A sentimental journey through representation theory: from finite groups to quivers (via algebras)

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Preface

Representation theory is a very active research topic in mathematics nowadays.

There are representations associated to several algebraic structures, representations of algebras, groups (of finite or infinite cardinal). Roughly speaking, a representation is a vector space equipped with a linear action of the algebraic structure. For example, the algebra of $n \times n$ matrices acts on the vector space $\mathbb{C}^n$. A slightly more complicated example is the action of the group $GL(n, \mathbb{C})$ in the set of $n \times n$-matrices, the group acting by conjugation.

In the beginning, there was no tendency to classify all the representations of a given object. The first result in this direction is due to Frobenius, who was interested in the general theory of finite groups. Let G be a finite group, a representation V of G is a complex vector space V together with a morphism of groups $\rho : G \to GL(V)$. One says V is irreducible if there exists no proper subspace $W \subset V$ such that W is stable under all $\rho(g), g \in G$. Frobenius showed there is finitely many irreducible representations of G and that they are completely determined by their characters: the character of V is the complex function $g \in G \mapsto Tr(\rho(g))$ where Tr is the trace of the endomorphism. These characters form a basis of the complex valued functions on G invariant under conjugation. Then Frobenius proceeded to compute the characters of symmetric groups in general. His results inspired Schur, who was able to relate them to the theory of complex finite dimensional representations of $GL(n, \mathbb{C})$ through the Schur-Weyl duality. In both cases, every finite dimensional representation of the group is a direct sum of irreducible representations (we say that the representations are completely reducible).

Most of the results about representations of finite groups can be generalized to compact groups. In particular, once more, the complex finite dimensional representations of a compact groups are completely reducible, and the regular representation in the space of continuous functions on the compact group has the similar structure. This theory was developed by H. Weyl and the original motivation came from quantum mechanics. The first examples of compact groups are the group $SO(2)$ of rotations of the plane (the circle) and the group $SO(3)$ of rotations of the 3-dimensional space. In the former case, the problem of computing the Fourier series for a function on the circle is equivalent to the decomposition of the regular representation. More generally, the study of complex representations of compact groups helps to understand Fourier analysis on such groups.

If a topological group is not compact, for example, the group of real numbers with operation of addition, the representation theory of such a group involves more complicated analysis (Fourier transform instead of Fourier series). The representation theory of real non-compact groups was initiated by Harish-Chandra and by the Russian school leaded by Gelfand. Here emphasis is on the classification of unitary representations due to applications from physics. It is also worth mentioning that this theory is closely related to harmonic analysis, and many special functions (such as Legendre polynomials) naturally appear in the context of representation theory.

In the theory of finite groups one can drop the assumption that the characteristic of the ground field is zero. This leads immediately to the loss of complete reducibility. This representation theory was initiated by Brauer and it is more algebraic. If one turns to algebras, a representation of an algebra is, by definition, the same as a module over this algebra. Let k be a field. Let A be a k-algebra which is finite dimensional as a vector space. It is a well-known fact that A-modules are not, in general, completely reducible: for instance, if $A = k[X]/X^2$ and $M = A$, the module M contains kX as a submodule which has no A-stable complement. An indecomposable A-module is a module which has no non-trivial decomposition as a direct sum. It is also interesting to attempt a classification of A-modules. It is a very difficult task in general. Nevertheless, the irreduducible A-modules are in finite number. The radical R of A is defined as the ideal of A which annihilates each of those irreducible modules, it is a nilpotent ideal. Assume k is algebraically closed, the quotient ring A/R is a product of matrix algebras over k, $A/R = \Pi_i End_k(S_i)$ where S_i runs along the irreducible A-modules.

If G is a finite group, the algebra $k(G)$ of k-valued functions on G, the composition law being the convolution, is a finite dimensional k-algebra, with a zero radical as long as the characteristic of the field k does not divide the cardinal of G. The irreducible modules of $k(G)$ are exactly the finite dimensional representations of the group G, the action of G extends linearly to $k(G)$. This shows that all $k(G)$-modules are completely reducible (Maschke's theorem).

In order to study finite dimensional k-algebras representations more generally, it is useful to introduce quivers. Let A be a finite dimensional k-algebra, denote $S_1, \ldots, S_n$ its irreducible representations, and draw the following graph, called the quiver associated to A: the vertices are labelled by the S_is and we put l arrows between S_i and S_j, pointing at S_j, if $Ext^1(S_i, S_j)$ is of dimension l (the explicit definition of Ext^1 requires some homological algebra which is difficult to summarize in such a short introduction).

More generally, a quiver is an oriented graph with any number of vertices. Let Q be a quiver, a representation of Q is a set of vector spaces indexed by the vertices of Q together with linear maps associated to the arrows of Q. Those objects were first systematically used by Gabriel in the early 70's, and studied by a lot of people ever since. The aim is to characterize the finitely represented algebras, or in other terms the algebras with a finite number of indecomposable modules (up to isomorphism).

Today the representation theory has many flavors. In addition to the above mentioned, one should add representations over non-archimedian local fields with its applications to number theory, representations of infinite-dimensional Lie algebras with applications to number theory and physics and representations of quantum groups. However, in all these theories certain main ideas appear again and again very often in disguise. Due to technical details it may be difficult for a neophyte to recognize them. The goal of this book is to present some of these ideas in their most elementary incarnation.

We will assume that the reader is familiar with usual linear algebra (including the theory of Jordan forms and tensor products of vector spaces) and basic theory of groups and rings.

CHAPTER 1

Introduction to representation theory of finite groups.

1. Definitions and examples

Let k be a field, V be a vector space over k. By $\mathrm{GL}(V)$ we denote the group of all invertible linear operators in V. If $\dim V = n$, then $\mathrm{GL}(V)$ is isomorphic to the group of invertible $n \times n$ matrices with entries in k.

A *(linear) representation* of a group G in V is a group homomorphism

$$\rho : G \to \mathrm{GL}(V),$$

$\dim V$ is called the *degree* or the *dimension* of the representation ρ (it may be infinite). For any $g \in G$ we denote by ρ_g the image of g in $\mathrm{GL}(V)$ and for any $v \in V$ we denote by $\rho_g v$ the image of v under the action of ρ_g. The following properties are direct consequences of the definition

- $\rho_g \rho_h = \rho_{gh}$;
- $\rho_1 = \mathrm{Id}$;
- $\rho_g^{-1} = \rho_{g^{-1}}$;
- $\rho_g (xv + yw) = x\rho_g v + y\rho_h w$.

Example 1.1. (1) Let us consider the abelian group of integers $\mathbb{Z}$ with operation of addition. Let V be the plane $\mathbb{R}^2$ and for every $n \in \mathbb{Z}$, we set $\rho_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. The reader can check that this defines a representation of degree 2 of $\mathbb{Z}$.

(2) Let G be the symmetric group S_n, $V = k^n$. For every $s \in S_n$ and $(x_1, \dots, x_n) \in k^n$ set

$$\rho_s (x_1, \dots, x_n) = \left(x_{s(1)}, \dots, x_{s(n)}\right).$$

In this way we obtain a representation of the symmetric group S_n which is called the *permutation representation*.

(3) For any group G (finite or infinite) the *trivial representation* is the homomorphism $\rho : G \to k^*$ such that $\rho_s = 1$ for all $s \in G$.

(4) Let G be a group and

$$\mathcal{F}(G) = \{\varphi : G \to k\}$$

be the space of functions on G with values in k. For any $g, h \in G$, $\varphi \in \mathcal{F}(G)$ and let

$$\rho_g \varphi (h) = \varphi (hg).$$

Then $\rho : G \to \mathrm{GL}\left(\mathcal{F}\left(G\right)\right)$ is a linear representation.

(5) Recall that the *group algebra* $k\left(G\right)$ is the vector space of all finite linear combinations $\sum c_g g,\ c_g \in k$ with natural multiplication. We define the *regular representation* $R : G \to \mathrm{GL}\left(k\left(G\right)\right)$ in the following way

$$R_s\left(\sum c_g g\right) = \sum c_g sg.$$

Definition 1.2. Two representations of a group G, $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are called *equivalent* or *isomorphic* if there exists an invertible linear operator $T : V \to W$ such that $T \circ \rho_g = \sigma_g \circ T$ for any $g \in G$.

