = \bigcup_{w \in W} \mathfrak{t}_\lambda^w.$$

Proof. Easily follows from the description of the Z formulated in [12] and proven [4] and in [9]. $\square$

Let us fix a central character χ. For each $\lambda \in \mathfrak{h}_\chi$ define $S_\lambda \subset S$ by the following

$$S_\lambda = \left\{ \mathrm{A} \in S \mid \mathrm{A} \subset (\lambda + \rho)^\perp \right\}.$$

Put

$$S_\chi = \cup_{\lambda \in \mathfrak{h}_\chi} S_\lambda.$$

Lemma 5.2. *There exists a number k such that $S_\chi = \bigcup_{i \le k} S_i$.*

Proof. It follows easily from Lemma 5.1 that S_χ is W-invariant. Furthermore, if $\mathrm{A} \in S_\chi$ and A' is obtained from A by multiplication of some roots in A on -1, then $\mathrm{A}' \in S_\chi$. Hence the statement follows from Lemma 4.4 (1) and (2). $\square$

The number k is called the *degree of atypicality* of χ. The degree of atypicality of λ is by definition the degree of atypicality of χ_λ. If $k = 0$, then χ is called *typical*. It is clear that the degree of atypicality of χ is not bigger than the defect of $\mathfrak{g}$.

Let $X_k = \Phi(S_k)$, $\bar{X}_k$ denote the closure of X_k. Lemma 4.4 (3) implies that

$$\bar{X}_k = \bigcup_{i=0}^{k} \Phi(X_i).$$

Theorem 5.3. *Let $\mathfrak{g}$ be a contragredient simple Lie superalgebra, M be a $\mathfrak{g}$-module which admits central character χ, the degree of atypicality of χ be equal to k. Then $X_M \subset \bar{X}_k$.*

Theorem 5.4. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ or $\mathfrak{sl}(m|n)$. For any integral dominant $\lambda \in \mathfrak{h}^*$ with degree of atypicality k, $X_{L_\lambda} = \bar{X}_k$.*

Conjecture 5.5. Let $\mathfrak{g}$ be a contragredient simple Lie superalgebra. For any integral dominant $\lambda \in \mathfrak{h}^*$ with degree of atypicality k, $X_{L_\lambda} = \bar{X}_k$.

First, observe that the conjecture is true for the typical character.

Theorem 5.6. *If λ is typical, then $X_{L_\lambda} = \{0\}$.*

Proof. If λ is typical, then L_λ is a direct summand of some induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M_0$(see [3]). Therefore Theorem follows from Lemma 2.2 (1) and (3). ☐

6. The structure of a generic fiber and the proof of Theorem 5.3

In this section we discuss properties of the fiber M_x over a point $x \in X_M$. Let $C_{\mathfrak{g}}(x)$ be the centralizer of $x \in X$, then by definition $\mathfrak{g}_x = C_{\mathfrak{g}}(x) / [x, \mathfrak{g}]$.

Lemma 6.1. *The subspace $[x, \mathfrak{g}]$ is an ideal in $C_{\mathfrak{g}}(x)$. Let $\mathfrak{m}^\perp$ denote the orthogonal complement to $\mathfrak{m}$ with respect to the invariant form on $\mathfrak{g}$. Then $[x, \mathfrak{g}]^\perp = C_{\mathfrak{g}}(x)$.*

Proof. Let $u \in C_{\mathfrak{g}}(x)$, $v \in [x, \mathfrak{g}]$. Then $v = [x, z]$ and

$$[u, [x, z]] = (-1)^{p(u)} [x, [u, z]] \in [x, \mathfrak{g}].$$

The second statement follows from the identity

$$(u, [x, z]) = -([u, x], z).$$

☐

Lemma 6.2. *M_x is a $C_{\mathfrak{g}}(x)$-module trivial over $[x, \mathfrak{g}]$.*

Proof. Let $m \in \operatorname{Ker} x$, $v = [x, z] \in [x, \mathfrak{g}]$. Then

$$vm = xzm - (-1)^{p(z)} zxm = xzm \in xM.$$

☐

In the case of contragredient finite-dimensional superalgebra we can describe $\mathfrak{g}_x$ precisely. Let $\mathrm{A} = \{\alpha_1, \dots, \alpha_k\} \in S$, $x \in X$, and $x = x_1 + \dots + x_k$, where $x_i \in \mathfrak{g}_{\alpha_i}$, $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Define $\mathrm{A}' = \mathrm{A}^\perp \cap \Delta \setminus (\mathrm{A} \cup -\mathrm{A})$, $\mathfrak{h}_{\mathrm{A}} = \mathfrak{h}_{\alpha_1} \oplus \dots \oplus \mathfrak{h}_{\alpha_k}$.

Lemma 6.3. *If $\mathfrak{g}$ is finite-dimensional contragredient superalgebra, $A = \{\alpha_1, \dots, \alpha_k\} \in S$, $x \in X$, and $x = x_1 + \dots + x_k$, where $x_i \in \mathfrak{g}_{\alpha_i}$. Then $C_{\mathfrak{g}}(x)$ can be decomposed in a semidirect sum $\mathfrak{g}_x + [x, \mathfrak{g}]$, where $\mathfrak{g}_x$ is spanned by the root spaces $\mathfrak{g}_\alpha$ for all $\alpha \in A'$ and $\mathfrak{h}_x \subset \mathfrak{h}_{\mathrm{A}}^{\perp}$ is such that $\mathfrak{h}_x \oplus \mathfrak{h}_A = \mathfrak{h}_{\mathrm{A}}^{\perp}$. Furthermore, def $\mathfrak{g}_x =$def $\mathfrak{g} - k$.*

Proof. We use the same argument as in the proof of Theorem 4.5. Let h and $\mathfrak{g}^{\mu}$ be as in this proof. First, there is an isomorphism

$$\mathfrak{g}_x \cong \mathfrak{g}^0 \cap C_{\mathfrak{g}}(x) / \mathfrak{g}^0 \cap [x, \mathfrak{g}] .$$

