TENSOR PRODUCT EXERCISES
MATH 252

1. A tensor product $V \otimes W$ of two vector spaces V and W is a vector space equipped with bilinear map $f : V \times W \to V \otimes W$ such that for any linear map $\beta : V \times W \to U$ there exists a linear map $\phi : V \otimes W \to U$ such that $\beta = \phi \circ f$. The image $f(v, w)$ is denoted by $v \otimes w$.

(a) Prove existence and uniqueness (up to isomorphism)of tensor product.

(b) Show that dim $V \otimes W$ =dim$V$dim$W$.

2. Consider the natural map

$$\varphi : V^* \otimes W \to \mathrm{Hom}_k(V, W)$$

given by

$$\varphi(\alpha \otimes w)(v) = \langle \alpha, v \rangle w$$

for any $\alpha \in V^*$, $v \in V$ and $w \in W$. Show that φ is injective. Show that if V is finite-dimensional then φ is an isomorphism. Describe the image of φ for arbitrary V.

3. Construct a canonical (independent on a choice of basis) isomorphism

$$(V \otimes W) \otimes U \simeq V \otimes (W \otimes U).$$

4. Let X be a linear operator in V and Y be a linear operator in W. Define $X \otimes Y : V \otimes W \to V \otimes W$ by

$$X \otimes Y(v \otimes w) = Xv \otimes Yw.$$

Show that

$$\mathrm{tr}(X \otimes Y) = \mathrm{tr}(X)\,\mathrm{tr}(Y).$$

5. Let $V^{\otimes n}$ denote the tensor product of n copies of V and let $T(V) = \oplus_{n=0}^{\infty} V^{\otimes n}$. Define the associative multiplication on $T(V)$ via tensor product. $T(V)$ is called tensor algebra. If $\{v_i\}$ is a basis of V, then $T(V)$ is a free associative algebra with generators $\{v_i\}$.

6. (Symmetric algebra.) Let $S(V)$ be the quotient of $T(V)$ by the ideal I generated by $v \otimes w - w \otimes v$ for all $v, w \in V$.

(a) Show that $S(V) = \oplus_{n=0}^{\infty} S^n(V)$, where $S^n(V) = V^{\otimes n}/(I \cap V^{\otimes n})$, is a graded commutative algebra isomorphic to the polynomial algebra in d variables where $d = \dim V$. Find $\dim S^n(V)$ as a function of d.

(b) Assume that the ground field has characteristic 0. Show that the symmetrization map $\mathrm{Sym} : V^{\otimes n} \to V^{\otimes n}$ defined by

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$$\mathrm{Sym}(v_1 \otimes \cdots \otimes v_n) = \frac{1}{n!} \sum_{s \in S_n} v_{s(1)} \otimes \cdots \otimes v_{s(n)}$$

is a projector and the image of Sym is isomorphic to $S^n(V)$. Moreover, $\mathrm{Ker}(\mathrm{Sym}) = I \cap V^{\otimes n}$.

7. (Exterior algebra.) Assume that the characteristic of the ground field is not equal to 2. Let $\Lambda(V)$ be the quotient of $T(V)$ by the ideal J generated by $v \otimes w + w \otimes v$ for all $v, w \in V$.

(a) Show that $\Lambda(V) = \oplus_{n=0}^{d} \Lambda^n(V)$, where $\Lambda^n(V) = V^{\otimes n}/(J \cap V^{\otimes n})$, is a graded algebra. Find $\dim \Lambda^n(V)$ as a function of d.

(b) Assume that the ground field has characteristic 0. Show that the map $\mathrm{Alt} : V^{\otimes n} \to V^{\otimes n}$ defined by

$$\mathrm{Alt}(v_1 \otimes \cdots \otimes v_n) = \frac{1}{n!} \sum_{s \in S_n} \mathrm{sgn}(s) v_{s(1)} \otimes \cdots \otimes v_{s(n)}$$

is a projector and the image of Alt is isomorphic to $\Lambda^n(V)$. Moreover, $\mathrm{Ker}(\mathrm{Alt}) = J \cap V^{\otimes n}$.

8. A linear operator $X \in \mathrm{End}_k(V)$ induces linear operators in $V^{\otimes n}$, $S^n(V)$ and $\Lambda^n(V)$. Let

$$\det(X - t\,\mathrm{id}) = a_0 + a_1 t + \cdots + (-1)^d t^d$$

be the characteristic polynomial of X. Show that $(-1)^{d-k} a_k$ equals the trace of the corresponding linear operator in $\Lambda^k(V)$.