Example 1.3. If G is a finite group, then the representations in examples 4 and 5 are equivalent. Indeed, define $T : \mathcal{F}\left(G\right) \to k\left(G\right)$ by the formula

$$T\left(\varphi\right) = \sum_{x \in G} \varphi\left(x\right) x^{-1}.$$

Then for any $\varphi \in \mathcal{F}\left(G\right)$ and $g \in G$ we have

$$T\left(\rho_g \varphi\right) = \sum_{x \in G} \rho_g \varphi\left(x\right) x^{-1} = \sum_{x \in G} \varphi\left(xg\right) x^{-1} = \sum_{y \in G} \varphi\left(y\right) g y^{-1} = R_g\left(T\varphi\right).$$

Let a group G act on a set X on the right. Let $\mathcal{F}(X)$ be the set of k-valued functions on X. Then there is a representation of G in $\mathcal{F}(X)$ defined by

$$\rho_g \varphi(x) := \varphi(x \cdot g)$$

.

Exercise 1.4. Consider a *left* action $l : G \times X \to X$ of G on X. For every $\varphi \in \mathcal{F}(X)$, $g \in G$ and $x \in X$ set

$$\sigma_g \varphi(x) = \varphi(g^{-1} \cdot x).$$

(a) Prove that σ is a representation of G in $\mathcal{F}(X)$.

(b) Define a right action $r : X \times G \to X$ by

$$x \cdot g := g^{-1} \cdot x,$$

and consider the representation ρ of G in $\mathcal{F}(X)$ associated with this action. Check that ρ and σ are equivalent representations.

Remark 1.5. As one can see from the previous exercise, there is a canonical way to go between right and left action and between corresponding representations.

2. Ways to produce new representations

Let G be a group.

Restriction. If H is a subgroup of G and $\rho : G \to \mathrm{GL}(V)$ is a representation of G, the restriction of homomorphism ρ to H gives a representation of H which we call the *restriction* of ρ to H. We denote by $\mathrm{Res}_H \rho$ the restriction of ρ on H.

Lift. Let $p : G \to H$ be a homomorphism of groups. Then for every representation $\rho : H \to \mathrm{GL}(V)$, the composite homomorphism $\rho \circ p : G \to \mathrm{GL}(V)$ gives a representation of G on V. This construction is frequently used in the following case: let N be a normal subgroup of G, H denote the quotient group G/N and p be the natural projection. In this case p is obviously surjective. Note that in the general case we do not require p to be surjective.

Direct sum. If we have two representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$, then we can define $\rho \oplus \sigma \colon G \to \mathrm{GL}(V \oplus W)$ by the formula

$$(\rho \oplus \sigma)_g (v, w) = (\rho_g v, \sigma_g w).$$

Tensor product. The tensor product of two representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ is defined by

$$(\rho \otimes \sigma)_g (v \otimes w) = \rho_g v \otimes \sigma_g w.$$

Exterior tensor product. Let G and H be two groups. Consider representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : H \to \mathrm{GL}(W)$ of G and H respectively. One defines their *exterior tensor product* $\rho \boxtimes \sigma : G \times H \to \mathrm{GL}(V \otimes W)$ by the formula

$$(\rho \boxtimes \sigma)_{(g,h)} v \otimes w = \rho_g v \otimes \sigma_h w.$$

Exercise 2.1. If $\delta : G \to G \times G$ is the diagonal embedding, show that for any representations ρ and σ of G

$$\rho \otimes \sigma = (\rho \boxtimes \sigma) \circ \delta.$$

Dual representation. Let V^* denote the dual space of V and $\langle \cdot, \cdot \rangle$ denote the natural pairing between V and V^*. For any representation $\rho : G \to \mathrm{GL}(V)$ one can define the dual representation $\rho^* : G \to \mathrm{GL}(V^*)$ by the formula

$$\langle \rho_g^* \varphi, v \rangle = \langle \varphi, \rho_g^{-1} v \rangle$$

for every $v \in V, \varphi \in V^*$.

Let V be a finite-dimensional representation of G with a fixed basis. Let A_g for $g \in G$ be the matrix of ρ_g in this basis. Then the matrix of ρ_g^* in the dual basis of V^* is equal to $(A_g^t)^{-1}$.

Exercise 2.2. Show that if G is finite, then its regular representation is self-dual (isomorphic to its dual).

More generally, if $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are two representations, then one can naturally define a representation τ of G on $\mathrm{Hom}_k(V, W)$ by the formula

$$\tau_g \varphi = \sigma_g \circ \varphi \circ \rho_g^{-1},\ g \in G,\ \varphi \in \mathrm{Hom}_k(V, W).$$

Exercise 2.3. Show that if V and W are finite dimensional, then the representation τ of G on $\mathrm{Hom}_k(V, W)$ is isomorphic to $\rho^* \otimes \tau$.

Intertwining operators. A linear operator $T : V \to W$ is called an *intertwining* operator if $T \circ \rho_g = \sigma_g \circ T$ for any $g \in G$. The set of all intertwining operators will be denoted by $\mathrm{Hom}_G(V, W)$. It is clearly a vector space. Moreover, if $\rho = \sigma$, then $\mathrm{End}_G(V) := \mathrm{Hom}_G(V, V)$ has a natural structure of associative k-algebra with multiplication given by composition.

Exercise 2.4. Consider the regular representation of G in $k(G)$. Prove that the algebra of intertwiners $\mathrm{End}_G(k(G))$ is isomorphic to $k(G)$. (Hint: $\varphi \in \mathrm{End}_G(k(G))$ is completely determined by $\varphi(1)$.)

3. Invariant subspaces and irreducibility

3.1. Invariant subspaces and subrepresentations. Consider a representation $\rho : G \to \mathrm{GL}(V)$. A subspace $W \subset V$ is called G-*invariant* if $\rho_g(W) \subset W$ for any $g \in G$.

If W is a G-invariant subspace, then there are two representations of G naturally associated with it: the representation in W which is called a *subrepresentation* and the representation in the quotient space V/W wjich is called a *quotient* representation.

Exercise 3.1. Let $\rho : S_n \to \mathrm{GL}(k^n)$ be the permutation representation, then

$$W = \{x(1, \dots, 1) \mid x \in k\}$$

and

$$W' = \{(x_1, \dots, x_n) \mid x_1 + x_2 + \dots + x_n = 0\}$$

are invariant subspaces.

Exercise 3.2. Let G be a finite group of order $|G|$. Prove that any representation of G contains an invariant subspace of dimension less or equal than $|G|$.

3.2. Maschke's theorem.

Theorem 3.3. *(Maschke) Let G be a finite group such that* char k *does not divide $|G|$. Let $\rho : G \to \mathrm{GL}(V)$ be a representation and $W \subset V$ be a G-invariant subspace. Then there exists a complentary G-invariant subspace, i.e. a G-invariant subspace $W' \subset V$ such that $V = W \oplus W'$.*

PROOF. Let W'' be a subspace (not necessarily G-invariant) such that $W \oplus W'' = V$. Consider the projector $P : V \to V$ onto W with kernel W''. One has $P^2 = P$. Now we construct a new operator

$$\bar{P} := \frac{1}{|G|} \sum_{g \in G} \rho_g \circ P \circ \rho_g^{-1}.$$

An easy calculation shows that $\rho_g \circ \bar{P} \circ \rho_g^{-1} = \bar{P}$ for all $g \in G$, and therefore $\rho_g \circ \bar{P} = \bar{P} \circ \rho_g$. In other words, $\bar{P} \in \mathrm{End}_G(V)$.

On the other hand, $\bar{P}_{|W} = \mathrm{Id}$ and $\mathrm{Im}\,\bar{P} = W$. Hence $\bar{P}^2 = \bar{P}$.

Let $W' = \mathrm{Ker}\,\bar{P}$. First, we claim that W' is G-invariant. Indeed, let $w \in W'$, then $\bar{P}(\rho_g w) = \rho_g \left(\bar{P} w\right) = 0$ for all $g \in G$, hence $\rho_g w \in \mathrm{Ker}\,\bar{P} = W'$.