Then we notice that

$$\mathfrak{g}^0 \cap C_{\mathfrak{g}}(x) = \mathfrak{h}_{\mathrm{A}}^{\perp} \oplus \oplus_{\alpha \in \mathrm{A}^{\perp} \cap \Delta \setminus -\mathrm{A}} \mathfrak{g}_\alpha, \ \mathfrak{g}^0 \cap [x, \mathfrak{g}] = \mathfrak{h}_{\mathrm{A}} \oplus \mathfrak{g}_{\alpha_1} \oplus \dots \oplus \mathfrak{g}_{\alpha_k}.$$

Choose $\mathfrak{h}_x$ in such a way that $\mathfrak{g}_x = (\mathfrak{h}_x \oplus \oplus_{\alpha \in \mathrm{A}'} \mathfrak{g}_\alpha)$ is a subalgebra, then

$$\mathfrak{g}^0 \cap C_{\mathfrak{g}}(x) = \mathfrak{g}_x \oplus \mathfrak{g}^0 \cap [x, \mathfrak{g}] .$$

□

Remark 6.4. If $\mathfrak{g} = \mathfrak{gl}(m|n)$, then $\mathfrak{g}_x \cong \mathfrak{gl}(m-k|n-k)$. If $\mathfrak{g} = \mathfrak{osp}(m|2n)$, then $\mathfrak{g}_x \cong \mathfrak{osp}(m-2k|2n-2k)$. If $\mathfrak{g} = D(\alpha)$ and $x \neq 0$, then $\mathfrak{g}_x \cong \mathbb{C}$. For $\mathfrak{g} = G_3$ or F_4 for a non-zero $x \in X$, $\mathfrak{g}_x$ is isomorphic to $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ respectively.

Lemma 6.5. *Let $x : V \to V$ be an odd linear operator such that $x^2 = 0$. Assume that $V = W \oplus U$, where W is a trivial $\mathbb{C}[x]$-submodule and U is a free $\mathbb{C}[x]$-module. Let $S(V)^x$ denote the space of x-invariants in $S(V)$. Then $S(V)^x = S(W) \otimes S(U)^x$ and $S(U)^x \subset S(U)\, U^x$.*

Let $U(\mathfrak{g})^{\mathrm{ad}(x)}$ denote the subalgebra of $\mathrm{ad}(x)$-invariants in $U(\mathfrak{g})$, I_x be the left ideal in $U(\mathfrak{g})$ generated by $[x, \mathfrak{g}]$. One has the following sequence

$$U(\mathfrak{g}_x) \xrightarrow{\iota} U(\mathfrak{g})^{\mathrm{ad}(x)} \xrightarrow{\pi} U(\mathfrak{g})^{\mathrm{ad}(x)} / I_x \cap U(\mathfrak{g})^{\mathrm{ad}(x)} .$$

Let $\phi = \pi \circ \iota$.

Lemma 6.6. *The map $\phi : U(\mathfrak{g}_x) \to U(\mathfrak{g})^{\mathrm{ad}(x)} / I_x \cap U(\mathfrak{g})^{\mathrm{ad}(x)}$ is an isomorphism of vector spaces.*

Proof. Since $I_x \cap U(\mathfrak{g}_x) = \{0\}$, ϕ is injective. To prove surjectivity of ϕ use PBW and the corresponding sequence for symmetric algebras

$$S(\mathfrak{g}_x) \to S(\mathfrak{g})^{\mathrm{ad}(x)} \to S(\mathfrak{g})^{\mathrm{ad}(x)} / J_x \cap S(\mathfrak{g})^{\mathrm{ad}(x)} ,$$

where $J_x = [x, \mathfrak{g}]\, S(\mathfrak{g})$. Apply Lemma 6.5 with $V = \mathfrak{g}$. Then $W = \mathfrak{g}_x$, $U^x = [x, \mathfrak{g}]$, and we obtain $S(\mathfrak{g})^{\mathrm{ad}(x)} = S(\mathfrak{g}_x) \otimes S(U)^x$ and $S(U)^x \subset [x, \mathfrak{g}]\, S(U)$. Thus, $gr\phi$ is an isomorphism. Hence ϕ is an isomorphism. □

Define the map $\eta : U(\mathfrak{g})^{\mathrm{ad}(x)} \to U(\mathfrak{g}_x)$ by putting $\eta = \phi^{-1} \circ \pi$. As follows from Lemma 6.2 for any $u \in U(\mathfrak{g})^{\mathrm{ad}(x)}$, $m \in M_x$

$$um = \eta(u)\, m \tag{6.1}$$

Note that ι, π are homomorphisms of $\mathfrak{g}_x$-modules (with respect to the adjoint action). The center Z of $U(\mathfrak{g})$ obviously is a subalgebra in $U(\mathfrak{g})^{\mathrm{ad}(x)}$. Let $Z(\mathfrak{g}_x)$ be the center of $U(\mathfrak{g}_x)$. Since η is a homomorphism of $\mathfrak{g}_x$-modules, $\eta(Z) \subset Z(\mathfrak{g}_x)$. We are going to describe the dual map

$$\eta^* \colon \operatorname{Hom}(Z(\mathfrak{g}_x), \mathbb{C}) \to \operatorname{Hom}(Z, \mathbb{C}).$$

Choose a borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ such that $\alpha_1, \dots, \alpha_k$ are simple roots.

