FINAL EXAM
MATH 252

Please submit by email not later than December 15. Problems marked by * are harder. Do just one of them.

1. Let p be a prime number, $\mathbb{F}_p$ denote the field with p elements and G be the subgroup of $GL_3(\mathbb{F}_p)$ consisting of matrices of the form

$$\left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \quad | \quad a, b, c \in \mathbb{F}_p \right\}.$$

Classify irreducible representations of G over $\mathbb{C}$ and compute the character table.

2. Let A be a unital ring which has one up to isomorphism simple module.

(a) Prove that A has a unique proper maximal two-sided ideal $\mathfrak{m}$.

(b) Assume that A is artinian. Show that $A/\mathfrak{m}$ is isomorphic to the matrix algebra over some division ring.

(c*) Is (b) true without assumption that A is artinian?[1]

3. Let G be a compact group and V be some finite-dimensional representation of G.

(a) Let χ_n denote the character of $S^n(V)$. Show that for any $g \in G$

$$\frac{1}{\det(\mathrm{Id} - tg)} = \sum_{n=0}^{\infty} \chi_n(g) t^n.$$

(b) Let u_n denote the number of G-invariant homogeneous polynomials of degree n on V and dg denotes the Haar measure on G. Prove that

$$\int_G \frac{dg}{\det(\mathrm{Id} - tg)} = \sum_{n=0}^{\infty} u_n t^n.$$

4. Let V and W be finite-dimensional vector spaces over $\mathbb{C}$. If λ is a partition we denote $S_\lambda(V)$ (resp. $S_\lambda(W)$) the corresponding irreducible representation of $GL(V)$ (resp. of $GL(W)$).

(a) Show that $S_\lambda(V) \boxtimes S_\mu(W)$ is an irreducible representation of $GL(V) \times GL(W)$.

(b) Prove the identities

$$S^n(V \boxtimes W) = \bigoplus_{|\lambda|=n, r(\lambda) \leq \min(\dim V, \dim W)} S_\lambda(V) \boxtimes S_\lambda(W),$$

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[1] I do not know the answer, so skip this problem unless you have an idea how to approach it.

$$\Lambda^n(V \boxtimes W) = \bigoplus_{|\lambda|=n, r(\lambda)\le \dim V, r(\lambda^\perp)\le \dim W} S_\lambda(V) \boxtimes S_{\lambda^\perp}(W),$$

where $|\lambda|$ is the number of boxes in the Young diagram λ, $r(\lambda)$ is the number of rows and $\lambda^\perp$ denotes the conjugate partition.

Hint: Consider the action of $S_n \times S_n$ on $V^{\otimes n} \otimes W^{\otimes n}$ and use the Schur–Weyl duality to find the invariants of the diagonal subgroup $S_n \subset S_n \times S_n$.

5*. For two Young diagrams λ and μ we say $\mu \subset \lambda$ if μ is included in λ or equivalently if $\mu_i \le \lambda_i$ for all i. By λ' we denote the diagram obtained from λ by removing the first row.

(a) Let V be an n-dimensional vector space over $\mathbb{C}$, $W \subset V$ be a subspace of codimension one. Fix a decomposition $V = W \oplus \mathbb{C}$ and consider the corresponding embedding of the groups $GL(W) \subset GL(V)$. Prove that

$$\operatorname{Res}_{GL(W)} S_\lambda(V) = \bigoplus_{\lambda'\subset\mu\subset\lambda, r(\mu)\le n-1} S_\mu(W).$$

Hint: Consider the homomorphism between the rings of characters of $GL(V)$ and $GL(W)$ induced by the restriction functor and use the Jacobi–Trudi identity.

(b) Use the chain of subgroups $GL_1(\mathbb{C}) \subset GL_2(\mathbb{C}) \subset \cdots \subset GL_n(\mathbb{C}) = GL(V)$ and part (a) to construct a basis in the representation $S_\lambda(V)$ enumerated by all semi-standard tableaux of shape λ with entries $1, \ldots, n$. (This basis is called the Gelfand–Tsetlin basis).

6*. (Zelevinsky) Let $G_n := GL_n(\mathbb{F}_q)$. Let $\mathcal{H}_n$ denote the $\mathbb{Z}$-span of complex irreducible characters of G_n, $\mathcal{H}_0 := \mathbb{Z}$ and

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

We define the scalar product on $\mathcal{H}$ by setting

$$(\chi, \psi) = \delta_{m,n}(\chi, \psi)_{G_n} \quad \text{for all} \quad \chi \in \mathcal{H}_m, \psi \in \mathcal{H}_n.$$

(a) For every $r, s \in \mathbb{N}$ such $n = r + s$ denote by $P_{r,s}$ the subgroup of matrices of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ such that $A \in G_r, C \in G_s$ and B is the arbitrary $r \times s$ matrix. Show that $U_{r,s}$ consisting of matrices such that $A = \mathrm{Id}_r$ and $C = \mathrm{Id}_s$ is a normal subgroup of $P_{r,s}$ and the quotient $P_{r,s}/U_{r,s}$ is isomorphic to $G_r \times G_s$.

(b) Describe the double cosets $P_{r,s}\backslash G_n / P_{r',s'}$.

(c) For any representation ρ of $G_r \times G_s$ denote by $\tilde\rho$ the representation of $P_{r,s}$ obtained by pull back of ρ under the natural projection $P_{r,s} \to G_r \times G_s$. Let

$$I_{r,s}(\rho) := \operatorname{Ind}_{P_{r,s}}^{G_n} \tilde\rho.$$

For any representation τ of G_n and $r + s = n$ set

$$R_{r,s}(\tau) := \operatorname{Res}_{G_r\times G_s} \operatorname{Hom}_{U_{r,s}}(\mathrm{triv}, \tau).$$

By abuse of notation denote by $I_{r,s}$ and $R_{r,s}$ the corresponding $\mathbb{Z}$-linear maps

$$\mathcal{H}_r \otimes \mathcal{H}_s \to \mathcal{H}_n, \quad \mathcal{H}_n \to \mathcal{H}_r \otimes \mathcal{H}_s.$$

Next define

$$\mu : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}, \mu^* : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$$

by

$$\mu(\chi, \psi) := I_{r,s}(\chi \otimes \psi) \quad \text{for all} \quad \chi \in \mathcal{H}_r, \psi \in \mathcal{H}_s,$$

and

$$\mu^*(\varphi) := \sum_{r+s=n} R_{r,s}(\varphi) \quad \text{for all} \quad \varphi \in \mathcal{H}_n.$$

Check that $\mathcal{H}$ is a PSH algebra. (The most difficult part is to check that μ^* is a homomorphism of algebras.)

7. Classify indecomposable representations of the quiver

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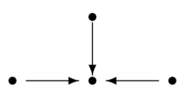

8. List typos and mistakes you noticed in the lecture notes.