Now we prove that $V = W \oplus W'$. Indeed, $W \cap W' = 0$, since $\bar{P}_{|W} = Id$. On the other hand, for any $v \in V$, we have $w = \bar{P}v \in W$ and $w' = v - \bar{P}v \in W'$. Thus, $v = w + w'$, and therefore $V = W + W'$. □

Remarks. If $\mathrm{char}\, k$ divides $|G|$ or G is infinite, the conclusion of Mashke's theorem does not hold anymore. Indeed, in the example of Exercise 3.1 W and W' are complementary if and only if $\mathrm{char}\, k$ does not divide n. Otherwise, $W \subset W' \subset V$, and one can show that neither W nor W' have a G-invariant complement.

In the case of an infinite group, consider the representation of $\mathbb{Z}$ in $\mathbb{R}^2$ as in the first example of Section 1. The span of $(1, 0)$ is the only G-invariant line. Therefore it can not have a G-invariant complement in $\mathbb{R}^2$. direct sum of two proper invariant subspaces.

3.3. Irreducible representations and Schur's lemma.

DEFINITION 3.4. A non-zero representation is called *irreducible* if it does not contain any proper non-zero G-invariant subspace.

EXERCISE 3.5. Show that the dimension of any irreducible representation of a finite group G is not bigger than its order $|G|$.

The following elementary statement plays a key role in representation theory.

LEMMA 3.6. *(Schur) Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be two irreducible representations. If $T \in \mathrm{Hom}_G(V, W)$, then either $T = 0$ or T is an isomorphism.*

PROOF. Note that $\mathrm{Ker}\, T$ and $\mathrm{Im}\, T$ are G-invariant subspaces of V and W, respectively. Then by irreducibility of ρ, either $\mathrm{Ker}\, T = V$ or $\mathrm{Ker}\, T = 0$, and by irreducibility of σ, either $\mathrm{Im}\, T = W$ or $\mathrm{Im}\, T = 0$. Hence the statement. □

COROLLARY 3.7. *(a) Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation. Then $\mathrm{End}_G(V)$ is a division ring.*

(b) If the characteristic of k does not divide $|G|$, $\mathrm{End}_G(V)$ is a division ring if and only if ρ is irreducible.

(c) If k is algebraically closed and ρ is irreducible, then $\mathrm{End}_G(V) = k$

PROOF. (a) is an immediate consequence of Schur's Lemma.

To prove (b) we use Maschke's theorem. Indeed, if V is reducible, then $V = V_1 \oplus V_2$ for some proper subspaces V_1 and V_2. Let p_1 be the projector on V_1 with kernel V_2 and p_2 be the projector onto V_2 with kernel V_1. Then $p_1, p_2 \in \mathrm{End}_G(V)$ and $p_1 \circ p_2 = 0$. Hence $\mathrm{End}_G(V)$ has zero divisors.

Let us prove (c). Consider $T \in \mathrm{End}_G(V)$. Then T has an eigenvalue $\lambda \in k$ and $T - \lambda \, \mathrm{Id} \in \mathrm{End}_G(V)$. Since $T - \lambda \, \mathrm{Id}$ is not invertible, it must be zero by (a). Therefore $T = \lambda \, \mathrm{Id}$. □

3.4. Complete reducibility.

DEFINITION 3.8. A representation is called *completely reducible* if it splits into a direct sum of irreducible subrepresentations. (This direct sum might be infinite.)

THEOREM 3.9. *Let $\rho : G \to GL(V)$ be a representation of a group G. The following conditions are equivalent.*

(a) ρ is completely reducible;

(b) For any G-invariant subspace $W \subset V$ there exists a complementary G-invariant subspace W'.

PROOF. This theorem is easier in the case of finite-dimensional V. To prove it for arbitrary V and G we need Zorn's lemma. First, note that if V is finite dimensional, then it always contains an irreducible subrepresentation. Indeed, we can take a subrepresentation of minimal positive dimension. If V is infinite dimensional then this is not true in general.

LEMMA 3.10. *If ρ satisfies (b), any subrepresentation and any quotient of ρ also satisfy (b).*

PROOF. To prove that any subrepresentation satisfies (b) consider a flag of G-invariant subspaces $U \subset W \subset V$. Let $U' \subset V$ and $W' \subset V$ be G-invariant subspaces such that $U \oplus U' = V$ and $W \oplus W' = V$. Let P be the projector on W with kernel W'. Then $W = U \oplus P(U')$.

The statement about quotients is dual and we leave it to the reader as an exercise. □

LEMMA 3.11. *Let ρ satisfy (b). Then it contains an irreducible subrepresentation.*

PROOF. Pick up a non-zero vector $v \in V$ and let V' be the span of $\rho_g v$ for all $g \in G$. Consider the set of G-invariant subspaces of V' which do not contain v, with partial order given by inclusion. For any linearly ordered subset $\{X_i\}_{i \in I}$ there exists a maximal element, given by the union $\bigcup_{i \in I} X_i$. Hence there exists a proper maximal G-invariant subspace $W \subset V'$, which does not contain v. By the previous lemma one can find a G-invariant subspace $U \subset V'$ such that $V' = W \oplus U$. Then U is isomorphic to the quotient representation V'/W, which is irreducible by the maximality of W in V'. □

Now we will prove that (a) implies (b). We write

$$V = \bigoplus_{i \in I} V_i$$

for a family of irreducible G-invariant subspaces V_i. Let $W \subset V$ be some G-invariant subspace. By Zorn's lemma there exists a maximal subset $J \subset I$ such that

$$W \cap \bigoplus_{j \in J} V_j = 0.$$

We claim that $W' := \bigoplus_{j \in J} V_j$ is complementary to W. Indeed, it suffices to prove that $V = W + W'$. For any $i \notin J$ we have $(V_i \oplus W') \cap W \neq 0$. Therefore there exists a non-zero vector $v \in V_i$ equal to $w + w'$ for some $w \in W$ and $w' \in W'$. Hence $V_i \cap (W' + W) \neq 0$ and by irreducibility of V_i, we have $V_i \subset W + W'$. Therefore $V = W + W'$.

To prove that (b) implies (a) consider the family of all irreducible subrepresentations $\{W_k\}_{k \in K}$ of V. Note that $\sum_{k \in K} W_k = V$ because otherwise $\sum_{k \in K} W_k$ has a G-invariant complement which contains an irreducible subrepresentation. Again due to Zorn's lemma one can find a minimal $J \subset K$ such that $\sum_{j \in J} W_j = V$ Then clearly $V = \bigoplus_{j \in J} W_j$. □

The next statement follows from Maschke's theorem and Theorem 3.9.

PROPOSITION 3.12. *Let G be a finite group and k be a field such that* $\operatorname{char} k$ *does not divide $|G|$. Then every representation of G is completely reducible.*

4. Characters

4.1. Definition and main properties. For a linear operator T in a finite-dimensional vector space V we denote by $\operatorname{tr} T$ the trace of T.

For any finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ the function $\chi_\rho : G \to k$ defined by

$$\chi_\rho(g) = \operatorname{tr} \rho_g.$$

is called the *character* of the representation ρ.

EXERCISE 4.1. Check the following properites of characters.

(1) $\chi_\rho(1) = \dim \rho$;
(2) if $\rho \cong \sigma$, then $\chi_\rho = \chi_\sigma$;
(3) $\chi_{\rho \oplus \sigma} = \chi_\rho + \chi_\sigma$;
(4) $\chi_{\rho \otimes \sigma} = \chi_\rho \chi_\sigma$;
(5) $\chi_{\rho^*}(g) = \chi_\rho(g^{-1})$;

(6) $\chi_\rho(ghg^{-1}) = \chi_\rho(h)$.

EXERCISE 4.2. Calculate the character of the permutation representation of S_n (see the first example of Section 1).

EXAMPLE 4.3. If R is the regular representation of a finite group, then $\chi_R(g) = 0$ for any $s \neq 1$ and $\chi_R(1) = |G|$.

EXAMPLE 4.4. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of dimension n and assume $\operatorname{char} k \neq 2$. Consider the representation $\rho \otimes \rho$ in $V \otimes V$ and the decomposition

$$V \otimes V = S^2 V \oplus \Lambda^2 V.$$

The subspaces S^2V and $\Lambda^2 V$ are G-invariant. Denote by sym and alt the subrepresentations of G in S^2V and Λ^2V respectively. Let us compute the characters χ_{sym} and χ_{alt}.