Lemma 6.7. *Let $\lambda \in \mathfrak{h}^*$ satisfy $(\lambda+\rho, \alpha_1) = \dots = (\lambda+\rho, \alpha_k) = 0$. Then $(L_\lambda)_x \neq 0$. In particular the highest vector v belongs to $(L_\lambda)_x$.*

Proof. Clearly, $v \in \operatorname{Ker} x$. If $v = xw$, then one can choose w with weight $\lambda - \alpha_i$ for some i. However, L_λ does not have vectors of such weight. □

Corollary 6.8. *Let λ be as in Lemma 6.7 and μ be the restriction of λ to $\mathfrak{h}_x$. Let $\chi_\mu \in \operatorname{Hom}(Z(\mathfrak{g}_x), \mathbb{C})$ be induced by μ and $\chi_\lambda \in \operatorname{Hom}(Z, \mathbb{C})$ be induced by λ via Harish-Chandra homomorphism. Then $\eta^*(\chi_\mu) = \chi_\lambda$.*

Corollary 6.9. *Let $\chi \in \operatorname{Hom}(Z(\mathfrak{g}_x), \mathbb{C})$ and have the degree of atypicality s. Then the degree of atypicality of $\eta^*(\chi)$ equals $s+k$.*

Corollary 6.9 implies Theorem 5.3. It also implies the following

Theorem 6.10. *Let M admit a central character with degree of atypicality k, and $x \in X_k$. Then $\mathfrak{g}_x$-module M_x admits a typical central character. In particular, if M_x is finite dimensional, it is semi-simple over $\mathfrak{g}_x$, and therefore over $C_{\mathfrak{g}}(x)$.*

Theorem 6.11. *If $\mathfrak{g} \neq \mathfrak{osp}(2l|2n)$ or $D(\alpha)$, then η^* is injective, and therefore η is surjective. If $\mathfrak{g} = \mathfrak{osp}(2l|2n)$ or $D(\alpha)$, then a preimage of η^* has at most two elements.*

Proof. Let $\mathrm{A} = \{\alpha_1, \dots, \alpha_k\}$, $x = x_1 + \dots + x_k$, $x_i \in \mathfrak{g}_{\alpha_i}$, $\mathfrak{b}$ be such that $\alpha_1, .., \alpha_k$ are simple. Let

$$W' = \{w \in W \mid w(\mathrm{A}) \subset \mathrm{A} \cup -\mathrm{A}\},$$

and $W(\mathfrak{g}_x)$ denote the Weyl group of $\mathfrak{g}_x$. Clearly, $W(\mathfrak{g}_x) \subset W$. One can show that if $\mathfrak{g} \neq \mathfrak{osp}(2l|2n)$ or $D(\alpha)$, then $W' = W(\mathfrak{g}_x) \times W''$, where W'' consists of all elements which act trivially on $\mathfrak{g}_x$.

Let $\lambda \in \mathfrak{h}^*$, $(\lambda+\rho, \alpha_i) = 0$ for all $i = 1, \dots, k$. Then[1]

$$\mathfrak{h}_{\chi_\lambda} \cap \left(\mathfrak{h}_{\mathrm{A}}^{\perp}\right)^* = \bigcup_{w \in W'} \mathfrak{t}_\lambda^w, \ \mathfrak{h}_{\chi_\lambda} \cap \mathfrak{h}_x = \bigcup_{w \in W'} \left(\mathfrak{t}_\lambda^w \cap \mathfrak{h}_x^*\right).$$

[1]We also use the fact that $\rho_x = \frac{1}{2}\Sigma_{\alpha \in \Delta_0(\mathfrak{g}_x)}\alpha - \frac{1}{2}\Sigma_{\alpha \in \Delta_1(\mathfrak{g}_x)}\alpha = \rho_{|\mathfrak{h}_x}$. Hence the shifted action of $W(\mathfrak{g}_x)$ is the same.

Let $\mathfrak{g} \neq \mathfrak{osp}(2l|2n)$ or $D(\alpha)$ and μ be the restriction of λ on $\mathfrak{h}_x$. Then

$$\mathfrak{h}_{\chi_\lambda} \cap \mathfrak{h}_x = \bigcup_{w \in W(\mathfrak{g}_x)} (\mathfrak{t}_\lambda \cap \mathfrak{h}_x^*)^w = \bigcup_{w \in W(\mathfrak{g}_x)} \mathfrak{t}_\mu^w = (\mathfrak{h}_x)_{\chi_\mu},$$

that shows $(\eta^*)^{-1}(\chi_\lambda) = \chi_\mu$.

In case $\mathfrak{g} = \mathfrak{osp}(2l|2n)$ or $D(\alpha)$, $W(\mathfrak{g}_x) \times W''$ has index 2 in W'. Take $u \in W'$, $u \notin W(\mathfrak{g}_x) \times W''$, let μ be the restriction of λ on $\mathfrak{h}_x$ and μ' be the restriction of λ^u on $\mathfrak{h}_x$. Then

$$\mathfrak{h}_{\chi_\lambda} \cap \mathfrak{h}_x = \bigcup_{w \in W'} (\mathfrak{t}_\lambda \cap \mathfrak{h}_x^*)^w = \bigcup_{w \in W(\mathfrak{g}_x)} \left(\mathfrak{t}_\mu^w \cup \mathfrak{t}_{\mu'}^w\right) = (\mathfrak{h}_x)_{\chi_\mu} \cup (\mathfrak{h}_x)_{\chi_{\mu'}}.$$

Therefore $(\eta^*)^{-1}(\chi_\lambda) = \{\chi_\mu, \chi_{\mu'}\}$. □

Assume that M is finite-dimensional and has central character χ with degree of atypicality k. Let $x \in \bar{X}_k$. Let

$$Y_x = \{y \in (\mathfrak{g}_x)_1 \mid [y,y] = 0\}.$$

Then

$$x + Y_x \subset X \tag{6.2}$$

Define the coherent sheaf $\mathcal{N}$ on Y_x as the cohomology of

$$\partial : \mathcal{O}_{Y_x} \otimes M_x \to \mathcal{O}_{Y_x} \otimes M_x.$$

Let $\mathcal{N}(x)$ be the image of the fiber $\mathcal{N}_x$ in M_x under the evaluation map.

Theorem 6.12. $\mathcal{M}(x) = \mathcal{N}(0)$.

Proof. Obviously $\mathcal{M}(x) \subset \mathcal{N}(0)$. We have to show that $\mathcal{M}(x) = \mathcal{N}(0)$. Let $m \in \mathcal{N}(0)$. There exists an open $\mathcal{V} \subset Y_x$, $0 \in \mathcal{V}$, $\varphi \in \mathcal{O}(\mathcal{V}) \otimes M_x$ such that $\partial\varphi = 0$ and $\varphi(0) = m$. We have to extend φ to some open set $\mathcal{U} \subset X$. Let $\mathfrak{g} = C_{\mathfrak{g}}(x) \oplus \mathfrak{l}$ as $\mathfrak{g}_x$-module. Define the map

$$\tau : \mathfrak{l}_0 \times Y_x \to X$$

by the formula

$$\tau(l,y) = \exp \operatorname{ad}(l)(x+y),$$

for any $y \in Y_x$, $l \in \mathfrak{l}_0$. Then τ is a local isomorphism. Hence in some neighborhood $\mathcal{U} \subset X$, $x \in \mathcal{U}$, $x = \tau(l,y)$ and one can define

$$\psi(\tau(l,y)) = \exp l\varphi(y).$$

Then $\partial\psi = 0$ and $\psi(x) = m$. Theorem is proven. □

7. Application to supercharacters

The properties of M_x allow one to say something about the superdimension and supercharacter of M. First, we recall that sdim M_x =sdim M. Therefore

Lemma 7.1. *If $X_M \neq X$, then sdim $M = 0$. In particular, if a finite-dimensional module M admits a central character whose degree of atypicality is less than the defect of $\mathfrak{g}$, then sdim $M = 0$.*

Now let M be a finite-dimensional $\mathfrak{g}$-module and $h \in \mathfrak{h}$. Write

$$\mathrm{ch}_M(h) = \mathrm{str}_M\left(e^h\right).$$

Obviously, ch_M is W-invariant analytic function on $\mathfrak{h}$. We can write Taylor series for ch_M at $h = 0$

$$\mathrm{ch}_M(h) = \sum_{i=0}^{\infty} p_i(h),$$

where $p_i(h)$ is a homogeneous polynomial of degree i on $\mathfrak{h}$. The order of ch_M at zero is by definition the minimal i such that $p_i \not\equiv 0$.

Theorem 7.2. *Assume that all odd roots of $\mathfrak{g}$ are isotropic. Let M be a finite-dimensional $\mathfrak{g}$-module, s be the codimension of X_M in X. The order of ch_M at zero is greater or equal than s. Moreover, the polynomial $p_s(h)$ in Taylor series for ch_M is determined uniquely up to proportionality.*

Proof. The proof is based on the following Lemma, the proof of this Lemma is similar to the proof of Lemma 2.2 (6). We leave it to the reader.

Lemma 7.3. *Let $x \in X$, $h \in \mathfrak{g}_0$ and $[h, x] = 0$. Then $\mathrm{Ker}\, x$ and xM are h-invariant and $str_M h$ =$str_{M_x} h$.*

If $X_M = X$, the statement of theorem is trivial. Let $X_M \neq X$. By Theorem 5.3 there exists k less than the defect of $\mathfrak{g}$ such that

$$X_M \subset \cup_{\mathrm{A} \in S,\ |\mathrm{A}| \leq k} \Phi(\mathrm{A}).$$

Let $\mathrm{A} = \{\alpha_1, \dots, \alpha_{k+1}\} \in S$, $x = x_1 + \dots + x_{k+1}$ for some nonzero $x_i \in \mathfrak{g}_{\alpha_i}$. Then $M_x = \{0\}$. If $h \in \mathfrak{h}$ satisfies $\alpha_1(h) = \dots = \alpha_{k+1}(h) = 0$, then $[h, x] = 0$. Hence by Lemma 7.3 $\mathrm{str}_M h$ =$\mathrm{str}_{M_x} h = 0$. Hence we just have proved the following property

$$\mathrm{ch}_M\left(\mathfrak{h}_{\mathrm{A}}^{\perp}\right) = 0 \text{ for all } \mathrm{A} \in S, |\mathrm{A}| = k+1. \tag{7.1}$$

Let p_i be the first non-zero polynomial in the Taylor series for ch_M at zero. Then p_i also satisfies (7.1). Let $\mathrm{B} = \{\alpha_1, \dots, \alpha_k\} \in S$ and $\bar{p}_i$ be the restriction of p_i to $\mathfrak{h}_{\mathrm{B}}^{\perp}$. If $\bar{p}_i \neq 0$, then degree of $\bar{p}_i$ is i. Since $p_i\left(\mathfrak{h}_{\mathrm{B}\cup\alpha}^{\perp}\right) = 0$ for any $\alpha \neq \pm\alpha_i$, $\alpha \in \mathrm{B}^{\perp}$, then α divides $\bar{p}_i$. That gives the estimate on i. Indeed, i is not less the number of all possible α up to proportionality, i.e. $\frac{|\Delta_1 \cap \mathrm{B}^{\perp}|}{2} - |\mathrm{B}|$. By Corollary 4.9 the latter number is the codimension s of X_M in X. Hence $i \geq s$.

To prove the second statement we need to show that if two homogeneous W-invariant polynomials p and q of degree s satisfy (7.1), then $p = cq$ for some $c \in \mathbb{C}$. After restriction on $\mathfrak{h}_{\mathrm{B}}^{\perp}$

$$\bar{p} = a\Pi_{\alpha\in\left(\Delta^+\cap \mathrm{B}^{\perp}\right)\setminus\pm \mathrm{B}}\alpha,\ \bar{q} = b\Pi_{\alpha\in\left(\Delta^+\cap \mathrm{B}^{\perp}\right)\setminus\pm \mathrm{B}}\alpha$$

for some constants a and b. Therefore there exists $f = p - cq$ such that $f\left(\mathfrak{h}_{\mathrm{B}}^{\perp}\right) = 0$. Thus, f satisfies (7.1) for k instead of $k+1$. Then the degree of f is bigger than s, which implies $f = 0$. □

8. Translation functor

In this section we introduce translation functors, we use these functors in the proof of Theorem 5.4. A translation functor is a superanalogue of similar functor in category $\mathcal{O}$ (see[6]). For superalgebras translation functors were used in [17] and [10].

Let V be a $\mathfrak{g}$-module, on which the center Z of the universal enveloping algebra acts locally finitely. Then $V = \oplus V^{\chi}$, where

$$V^{\chi} = \left\{ v \in V \mid (z - \chi(z))^N v = 0, z \in Z \right\}.$$

Let $\mathcal{B}$ be the category of all finitely generated $\mathfrak{g}$-modules with finite Z-action. Then $\mathcal{B}$ has a decomposition

$$\mathcal{B} = \oplus \mathcal{B}^{\chi},$$

where $\mathcal{B}^{\chi}$ denotes the subcategory of all $V \in \mathcal{B}$ such that $V^{\chi} = V$.