Let $g \in G$ and denote by $\lambda_1, \dots, \lambda_n$ the eigenvalues of ρ_g (taken with multiplicities). Then the eigenvalues of alt_g are the products $\lambda_i \lambda_j$ for all $i < j$ while the eigenvalues of sym_g are $\lambda_i\lambda_j$ for $i \leq j$. This leads to

$$\chi_{\mathrm{sym}}(g) = \sum_{i \leq j} \lambda_i \lambda_j,$$

$$\chi_{\mathrm{alt}}(g) = \sum_{i < j} \lambda_i \lambda_j.$$

Hence

$$\chi_{\mathrm{sym}}(g) - \chi_{\mathrm{alt}}(g) = \sum_i \lambda_i^2 = \operatorname{tr} \rho_{g^2} = \chi_\rho\left(g^2\right).$$

On the other hand by properties (3) and (4)

$$\chi_{\mathrm{sym}}(g) + \chi_{\mathrm{alt}}(g) = \chi_{\rho\otimes\rho}(g) = \chi_\rho^2(g).$$

Thus, we get

$$\chi_{\mathrm{sym}}(g) = \frac{\chi_\rho^2(g) + \chi_\rho(g^2)}{2}, \ \chi_{\mathrm{alt}}(g) = \frac{\chi_\rho^2(g) - \chi_\rho(g^2)}{2}. \tag{1.1}$$

LEMMA 4.5. *If $k = \mathbb{C}$ and G is finite, then for any finite-dimensional representation ρ and any $g \in G$ we have*

$$\chi_\rho(g) = \overline{\chi_\rho(g^{-1})}.$$

PROOF. Indeed, $\chi_\rho(g)$ is the sum of all the eigenvalues of ρ_g. Since g has finite order, every eigenvalue of ρ_g is a root of 1. Therefore the eigenvalues of $\rho_{g^{-1}}$ are the complex conjugates of the eigenvalues of ρ_g. $\square$

4.2. Orthogonality relations. In this subsection we assume that G is finite and the characteristic of the ground field k is zero. Introduce a non-degenerate symmetric bilinear form on the space of functions $\mathcal{F}(G)$ by the formula

$$(\varphi,\psi)=\frac{1}{|G|}\sum_{g\in G}\varphi\left(s^{-1}\right)\psi\left(s\right). \tag{1.2}$$

If $\rho : G \to \mathrm{GL}(V)$ is a representation, then we denote by V^G the subspace of G-invariant vectors, i.e.

$$V^G=\{v\in V|\rho_g(v)=v,\forall g\in G\}.$$

Lemma 4.6. *If $\rho : G \to \mathrm{GL}(V)$ is a representation, then*

$$\dim V^G=(\chi_\rho,\chi_{triv}),$$

where χ_{triv} denotes the character of the trivial representation, i.e. $\chi_{triv}(g)=1$ for all $g\in G$.

Proof. Consider the linear operator $P\in \mathrm{End}_G(V)$ defined by the formula

$$P=\frac{1}{|G|}\sum_{g\in G}\rho_g.$$

Note that $P^2 = P$ and $\operatorname{Im} P = V^G$. Thus, P is a projector on V^G. Since $\operatorname{char} k = 0$ we have

$$\operatorname{tr}P=\dim\operatorname{Im}P=\dim V^G.$$

On the other hand, by direct calculation we get $\operatorname{tr} P = (\chi_\rho, \chi_{triv})$, and the lemma follows. □

Note that for two representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ we have

$$\mathrm{Hom}_k(V,W)^G=\mathrm{Hom}_G(V,W)=(V^*\otimes W)^G. \tag{1.3}$$

Therefore we have the following

Corollary 4.7. *One has*

$$\dim \mathrm{Hom}_G(V,W)=(\chi_\rho,\chi_\sigma).$$

Proof. The statement is a consequence of the following computation:

$$(\chi_\rho,\chi_\sigma)=\frac{1}{|G|}\sum_{g\in G}\chi_\rho(g^{-1})\chi_\sigma(g)=\frac{1}{|G|}\sum_{g\in G}\chi_{\rho^*\otimes\sigma}(g)=(\chi_{\rho^*\otimes\sigma},\chi_{triv}).$$

□

The following theorem is usually called the *orthogonality relations* for characters.

THEOREM 4.8. *Let ρ, σ be irreducible representations over a field of characteristic zero.*

(a) If $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$ are not isomorphic, then $(\chi_\rho, \chi_\sigma) = 0$.

(b) Assume that the ground field is algebraically closed. If ρ and σ are equivalent, then $(\chi_\rho, \chi_\sigma) = 1$.

PROOF. By Schur's lemma

$$\mathrm{Hom}_G(V, W) = 0$$

Therefore Corollary 4.7 implies (a).

Assertion (b) follows form Corollary 3.7 (c) and Corollary 4.7. □

This theorem has several important corollaries.

COROLLARY 4.9. *Let*

$$\rho = m_1\rho_1 \oplus \cdots \oplus m_r\rho_r$$

be a decomposition into a sum of irreducible representations, where $m_i\rho_i$ is the direct sum of m_i copies of ρ_i. Then $m_i = \frac{(\chi_\rho, \chi_{\rho_i})}{(\chi_{\rho_i}, \chi_{\rho_i})}$.

The number m_i is called the *multiplicity* of an irreducible representation ρ_i in ρ.

COROLLARY 4.10. *Two finite-dimensional representations ρ and σ are equivalent if and only if their characters coincide.*

In the rest of this section we assume that the ground field is algebraically closed.

COROLLARY 4.11. *A representation ρ is irreducible if and only if $(\chi_\rho, \chi_\rho) = 1$.*

EXERCISE 4.12. Let ρ and σ be irreducible representations of finite groups G and H respectively.

(a) If the ground field is algebraically closed, then the exterior product $\rho \boxtimes \sigma$ is an irreducible representation of $G \times H$.

(b) Give a counterexample to (a) in the case when the ground field is not algebraically closed.

THEOREM 4.13. *Every irreducible representation ρ appears in the regular representation with multiplicity* $\dim \rho$.

PROOF. The statement is a direct consequence of the following computation

$$(\chi_\rho, \chi_R) = \frac{1}{|G|}\chi_\rho(1)\,\chi_R(1) = \dim \rho.$$

□

COROLLARY 4.14. *Let $\rho_1, \ldots, \rho_r$ be all (up to isomorphism) irreducible representations of G and $n_i = \dim \rho_i$. Then*

$$n_1^2 + \cdots + n_r^2 = |G|.$$

PROOF. Indeed,

$$\dim R = |G| = \chi_R(1) = \sum_{i=1}^{r} n_i \chi_{\rho_i}(1) = \sum_{i=1}^{r} n_i^2.$$

$\square$

EXAMPLE 4.15. Let G act on a finite set X and

$$k\,(X) = \left\{ \sum_{x \in X} b_x x \mid b_x \in k \right\}.$$

Define $\rho : G \to \mathrm{GL}\,(k\,(X))$ by

$$\rho_g \left(\sum_{x \in X} b_x x \right) = \sum_{x \in X} b_x g \cdot x.$$

It is easy to check that ρ is a representation and

$$\chi_\rho\,(g) = |\,\{x \in X \mid g \cdot x = x\}\,|.$$

Clearly, ρ contains the trivial subrepresentation. To find the multiplicity of the trivial representation in ρ we have to calculate $(1, \chi_\rho)$:

$$(1, \chi_\rho) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho\,(g) = \frac{1}{|G|} \sum_{g \in G} \sum_{g \cdot x = x} 1 = \frac{1}{|G|} \sum_{x \in X} \sum_{g \in G_x} 1 = \frac{1}{|G|} \sum_{x \in X} |G_x|,$$

where

$$G_x = \{g \in G \mid g \cdot x = x\}\,.$$

Let $X = X_1 \cup \cdots \cup X_m$ be the disjoint union of orbits. Then $|G_x| = \frac{|G|}{|X_i|}$ for each $x \in X_i$ and therefore

$$(1, \chi_\rho) = \frac{1}{|G|} \sum_{i=1}^{m} \sum_{x \in X_i} \frac{|G|}{|X_i|} = m.$$

Now let us evaluate (χ_ρ, χ_ρ):

$$(\chi_\rho, \chi_\rho) = \frac{1}{|G|} \sum_{g \in G} \left(\sum_{g \cdot x = x} 1 \right)^2 = \frac{1}{|G|} \sum_{g \in G} \sum_{g \cdot x = x, g \cdot y = y} 1 = \frac{1}{|G|} \sum_{(x,y) \in X \times X} |G_{(x,y)}|.$$

Let σ be the representation associated with the action of G on $X \times X$. Then the last formula implies

$$(\chi_\rho, \chi_\rho) = (1, \chi_\sigma)\,.$$

Thus, ρ is irreducible if and only if $|X| = 1$, and ρ has two irreducible components if and only if the action of G on $X \times X$ with removed diagonal is transitive.