Let E be a finite-dimensional $\mathfrak{g}$-module. A *translation functor* T_E^{χ} is a functor in the category $\mathcal{B}$, defined by

$$T_E^{\chi}(V) = (V \otimes E)^{\chi}.$$

To simplify the notation we also will write T_E^{λ} instead of $T_E^{\chi_\lambda}$.

Lemma 8.1. *T_E^{χ} is an exact functor.*

Proof. Both tensoring with finite-dimensional vector space and the projection on the component with a given central character are obviously exact functors. □

Denote by $P(E)$ the set of all weights of E counted with multiplicities.

Lemma 8.2. (1) *For the Verma module M_λ, $M_\lambda \otimes E$ has a finite filtration $\{0\} = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_q = (M_\lambda \otimes E)$ of length $q = \dim E$ such that $\mathcal{F}_{i+1}/\mathcal{F}_i$ is ismorphic to $M_{\lambda+\nu}$, $\nu \in P(E)$;*

(2) *If V is a module generated by a highest vector of weight λ, then $T_E^{\chi}(V)$ has a finite filtration $\{0\} = \mathcal{V}_0 \subset \cdots \subset \mathcal{V}_r = T_E^{\chi}(V)$ such that $\mathcal{V}_i/\mathcal{V}_{i+1}$ is a highest weight module of weight $\lambda + \nu \in \mathfrak{h}_{\chi}$ for some $\nu \in P(E)$.*

Proof. The first statement can be found in [6]. The second one follows from the first and Lemma 8.1.

□

Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, V be a $\mathfrak{g}$-module. A vector $v \in V$ is $\mathfrak{b}$-*primitive* if $\mathfrak{b}v \in \mathbb{C}v$.

Lemma 8.3. *If v is a $\mathfrak{b}$-primitive vector of $(L_\lambda \otimes E)$ then the weight of v equals $\lambda + \nu$ for some $\nu \in P(E)$.*

Proof. Introduce the order on $\mathfrak{h}^*$ by putting $\mu \leq \nu$ if $\nu = \mu + \Sigma n_\alpha \alpha$ for some $\alpha \in \Delta^+$ and $n_\alpha \in \mathbb{Z}_{\geq 0}$. Choose a maximal weight γ of L_λ such that

$$v = v_1 \otimes w_1 + \cdots + v_r \otimes w_r + v'_1 \otimes w'_1 + \cdots + v'_t \otimes w'_t$$

for some linearly independent $v_1, \ldots, v_r \in L_\lambda$ of weight γ, linearly independent weight vectors $w_1, \ldots, w_r \in E$ and some linearly independent weight vectors $v'_1, \ldots, v'_t \in L_\lambda$ of weights different from γ, $w'_1, \ldots, w'_t \in E$. For any simple root element $e \in \mathfrak{n}$ the condition $ev = 0$ implies

$$ev_1 \otimes w_1 + \cdots + ev_r \otimes w_r = 0.$$

Since $w_1, \ldots, w_r$ are linearly independent, we must have $ev_i = 0$. Therefore all v_i are $\mathfrak{b}$-primitive. But L_λ has a unique up to proportionality $\mathfrak{b}$-primitive vector. Therefore $\gamma = \lambda$, $r = 1$ and the weight of v is the sum of λ and the weight of w_1. $\square$

For any $\lambda \in \mathfrak{h}^*$ put

$$\mathfrak{h}_\lambda = \mathfrak{h}_{\chi_\lambda},\ \Sigma_\lambda = \mathfrak{h}_\lambda \cap \Sigma.$$

Lemma 8.4. *Let $\lambda, \mu \in \Sigma$ satisfy the conditions*

$$(\lambda + P(E)) \cap \Sigma_\mu = \{\mu\} \tag{8.1}$$

$$(\mu - P(E)) \cap \Sigma_\lambda = \{\lambda\} \tag{8.2}$$

and λ is minimal in $(\mu - P(E)) \cap \mathfrak{h}_\lambda$. Then

$$T_E^\mu(L_\lambda) = L_\mu,\ T_{E^*}^\lambda(L_\mu) = L_\lambda.$$

Proof. By Lemma 8.2 (2) and (8.1) $T_E^\mu(L_\lambda)$ is a highest weight module with highest weight μ. By Lemma 8.3 and (8.1) $T_E^\mu(L_\lambda)$ has a unique up to proportionality $\mathfrak{b}$-primitive vector. Therefore either $T_E^\mu(L_\lambda) = L_\mu$ or $T_E^\mu(L_\lambda) = \{0\}$. In the same way either $T_E^\lambda(L_\mu) = L_\lambda$ or $T_E^\lambda(L_\mu) = \{0\}$.

Our next observation is

$$\mathrm{Hom}_{\mathfrak{g}}(M \otimes E^*, N) \cong \mathrm{Hom}_{\mathfrak{g}}(M, N \otimes E), \tag{8.3}$$

hence, in particular

$$\mathrm{Hom}_{\mathfrak{g}}\left(T_{E^*}^\lambda(L_\mu), L_\lambda\right) \cong \mathrm{Hom}_{\mathfrak{g}}\left(L_\mu, T_E^\mu(L_\lambda)\right). \tag{8.4}$$

Therefore $T_E^\mu(L_\lambda) = \{0\}$ iff $T_E^\mu(L_\mu) = \{0\}$. Let us prove that $T_E^\mu(L_\lambda) \neq \{0\}$. Note that by Lemma 8.2 (1) and (8.2), $T_{E^*}^\lambda(M_\mu)$ has a subquotient isomorphic to M_λ.

Moreover, since λ is a minimal weight in $(\mu - P(E)) \cap \mathfrak{h}_\lambda$, there is a quotient in $T^{\lambda}_{E^*}(M_\mu)$ isomorphic to M_λ, hence there is a quotient isomorphic to L_λ. Therefore

$$\operatorname{Hom}_{\mathfrak{g}}\left(T^{\lambda}_{E^*}(M_\mu), L_\lambda\right) \neq \{0\}.$$

But then using (8.3)

$$\operatorname{Hom}_{\mathfrak{g}}\left(M_\mu, T^{\mu}_{E}(L_\lambda)\right) \cong \operatorname{Hom}_{\mathfrak{g}}\left(T^{\lambda}_{E^*}(M_\mu), L_\lambda\right) \neq 0.$$

Therefore $T^{\mu}_{E}(L_\lambda) \neq \{0\}$. Finally by (8.4) $T^{\lambda}_{E^*}(L_\mu) \neq \{0\}$. ☐

Lemma 8.5. *Let M be a finite-dimensional $\mathfrak{g}$-module and $N = T^{\chi}_{E}(M)$. Then $X_N \subset X_M$.*

Proof. Let $x \in X \backslash X_M$, then M is free over $\mathbb{C}[x]$, and $M \otimes E$ is also free over $\mathbb{C}[x]$. Since N is a direct summand of $M \otimes E$, then N is free over $\mathbb{C}[x]$. That implies $x \notin X_N$. ☐

9. Reduction to the stable case

Fix a set of simple roots and the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ generated by $\mathfrak{h}$ and simple roots. We say that a subalgebra $\mathfrak{q}$ is *admissible* if $\mathfrak{q}$ is generated by $\mathfrak{h}$, some subset of simple roots and their negatives. By $\Delta(\mathfrak{q})$ we denote the root system of $\mathfrak{q}$. We call λ *stable* with respect to $\mathfrak{q}$ if the following conditions hold for any isotropic $\alpha \in \Delta$, $(\lambda + \rho, \alpha) = 0$ implies $\alpha \in \Delta(\mathfrak{q})$.

In this section we assume that $\mathfrak{g} = \mathfrak{gl}(m|n)$. Then

$$\Delta_0 = \{\varepsilon_i - \varepsilon_j \mid i, j \leq m\} \cup \{\delta_i - \delta_j \mid i, j \leq n\}, \; \Delta_1 = \{\pm(\varepsilon_i - \delta_j) \mid i \leq m, j \leq n\}.$$

All odd roots are isotropic. The choice of the form on $\mathfrak{h}^*$ is such that $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $(\delta_i, \delta_j) = -\delta_{ij}$. The defect $d = \min(m, n)$. We choose a Borel subalgebra $\mathfrak{b}$ so that the simple roots are

$$\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m - \delta_1, \dots, \delta_{n-1} - \delta_n.$$

If $\lambda + \rho = a_1\varepsilon_1 + \dots + a_m\varepsilon_m + b_1\delta_1 + \dots b_n\delta_n$, then $\lambda \in \Sigma$ iff $a_i - a_{i+1}, b_j - b_{j+1} \in \mathbb{Z}_{>0}$ for all $i < m, j < n$. In other words, $\lambda \in \Sigma$ iff $\langle \lambda + \rho, \gamma^\vee \rangle \in \mathbb{Z}_{>0}$ for all $\gamma \in \Delta_0^+$. Since we consider only atypical λ we may assume that $a_i, b_j \in \mathbb{Z}$.

Lemma 9.1. *Let $\lambda + \rho = a_1\varepsilon_1 + \dots + a_m\varepsilon_m + b_1\delta_1 + \dots b_n\delta_n \in \Sigma$. If $a_i + b_j = 0$ implies $i > m-k$, then λ is stable for $\mathfrak{q}$ with simple roots $\varepsilon_{m-k+1} - \varepsilon_{m-k+2}, \dots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m - \delta_1, \dots, \delta_{n-1} - \delta_n$.*

Proof. Trivial. ☐

Theorem 9.2. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$. If $\lambda \in \Sigma$ and has the degree of atypicality k, then there exists a subalgebra $\mathfrak{q} \subset \mathfrak{g}$ of defect k, a stable $\mu \in \Sigma$ of degree atypicality k and translation functors $T_1, \dots, T_r$, $T_1^*, \dots, T_r^*$ such that*

$$L_\mu = T_1 \dots T_r(L_\lambda), \; L_\lambda = T_r^* \dots T_1^*(L_\mu).$$

Proof. Translation functors which we use are always related with E being the standard representation or its dual. We will provide a combinatorial algorithm, which constructs from a weight $\lambda \in \Sigma$ a new weight $\mu \in \Sigma_\lambda$ in such way that λ and μ satisfy the conditions of Lemma 8.4 and therefore $T_E^\mu(L(\lambda)) = L(\mu)$, $T_{E^*}^\lambda(L(\mu)) = L(\lambda)$. Applying this algorithm several times we obtain a sequence of weights $\mu_1, \dots, \mu_r$ such that μ_r is stable. Let $\lambda + \rho = a_1\varepsilon_1 + \dots + a_m\varepsilon_m + b_1\delta_1 + \dots b_n\delta_n$. Let g be maximal such that $a_i + b_j \neq 0$ for any $i \leq g$, $j \leq n$. If $g = m - k$, then λ is stable as in Lemma 9.1 and we can stop to apply the algorithm. Otherwise choose first $i > g$ such that $a_i + b_j \neq 0$ for all $j \leq n$. Construct μ depending on the following

(1) If $b_j \neq -a_i - 1$ for any $j \leq n$, then put $\mu = \lambda + \varepsilon_i$;
(2) If $b_j = -a_i - 1$ for some j look at a_{i-1}. If $a_{i-1} = a_i + 1$, put $\mu = \lambda + \delta_j$. Otherwise go to the next step;
(3) If $b_j = -a_i - 1$, $a_{i-1} \neq a_i + 1$ find the maximal p such that $b_{j+p} = b_j - p$. If $a_{i-1} + b_{j+p} > 0$, put $\mu = \lambda - \delta_{j+p}$. Otherwise go to the next step.
(4) If $a_{i-1} + b_{j+p} \leq 0$, then there exists $t \leq p$ such that $a_{i-1} + b_{j+t} = 0$. Put $\mu = \lambda - \varepsilon_{i-1}$.