4.3. The number of irreducible representations of a finite group.

DEFINITION 4.16. Let

$$\mathcal{C}(G) = \left\{\varphi \in \mathcal{F}(G) \mid \varphi\left(ghg^{-1}\right) = \varphi(h)\right\}.$$

Elements of $\mathcal{C}(G)$ are called *class functions*.

EXERCISE 4.17. Check that the restriction of $(\cdot,\cdot)$ on $\mathcal{C}(G)$ is non-degenerate.

THEOREM 4.18. *The characters of irreducible representations of G form an orthonormal basis of $\mathcal{C}(G)$.*

PROOF. We have to show that if $\varphi \in \mathcal{C}(G)$ and $(\varphi, \chi_\rho) = 0$ for any irreducible representation ρ, then $\varphi = 0$. The following lemma is straightforward.

LEMMA 4.19. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation, $\varphi \in \mathcal{C}(G)$ and*

$$T = \frac{1}{|G|}\sum_{g\in G}\varphi\left(g^{-1}\right)\rho_g.$$

Then $T \in \mathrm{End}_G V$ and $\mathrm{tr}\, T = (\varphi, \chi_\rho)$.

Thus, for any irreducible representation ρ we have

$$\frac{1}{|G|}\sum_{g\in G}\varphi\left(g^{-1}\right)\rho_g = 0. \tag{1.4}$$

But then the same is true for any representation ρ, since any representation is a direct sum of irreducible representations. Apply (1.4) to the case when $\rho = R$ is the regular representation. Then

$$\frac{1}{|G|}\sum_{g\in G}\varphi\left(g^{-1}\right)R_g(1) = \frac{1}{|G|}\sum_{g\in G}\varphi\left(g^{-1}\right)g = 0.$$

Hence $\varphi\left(g^{-1}\right) = 0$ for all $g \in G$, i.e. $\varphi = 0$ □

COROLLARY 4.20. *The number of isomorphism classes of irreducible representations equals the number of conjugacy classes in the group G.*

COROLLARY 4.21. *If G is a finite abelian group, then every irreducible representation of G is one-dimensional and the number of irreducible representations is the order of the group G.*

For any group G (not necessarily finite) let G^* denote the set of all one-dimensional representations of G.

EXERCISE 4.22. (a) Show that G^* is a group with respect to the operation of tensor product.

(b) Show that the kernel of any $\rho \in G^*$ contains the commutator $[G,G]$. Hence we have $G^* \simeq (G/[G,G])^*$.

(c) Show that if G is a finite abelian group, then $G^* \simeq G$. (This isomorphism is not canonical.)

EXERCISE 4.23. Consider the symmetric group S_n for $n \geq 2$.

(a) Prove that the commutator $[S_n, S_n]$ coincides with the subgroup A_n of all even permutation.

(b) Show that S_n has two up to isomorphism one-dimensional representations: the trivial and the sign representation $\epsilon : S_n \to \{1, -1\}$.

EXERCISE 4.24. Let ρ be a one-dimensional representation of a finite group G and σ is some other representation of G. Show that σ is irreducible if and only if $\rho \otimes \sigma$ is irreducible.

4.4. Isotypic components. Consider the decomposition of some representation $\rho : G \to GL(V)$ into a direct sum of irreducible representations

$$\rho = m_1\rho_1 \oplus \cdots \oplus m_r\rho_r.$$

The subspace $W_i \simeq V_i^{\oplus m_i}$ of the representation $m_i\rho_i$ is called the *isotypic component* of type ρ_i of V.

LEMMA 4.25. *Let n_i denote the dimension of the irreducible representation ρ_i and*

$$\pi_i := \frac{n_i}{|G|}\sum_{g\in G}\chi_i(g^{-1})\rho_g.$$

Then π_i is the projector on the isotypic component W_i of type ρ_i.

PROOF. Define a linear operator on V_j by the formula

$$\pi_{ij} := \frac{n_i}{|G|}\sum_{g\in G}\chi_i(g^{-1})(\rho_j)_g.$$

By construction $\pi_{ij} \in \operatorname{End}_G(V_j)$. Corollary 3.7 (c) implies that $\pi_{ij} = \lambda \operatorname{Id}$. By Theorem 4.8

$$\operatorname{tr}\pi_{ij} = n_i(\chi_i, \chi_j) = n_i\delta_{ij}.$$

Now we write

$$\pi_i = \sum_{j=1}^{r}\pi_{ij}.$$

Hence

$$\pi_i|_{W_j} = \delta_{ij}\operatorname{Id}.$$

The statement follows. □

5. Examples.

In the examples below we assume that the ground field is $\mathbb{C}$.

EXAMPLE 5.1. Let $G = S_3$. There are three conjugacy classes in G, each class is denoted by some element in this class: 1,(12),(123). Therefore there are three irreducible representations, denote their characters by χ_1, χ_2 and χ_3. It is not difficult to see that S_3 has the following table of characters

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

The characters of one-dimensional representations are given in the first and the second row (those are the trivial representation and the sign representation, see Exercise 4.23), the last character χ_3 can be obtained by using the identity

$$\chi_{\text{perm}} = \chi_1 + \chi_3, \tag{1.5}$$

where χ_{perm} stands for the character of the permutation representation, see Exercise 4.2.

EXAMPLE 5.2. Let $G = S_4$. In this case we have the following character table (in the first row we write the number of elements in each conjugacy class).

	1	6	8	3	6
	1	(12)	(123)	(12) (34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	-1	1
χ_5	2	0	-1	2	0

The first two rows are the characters of the one-dimensional representations. The third one can again be obtained from (1.5). When we take the tensor product $\rho_4 := \rho_2 \otimes \rho_3$ we get a new 3-dimensional irreducible representation, see Exercise 4.24 whose character χ_4 is equal to the product $\chi_2\chi_3$. The last character can be obtained through Theorem 4.8. An alternative way to describe ρ_5 is to consider S_4/K_4, where

$$K_4 = \{1, (12)\,(34)\,, (13)\,(24)\,, (14)\,(23)\}$$

is the Klein subgroup. Observe that $S_4/K_4 \cong S_3$, and therefore the two-dimensional representation σ of S_3 can be lifted to a representation of S_4 by

$$\rho_5 = \sigma \circ p,$$

where $p : S_4 \to S_3$ is the natural projection.

EXAMPLE 5.3. Now let $G = A_5$. There are 5 irreducible representations of G over $\mathbb{C}$. Here is the character table

	1	20	15	12	12
	1	(123)	(12) (34)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_5	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

To obtain χ_2 we use the permutation representation and (1.5) once more. In order to construct new irreducible representations we consider the characters χ_{sym} and χ_{alt} of the second symmetric and the second exterior powers of ρ_2 respectively. Using (1.1) we compute

	1	(123)	(12) (34)	(12345)	(12354)
χ_{sym}	10	1	2	0	0
χ_{alt}	6	0	-2	1	1

It is easy to check that

$$(\chi_{\text{sym}}, \chi_{\text{sym}}) = 3, \ (\chi_{\text{sym}}, \chi_1) = (\chi_{\text{sym}}, \chi_2) = 1.$$

Therefore

$$\chi_3 = \chi_{\text{sym}} - \chi_1 - \chi_2$$

is the character of another irreducible representation of dimension 5. We still miss two.

To find then we use χ_{alt}. We have

$$(\chi_{\text{alt}}, \chi_{\text{alt}}) = 2, \ (\chi_{\text{alt}}, \chi_1) = (\chi_{\text{alt}}, \chi_2) = (\chi_{\text{alt}}, \chi_3) = 0.$$

Therefore $\chi_{\text{alt}} = \chi_4 + \chi_5$ is the sum of two irreducible characters. First we compute the dimensions of ρ_4 and ρ_5 using

$$1^2 + 4^2 + 5^2 + n_4^2 + n_5^2 = 60.$$

We obtain $n_4 = n_5 = 3$.