Note that at some point one always arrives to the case 2, that decreases i and eventually increases g. In the end one will come to the stable weight. □

Theorem 9.2 and Lemma 8.5 imply

Theorem 9.3. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$. For any $\lambda \in \Sigma$ there exists a stable $\mu \in \Sigma$ with the same degree of atypicality such that $X_{L_\lambda} = X_{L_\mu}$.*

10. Proof of Theorem 5.4 for $\mathfrak{gl}(m|n)$

In this section $\mathfrak{g} = \mathfrak{gl}(m|n)$ and λ is an integral dominant weight with degree of atypicality k. As Theorem 5.3 is already proven we have to show only that if $\mathrm{A} \in S$, $|\mathrm{A}| = k$, then $(L_\lambda)_x \neq \{0\}$ for any $x \in \Phi(\mathrm{A})$. As follows from Theorem 9.3, we may assume that λ is stable with respect to $\mathfrak{q} = \mathfrak{gl}(k|n)$. It is easy to check that $\Phi(\mathrm{A}) \cap \mathfrak{q} \neq \varnothing$, and therefore one may assume that $x \in \mathfrak{q}$. On the other hand, $L_\lambda = L_\lambda(\mathfrak{q}) \oplus N$ as a module over $\mathfrak{q}$. Thus, it is sufficient to prove that $(L_\lambda(\mathfrak{q}))_x \neq \{0\}$. In other words, we reduce the theorem to the case of $\mathfrak{gl}(k|n)$. Using the isomorphism $\mathfrak{gl}(k|n) \cong \mathfrak{gl}(n|k)$ we can repeat the above argument and reduce the theorem to the case $\mathfrak{g} = \mathfrak{gl}(k|k)$. Summing up, Theorem 5.4 is equivalent to the following Lemma.

Lemma 10.1. *Let $\mathfrak{g} = \mathfrak{gl}(k|k)$ and λ be an integral dominant weight with degree of atypicality k. Then $(L_\lambda)_x \neq \{0\}$ for any $x \in X$.*

We prove Lemma 10.1 in several steps. We use the fact that $\mathfrak{g}$ has the $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ and $\mathfrak{g}(-1), \mathfrak{g}(1)$ are irreducible components of X. We have

$k+1$ open orbits on X. Choose a representative x on each orbit in the following way

$$\begin{matrix} 0 & x^+ \\ x^- & 0 \end{matrix}$$

where x^+ is the block matrix

$$\begin{matrix} 1_p & 0 \\ 0 & 0 \end{matrix}$$

and x^- is the block matrix

$$\begin{matrix} 0 & 0 \\ 0 & 1_q \end{matrix}$$

here $p+q=k$.

If $x \in \mathfrak{g}(1)$, then $x^- = 0$, if $x \in \mathfrak{g}(-1)$, then $x^+ = 0$. In both cases the stabilizer K of x in G_0 is isomorphic to $\mathrm{GL}(k)$ embedded diagonally in $G_0 = \mathrm{GL}(k) \times \mathrm{GL}(k)$. By $\mathfrak{k}$ we denote the Lie algebra of K.

Lemma 10.2. *If $x \in \mathfrak{g}(\pm 1)$ and M is a finite-dimensional $\mathfrak{g}$-module, then M_x is a trivial K-module.*

Proof. Follows from the fact $C_{\mathfrak{g}}(x) = [x, \mathfrak{g}]$ and Lemma 6.2. □

Lemma 10.3. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{b}$ is the Borel subalgebra containing $\mathfrak{g}(1)$, $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_\lambda$ be the Verma module. If α is a negative isotropic root such that $(\lambda+\rho, \alpha) = 0$, then M_λ contains a $\mathfrak{b}$-primitive vector of weight $\lambda + \alpha$.*

Proof. Let I_α be the set of all weights $\lambda \in \mathfrak{h}^*$ such that M_λ has a $\mathfrak{b}$-primitive vector of weight $\lambda + \alpha$. Then I_α is Zariski closed, see for example [2]. Let

$$H_\alpha = \{\lambda \in \mathfrak{h}^* \mid (\lambda+\rho, \alpha) = 0\}.$$

We want to show that $H_\alpha \subset I_\alpha$. Consider

$$H'_\alpha = \{\lambda \in H_\alpha \mid (\lambda+\rho, \beta) \neq 0, \beta \neq \pm\alpha, \beta \in \Delta\mathfrak{g}_1\}.$$

It suffices to show that $H'_\alpha \subset I_\alpha$. Consider $\mathfrak{b}' = \mathfrak{b}_0 + \mathfrak{g}(-1)$. If v is a highest vector of M_λ and $X_\beta \in \mathfrak{g}_\beta$, then $w = \Pi_{\beta \in \Delta(\mathfrak{g}(-1))} X_\beta v$ is a $\mathfrak{b}'$-primitive, and $u = \Pi_{\beta \in \Delta(\mathfrak{g}(1)) \setminus \{\alpha\}} X_\beta w$ is a $\mathfrak{b}$-primitive. Since the weight of u equals $\lambda + \alpha$, we obtain $H'_\alpha \subset I_\alpha$ as required. □

The $\mathbb{Z}$-grading on $\mathfrak{g}$ induces the $\mathbb{Z}$-grading on an irreducible $\mathfrak{g}$-module $M = M(0) \oplus M(-1) \oplus \cdots \oplus M(-k^2)$ in the following way

$$M(0) = \operatorname{Ker} \mathfrak{g}(1), \; M_i = \mathfrak{g}(-1) M(i+1).$$

Each $M(i)$ is a $\mathfrak{g}_0$-submodule of M.