Next, we use Theorem 4.8 to compute some other values of χ_4 and χ_5. The equations

$$(\chi_4, \chi_1 + \chi_2) = 0,\ (\chi_4, \chi_3) = 0$$

imply

$$\chi_4((123)) = 0,\ \chi_4((12)(34)) = -1.$$

The same argument applied to χ_5 gives

$$\chi_5((123)) = 0,\ \chi_5((12)(34)) = -1.$$

Finally let us denote

$$x = \chi_4((12345)),\ y = \chi_4((12354))$$

and write down the equation arising from $(\chi_4, \chi_4) = 1$:

$$\frac{1}{60}\left(9 + 15 + 12x^2 + 12y^2\right) = 1,$$

or more simply

$$x^2 + y^2 = 3. \tag{1.6}$$

On the other hand, $(\chi_4, \chi_1) = 0$, which gives

$$3 - 15 + 12(x + y) = 0,$$

or simply

$$x + y = 1. \tag{1.7}$$

The system (1.6), (1.7) has two solutions

$$x_1 = \frac{1+\sqrt{5}}{2}, y_1 = \frac{1-\sqrt{5}}{2}, \quad x_2 = \frac{1-\sqrt{5}}{2}, y_2 = \frac{1+\sqrt{5}}{2}.$$

They give the characters χ_4 and χ_5.

Now that we have the character of A_5 we would like to explain a geometric construction related to it. First, we observe that the previous constructions work over the grouond field $\mathbb{R}$ of real numbers. In particular, the representations ρ_4 and ρ_5 are defined over $\mathbb{R}$. Indeed, they are subrepresentation of the second exterior power of ρ_2 and by Lemma 4.25 the corresponding projectors are defined over $\mathbb{R}$. Therefore we have an action of A_5 in $\mathbb{R}^3$. Our next step is to show that this action preserves the scalar product. In a more general context we have the following result.

Lemma 5.4. *Let V be a finite-dimensional vector space over $\mathbb{R}$ and ρ be a representation of some finite group G in V. There exists a positive definite scalar product $B : V \times V \to \mathbb{R}$ such that $B(\rho_g u, \rho_g v) = B(u, v)$ for any $u, v \in V$ and $g \in G$.*

Remark. Such a scalar product is called *invariant*.

PROOF. Let $C : V \times V \to \mathbb{R}$ be some positive definite scalar product. Set

$$B(u, v) := \sum_{g \in G} C(\rho_g u, \rho_g v).$$

Then B satisfies the conditions of the lemma. $\square$

Dodecahedron. We have constructed two 3-dimensional irreducible representations of A_5, wecan use any of them to construct a dodecahedron, i.e. a regular polyhedron with 12 pentagonal faces and 20 vertices. For instance, let us take $\rho = \rho_4$. By Lemma 5.4 we may assume that for all $g \in G$, ρ_g acts on $\mathbb{R}^3$ by an orthogonal matrix. We claim that ρ_g preserves the orientation in $\mathbb{R}^3$, in other words the determinant $\det \rho_g$ is 1 fo all $g \in G$. We already know that $\det \rho_g = \pm 1$. Therefore if g is of odd order the determinant is necessarily 1. If g is of even order, it belongs to the conjugacy class of $(12)(34)$. Hence it is an involution with trace -1, thus a rotation by $180°$. Recall that any isometry in $\mathbb{R}^3$ preserving orientation ia a rotation.

Let $g = (123)$, then it is of order 3, hence ρ_g is a rotation by $120°$. Pick up a non-zero x fixed by ρ_g. Consider its orbit $S = \{\rho_g(x) | g \in A_5\}$. Since the stabilizer of x in A_5 is the cyclic group generated by ρ_g, we know that S has 20 points. Moreover, all points of S lie on a sphere, and hence the convex hull Δ of S is a polytope with vertices in S. We will show that Δ is a regular polytope whose faces are regular pentagons.

Let $h = (12345)$. Consider the subgroup $H \subset A_5$ generated by h. Since ρ_h is a rotation by $72°$. Without loss of generality we may assume the axis of ρ_h is vertical. Hence the different orbits of H in S lie on 4 horizontal planes. The top and the bottom plane sections are faces of Δ. Thus, we can conclude that at least some faces of Δ are regular pentagons.

Next, we claim that any vertex of Δ belongs to exactly three pentagonal faces. Indeed, it follows from the fact that the stabilizer of any $s \in S$ has order three and it acts on the set of pentagonal faces containing s.

Finally, assume there is a face f of Δ which is not a regular pentagon. Then at least one angle of f is not less than $60°$. Denote this angle by α and the corresponding vertex by s. Consider the stabilizer of s in A_5. It is a cyclic group of order 3 acting on the set of faces containing s. Thus, there are at least three plane angles at s which are equal to α. But then the total sum of all plane angles at s should be at least $3 \times 72° + 3\alpha$ which is bigger that $360°$, thus, a contradiction. Thus, all the faces of Δ are regular pentagons. Hence the total number of faces is 12.

Note that we have also proved that the group of rotations of a dodecahedron is isomorphic to A_5.

EXERCISE 5.5. Let D_4 denote the dihedral group of order 8 and H_8 denote the multiplicative subgroup of quaternions consisting of $\pm 1, \pm i, \pm j, \pm k$. Compute the character tables of both groups and verify that those tables coincide.

6. Invariant forms

We assume here that $\operatorname{char} k = 0$. Recall that a bilinear form on a vector space V is a map $B : V \times V \to k$ satisfying

(1) $B(cv, dw) = cdB(v, w)$;
(2) $B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$;
(3) $B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2)$.

One can also think about a bilinear form as a vector in $V^* \otimes V^*$ or as the homomorphism $B : V \to V^*$ given by the formula $B_v(w) = B(v, w)$. A bilinear form is symmetric if $B(v, w) = B(w, v)$ and skew-symmetric if $B(v, w) = -B(w, v)$. Every bilinear form can be uniquely written as a sum $B = B^+ + B^-$ where B^+ is symmetric and B^- skew-symmetric form,

$$B^{\pm}(v, w) = \frac{B(v, w) \pm B(w, v)}{2}.$$

Such a decomposition corresponds to the decomposition

$$V^* \otimes V^* = S^2 V^* \oplus \Lambda^2 V^*. \tag{1.8}$$

A bilinear form is *non-degenerate* if $B : V \to V^*$ is an isomorphism, in other words if $B(v, V) = 0$ implies $v = 0$.

Let $\rho : G \to \mathrm{GL}(V)$ be a representation. We say that a bilinear form B on V is *G-invariant* if

$$B(\rho_g v, \rho_g w) = B(v, w)$$

for any $v, w \in V$, $g \in G$. If there is no possible confusion we use the word invariant instead of G-invariant.

Exercise 6.1. Check the following

(1) If $W \subset V$ is an invariant subspace, then $W^{\perp} = \{v \in V \mid B(v, W) = 0\}$ is invariant. In particular, $\operatorname{Ker} B$ is invariant.
(2) $B : V \to V^*$ is invariant if and only if $B \in \operatorname{Hom}_G(V, V^*)$.
(3) If B is invariant, then B^+ and B^- are invariant.

Lemma 6.2. *Let $\rho : G \to GL(V)$ be an irreducible representation of G, then any invariant bilinear form on V is non-degenerate. If k is algebraically closed, then such a bilinear form is unique up to scalar multiplication.*

Remark. Lemma 6.2 holds for a field of arbitrary characteristic.

Proof. Follows from Exercise 6.1 (2) and Schur's lemma. □

Lemma 6.3. *Let $\rho : G \to GL(V)$ be an irreducible representation of G. Then it admits an invariant form if and only if $\chi_\rho(g) = \chi_\rho(g^{-1})$ for any $g \in G$.*

Proof. Since every invariant bilinear form establishes an isomorphism between ρ and ρ^*, the statement follows from Corollary 4.10. □

LEMMA 6.4. *(a) If k is algebraically closed, then every non-zero invariant bilinear form on an irreducible representation ρ is either symmetric or skew-symmetric.*

(b) Define

$$m_\rho = \frac{1}{|G|}\sum_{g\in G}\chi_\rho\left(g^2\right).$$

Then $m_\rho = 1, 0$ or -1.

(c) If $m_\rho = 0$, then ρ does not admit an invariant form. If $m_\rho = 1$ (resp. $m_\rho = -1$), then ρ admits a symmetric (resp. skew-symmetric) invariant form.

PROOF. First, (a) is a consequence of Lemma 6.2 and Exercise 6.1.

Let us prove (b) and (c). Recall that $\rho\otimes\rho = \rho_{\text{alt}}\oplus\rho_{\text{sym}}$. Using 1.1 we obtain

$$(\chi_{\text{sym}}, \chi_{\text{triv}}) = \frac{1}{|G|}\sum_{g\in G}\frac{\chi_\rho^2(g) + \chi_\rho\left(g^2\right)}{2},$$

$$(\chi_{\text{alt}}, \chi_{\text{triv}}) = \frac{1}{|G|}\sum_{g\in G}\frac{\chi_\rho^2(g) - \chi_\rho\left(g^2\right)}{2}.$$

Note that

$$\frac{1}{|G|}\sum_{g\in G}\chi_\rho^2(g) = (\chi_\rho, \chi_{\rho^*}).$$

Therefore

$$(\chi_{\text{sym}}, \chi_{\text{triv}}) = \frac{(\chi_\rho, \chi_{\rho^*}) + m_\rho}{2},$$

$$(\chi_{\text{alt}}, \chi_{\text{triv}}) = \frac{(\chi_\rho, \chi_{\rho^*}) - m_\rho}{2}.$$

We have the folowing thrichotomy

- ρ does not have an invariant form, if and only if ρ is not isomorphic to ρ^*. In this case $(\chi_\rho, \chi_{\rho^*}) = 0$ and $(\chi_{\text{sym}}, \chi_{\text{triv}}) = (\chi_{\text{sym}}, \chi_{\text{triv}}) = 0$. Therefore $m_\rho = 0$.
- ρ has a symmetric invariant form if and only if $(\chi_\rho, \chi_{\rho^*}) = 1$ and $(\chi_{\text{sym}}, \chi_{\text{triv}}) = 1$. This implies $m_\rho = 1$.
- ρ admits a skew-symmetric invariant if and only if $(\chi_\rho, \chi_{\rho^*}) = 1$ and $(\chi_{\text{alt}}, \chi_{\text{triv}}) = 1$. This implies $m_\rho = 1$.

□

Let $k = \mathbb{C}$. An irreducible representation of a finite group G is called *real* if $m_\rho = 1$, *complex* if $m_\rho = 0$ and *quaternionic* if $m_\rho = -1$.

Remark. Since $\chi_\rho\left(s^{-1}\right) = \bar{\chi}_\rho(s)$, then χ_ρ takes only real values for real and quaternionic representations. If ρ is complex there is at least one $g\in G$ such that $\chi_\rho(g)\notin\mathbb{R}$. This terminology will become clear in Section 8.

Exercise 6.5. Show that
(a) All irreducible representation of S_4 are real.
(b) All non-trivial irreducible representations of $\mathbb{Z}_3$ are complex.
(c) The two-dimensional representation of the quaternionic group H_8 is quaternionic (see Exercise 5.5).

Exercise 6.6. Assume that the order of G be odd. Show that all non-trivial irreducible representation of G are complex. (Hint: prove that $m_\rho = (\chi_\rho, \chi_{\text{triv}})$.)

7. Representations over $\mathbb{R}$

Let us recall that by Lemma 5.4 every representation of a finite group over $\mathbb{R}$ admits an invariant scalar product. Assume the representation $\rho : G \to GL(V)$ is irreducible. Denote by $B(\cdot, \cdot)$ an invariant scalar product and let $Q(\cdot, \cdot)$ denote another invariant symmetric form on V. These two forms can be silmultaneously diagonalized. Therefore there exists $\lambda \in \mathbb{R}$, such that $\operatorname{Ker}(Q - \lambda B) \neq 0$. Since $\operatorname{Ker}(Q - \lambda B) \neq 0$ is G-invariant and ρ is irreducible, this implies $Q = \lambda B$. There we have

Lemma 7.1. *Let $\rho : G \to GL(V)$ be an irreducible representation of G over $\mathbb{R}$. There is exactly one invariant symmetric form on V up to scalar multiplication.*

Theorem 7.2. *Let $\mathbb{R} \subset K$ be a division ring, which is finite-dimensional over $\mathbb{R}$. Then $\mathbb{K}$ is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.*

Proof. If K is a field, then $K \cong \mathbb{R}$ or $\mathbb{C}$, because $\mathbb{C} = \bar{\mathbb{R}}$ and $[\mathbb{C} : \mathbb{R}] = 2$.

Assume that K is not commutative. Then it contains a subfield isomorphic to $\mathbb{C}$ obtained by taking $x \in K \backslash \mathbb{R}$ and conidering $\mathbb{R}$. Therefore without loss of generality we may assume $\mathbb{R} \subset \mathbb{C} \subset K$.

Consider the involutive $\mathbb{C}$-linear automoprhism of K defined by the formula

$$f(x) = ixi^{-1}.$$

Look at the eigenspace decomposition of K with respect to f

$$K = K_1 \oplus K_{-1},$$

where

$$K_{\pm 1} = \{x \in K \mid f(x) = \pm x\}.$$

One can easily check the following inclusions

$$K_1 K_1 \subset K_1, \quad K_{-1} K_{-1} \subset K_1, \quad K_1 K_{-1} \subset K_{-1}, \quad K_{-1} K_1 \subset K_{-1}.$$

The eigenspace K_1 coincides with the centralizer of $\mathbb{C}$ in K. Therefore $K_1 = \mathbb{C}$.

Choose a non-zero $y \in K^-$. The left multiplication by y defines an isomorphism of $\mathbb{R}$ vector spaces K_1 and K_{-1}. Hence $\dim_{\mathbb{R}} K_1 = \dim_{\mathbb{R}} K_{-1} = 2$ and $\dim K = 4$.

For any $z = a + bi \in K_1$ and any $w \in K_{-1}$, we have

$$w\bar{z} = wa - wbi = aw + biw = zw.$$

Since $w^2 \in \mathbb{C}$ and commutes with w, we have $w^2 \in \mathbb{R}$. We claim that w^2 is negative since otherwise $w^2 = c^2$ for some real c and $(w-c)(w+c) = 0$, which is impossible, since K is a division ring. Set $j := \frac{w}{\sqrt{-w^2}}$. Then $j^2 = -1$ and $ij = -ji$. So if we set $k := ij$, then $1, i, j, k$ form the standard basis of $\mathbb{H}$. $\square$

LEMMA 7.3. *Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation over $\mathbb{R}$, then there are three possibilities:*

(1) *$\mathrm{End}_G(V) = \mathbb{R}$ and $(\chi_\rho, \chi_\rho) = 1$;*
(2) *$\mathrm{End}_G(V) \cong \mathbb{C}$ and $(\chi_\rho, \chi_\rho) = 2$;*
(3) *$\mathrm{End}_G(V) \cong \mathbb{H}$ and $(\chi_\rho, \chi_\rho) = 4$.*

PROOF. Corollary 3.7 and Theorem 7.2 imply that $\mathrm{End}_G(V)$ is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, $(\chi_\rho, \chi_\rho) = 1, 2$ or 4 as follows from Corollary 4.7.

$\square$

8. Relationship between representations over $\mathbb{R}$ and over $\mathbb{C}$

Hermitian invariant form. Recall that a Hermitian form is a binary additive form on a complex vector space satisfying the conditions

$$H(av, bw) = \bar{a}bH(v, w), \; H(w, v) = \bar{H}(v, w).$$

The following Lemma can be proved exactly as Lemma 7.1.

LEMMA 8.1. *Every representation of a finite group over $\mathbb{C}$ admits a positive-definite invariant Hermitian form. If the representation is irreducible, then any two invariant Hermitian forms on it are proportional.*

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of dimension n over $\mathbb{C}$. Denote by $V^{\mathbb{R}}$ the space V considered as a vector space over $\mathbb{R}$ of dimension $2n$. Denote by $\rho^{\mathbb{R}}$ the representation of G in $V^{\mathbb{R}}$.