Lemma 10.4. *Let $x \in \mathfrak{g}(\pm 1)$, $M = L_\lambda$ for a dominant integral λ of degree atypicality k. Then $M(0)$ contains one trivial K-submodule and $M(-1)$ does not have trivial K_x-submodules.*

Proof. Since the degree of atypicality of λ is k, one can write

$$\lambda = a_1\varepsilon_1 + \cdots + a_k\varepsilon_k - a_k\delta_1 - \cdots - a_1\delta_1.$$

We denote by $V(a_1, \ldots, a_k)$ the irreducible $\mathfrak{gl}(k)$-module with highest weight $(a_1, \ldots, a_k)$ and by $L_\lambda(\mathfrak{g}_0)$ the irreducible $\mathfrak{g}_0$-module with highest weight λ. Since $M(0)$ is isomorphic to $L_\lambda(\mathfrak{g}_0)$, then

$$M(0) \cong V(a_1, \ldots, a_k) \otimes V^*(a_1, \ldots, a_k)$$

as K-module, which has exactly one trivial component. Hence the first statement is true.

Obviously $M(-1)$ is a submodule in

$$L_\lambda \otimes \mathfrak{g}(-1) \subset \oplus_{\alpha \in \Delta(\mathfrak{g}(-1))} L_{\lambda+\alpha}(\mathfrak{g}_0).$$

However, $(\lambda + \rho, \varepsilon_i - \delta_{k+1-i}) = 0$, therefore by Lemma 10.3 $M(-1)$ does not contain the component $L_{\lambda+\delta_{k+1-i}-\varepsilon_i}(\mathfrak{g}_0)$ for all $i = 1, \ldots, k$. Hence $M(-1)$ is a K-submodule of the K-module

$$\oplus_{i \neq j} V(a_1, \ldots, a_i - 1, \ldots, a_k) \otimes V^*(a_1, \ldots, a_j - 1, \ldots, a_k).$$

Therefore $M(-1)$ does not contain K-trivial submodules. $\square$

Lemma 10.5. *Let $x \in \mathfrak{g}(\pm 1)$ belong to an open G_0-orbit, $M = L_\lambda$ for a dominant integral λ of degree of atypicality k and N be a trivial K-submodule in $M(0)$. Then $N \subset M_x$ and therefore $M_x \neq \{0\}$.*

Proof. If $x \in \mathfrak{g}(1)$, then $xN = 0$. Since $x : M(-1) \to M(0)$ is a homomorphism of $\mathfrak{g}_x$-modules and $M(-1)$ does not have trivial $\mathfrak{g}_x$-submodules, then N does not belong to $\operatorname{Im} x$. If $x \in \mathfrak{g}(-1)$, then N clearly is not in $\operatorname{Im} x$. Since $x : M(0) \to M(-1)$ is a homomorphism of $\mathfrak{g}_x$-modules and $M(-1)$ does not contain trivial $\mathfrak{g}_x$-submodules, then $xN = 0$. $\square$

Lemma 10.5 shows that $M_x \neq \{0\}$ in two special cases: $x \in \mathfrak{g}(1)$ or $x \in \mathfrak{g}(-1)$. Now we will show the same for each open G_0-orbit on C. Let y_c be an odd element in $\mathfrak{g}$ given by

$$\begin{matrix} 0 & y_c^+ \\ y_c^- & 0 \end{matrix}$$

where y_c^+ is the block matrix

$$\begin{matrix} 1_p & 0 \\ 0 & c1_q \end{matrix}$$

and y_c^- is the block matrix

$$\begin{matrix} c1_p & 0 \\ 0 & 1_q \end{matrix}$$

here $p+q=k$, $c \in \mathbb{C}$. Note that $y_c \notin X$ if $c \neq 0$, but $[y_c, y_c]$ lies in the center of $\mathfrak{g}$. If $M = L_\lambda$ has the degree atypicality k, then the center of $\mathfrak{g}$ acts by zero on M. Hence $M_{y_c} = \operatorname{Ker} y_c / \operatorname{Im} y_c$ is well defined. Lemma 10.5 implies $M_{y_1} \neq \{0\}$. If $c \neq 0$, then there exists $g \in G_0$ such that $y_c = c^{1/2} \operatorname{Ad}_g (y_1)$. Therefore $M_{y_c} \neq \{0\}$ for any $c \neq 0$. The continuity argument shows that $M_{y_0} \neq \{0\}$. But $y_0 \in X$ is an element on an open orbit. Therefore Lemma 10.1 and Theorem 5.4 are proven.

11. Application to $H(\mathfrak{g}(-1); M)$ for $\mathfrak{gl}(m|n)$

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, then $\mathfrak{g}(-1)$ is an abelian subalgebra and the cohomology $H(\mathfrak{g}(-1); M)$ determine the character of a finite-dimensional module M. On the other hand, $\mathfrak{g}(-1)$ is an irreducible component of X. The complex calculating $H(\mathfrak{g}(-1); M)$ is

$$\partial : \mathcal{O}(\mathfrak{g}(-1)) \otimes M \to \mathcal{O}(\mathfrak{g}(-1)) \otimes M,$$

where ∂ is the same as for the sheaf $\mathcal{M}$. One can consider the localization of this complex and the corresponding coherent sheaf $\mathcal{H}_M$ is the restriction of $\mathcal{M}$ on $\mathfrak{g}(-1)$.

Theorem 5.4 and Theorem 6.12 imply the following

Theorem 11.1. *Let M be an irreducible finite-dimensional module with central character χ and the degree of atypicality of χ equal k. Then* $\operatorname{supp} \mathcal{H}_M = \bar{X}_k \cap \mathfrak{g}(-1)$.

Lemma 11.2. *Let M be a typical finite-dimensional module. Then* $\operatorname{supp} \mathcal{H}_M = \{0\}$ *and $\mathcal{H}_M(0) = H^0(\mathfrak{g}(-1), M)$.*

Proof. Since M is typical, then M is a free $\mathfrak{g}(-1)$ module and $H^i(\mathfrak{g}(-1), M) = 0$ for $i > 0$. □

Theorem 11.3. *Let $x \in X_k \cap \mathfrak{g}(-1)$, $M = L_\lambda$, the degree of atypicality of λ be k, and $Z = G_0 x$. Then $\mathcal{H}_M(Z)$ is the sheaf of section of the G_0-vector bundle inuced by $(\mathfrak{g}_x)_0$-module $H^0(\mathfrak{g}_x \cap \mathfrak{g}(-1); M_x)$.*

Proof. Follows from Lemma 11.2, Theorem 6.12 and Theorem 6.10. □

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