EXERCISE 8.2. Show that

$$\chi_{\rho^{\mathbb{R}}} = \chi_\rho + \bar{\chi}_\rho.$$

The exercise implies that $(\chi_{\rho^{\mathbb{R}}}, \chi_{\rho^{\mathbb{R}}})$ is either 2 or 4. Hence $\dim \mathrm{End}_G\left(V^{\mathbb{R}}\right)$ is either 2 or 4. Moreover, $\mathbb{C}$ is a self-centralizing subalgebra in $\mathrm{End}_G\left(V^{\mathbb{R}}\right)$. Therefore $\mathrm{End}_G\left(V^{\mathbb{R}}\right)$ is isomorphic to $\mathbb{C}$, $\mathbb{H}$ or to the ring $M_2(\mathbb{R})$ of real matrices of size 2×2.

PROPOSITION 8.3. *Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation over $\mathbb{C}$. Then one of the follwoing three cases occur.*

(1) $\mathrm{End}_G\left(V^{\mathbb{R}}\right) \simeq M_2(\mathbb{R})$. *Then there exists a basis of V such that the matrices ρ_g for all $g \in G$ have real entries. In this case V admits an invariant scalar product.*
(2) $\mathrm{End}_G\left(V^{\mathbb{R}}\right) \simeq \mathbb{C}$. *Then ρ is complex, i.e. ρ does not admit any invariant bilinear form.*
(3) $\mathrm{End}_G\left(V^{\mathbb{R}}\right) \simeq \mathbb{H}$. *Then ρ admits an invariant skew-symmetric form.*

Proof. The statement (1) follows from Lemma 7.1. For (2) use Exercise 8.3. Since $\left(\chi_{\rho^{\mathbb{R}}}, \chi_{\rho^{\mathbb{R}}}\right) = 2$ by Lemma 7.3, then $\chi_\rho \neq \bar{\chi}_\rho$, and therefore ρ is complex.

Finally let us prove (3). Let $j \in \operatorname{End}_G\left(V^{\mathbb{R}}\right) = \mathbb{H}$, then $j\,(bv) = \bar{b}v$ for all $b \in \mathbb{C}$. Let H be a positive-definite invariant Hermitian form on V. Then

$$Q\,(v, w) = H\,(jw, jv)$$

is another invariant positive-definite Hermitian form. By Lemma 8.1 $Q = \lambda H$ and λ should be positive because Q is also positive definite. Since $j^2 = -1$, one has $\lambda^2 = 1$ and therefore $\lambda = 1$. Thus,

$$H\,(v, w) = H\,(jw, jv)\,.$$

Set

$$B\,(v, w) = H\,(jv, w)\,.$$

Then B is a bilinear invariant form, and

$$B\,(w, v) = H\,(jw, v) = H\left(jv, j^2 w\right) = -H\,(jv, w) = -B\,(v, w)\,,$$

hence B is skew-symmetric. □

Corollary 8.4. *Let σ be an irreducible representation of G over $\mathbb{R}$. There are three possibilities for σ*

1. $\chi_\sigma = \chi_\rho$ *for some real representation ρ of G over $\mathbb{C}$;*
2. $\chi_\sigma = \chi_\rho + \bar{\chi}_\rho$ *for some complex representation ρ of G over $\mathbb{C}$;*
3. $\chi_\sigma = 2\chi_\rho$ *for some quaternionic representation ρ of G over $\mathbb{C}$.*

Theorem 8.5. *Let G be a finite group, r denote the number of conjugacy classes and s denote the number of classes which are stable under inversion. Then $\frac{r+s}{2}$ is the number of irreducible representations of G over $\mathbb{R}$.*

Proof. Recall that $\mathcal{C}(G)$ is the space of complex valued class functions on G. Consider the involution $\theta : \mathcal{C}(G) \to \mathcal{C}(G)$ given by

$$\theta\varphi(g) = \varphi(g^{-1}).$$

An easy calculation shows $\dim \mathcal{C}(G)^\theta = \frac{r+s}{2}$.

Denote by $\chi_1, \dots, \chi_r$ the irreducible characters of G over $\mathbb{C}$. Recall that $\chi_1, \dots, \chi_r$ form a basis of $\mathcal{C}(G)$. Observe that for any character χ_ρ

$$\theta(\chi_\rho) = \chi_{\rho^*}.$$

Therefore θ permutes irreducible characters $\chi_1, \dots, \chi_r$. Corollary 8.4 implies that the number of irreducible representations of G over $\mathbb{R}$ equals the number of self-dual irreducible representations over $\mathbb{C}$ plus half the number of those which are not self-dual. Therefore this number is exactly the dimension of $\mathcal{C}(G)^\theta$. □

CHAPTER 2

Modules with applications to finite groups

1. Modules over associative rings

1.1. The notion of module.

Definition 1.1. Let R be an associative ring with identity element $1 \in R$. An abelian group M is called a *(left) R-module* if there is a map $R \times M \to M$, $(a, m) \mapsto am$ such that for all $a, b \in R$ and $m, n \in M$ we have

(1) $(ab)\,m = a\,(bm)$;
(2) $1m = m$;
(3) $(a + b)\,m = am + bm$;
(4) $a\,(m + n) = am + an$.

One can define in the similar way a right R-module. Unless otherwise stated we only consider left modules and we say module for left module.

Example 1.2. If R is a field then R-modules are vector spaces over R.

Example 1.3. Let G be a group and $k\,(G)$ be its group algebra over k. Then every $k(G)$-module V is a vector space over k equipped with a G-action. Set

$$\rho_g v := gv$$

for all $g \in G \subset k\,(G)$, $v \in V$. This defines a representation $\rho : G \to GL(V)$.

Conversely, if V is a vector space over k and $\rho\colon G \to \mathrm{GL}\,(V)$ is a representation, the formula

$$\left(\sum_{g \in G} a_g g\right) v := \sum_{g \in G} a_g \rho_g v.$$

defines a $k(G)$-module structure on V.

In other words, to study representations of G over k is exactly the same as to study $k(G)$-modules. Hence from now on we will talk indifferently of $k(G)$-modules, representations of G over k or just simply G-modules over k.

Definition 1.4. Let M be an R-module. A *submodule* $N \subset M$ is a subgroup which is invariant under the R-action. If $N \subset M$ is a submodule then the quotient M/N has a natural R-module structure. A non-zero module M is *simple* or *irreducible* if all submodules are either zero or M.

Remark 1.5. Sums and an intersections of submodules are submodules.

EXAMPLE 1.6. If R is an arbitrary ring, then R is a left R-module with action given by left multiplication. Its submodules are the left ideals.

Let $\{M_j\}_{j\in J}$ be a family of R-modules. We define the direct sum $\bigoplus_{j\in J} M_j$ and the direct product $\prod_{j\in J} M_j$ in the obvious way. An R-module is *free* if it is isomophic to a direct sum of I copies of R, (I can be infinite).

EXERCISE 1.7. If R is a division ring, then every non-zero R-module is free.

EXERCISE 1.8. Let $R = \mathbb{Z}$ be the ring of integers.
(a) Show that any simple $\mathbb{Z}$-module is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p.
(b) Let M be a $\mathbb{Z}$-module. We call $m \in M$ a *torsion* element if $rm = 0$ for some non-zero $r \in \mathbb{Z}$. Prove that the subset M^{tor} of all torsion elements is a submodule.
(c) We say M is a *torsion free* if $M^{\text{tor}} = 0$. Prove that M/M^{tor} is torsion free.
(d) Give an example of a non-zero torsion free $\mathbb{Z}$-module which is not free.

Let M and N be R-modules. In the same way as in the group case we define the abelian group $\operatorname{Hom}_R(M,N)$ of R-invariant homomorphisms from M to N and the ring $\operatorname{End}_R(M)$ of R-invariant endomorphisms of M. In particular if k is a filed and V is an n-dimensional vector space, then $\operatorname{End}_k(V)$ is the matrix ring $M_n(k)$.

In this context we have the following formulation of Schur's Lemma. Its proof is the same as in the group case.

LEMMA 1.9. *Let M and N be simple R-modules. If $\varphi \in \operatorname{Hom}_R(M,N)$ is not zero then it is an isomorphism.*

If M is a simple module, then $\operatorname{End}_R(M)$ is a division ring.

1.2. A group algebra is a product of matrix rings. Recall that for every ring R one defines R^{op} as the ring with the same abelian group structure together with the new multiplication $*$ given by

$$a * b = ba.$$

LEMMA 1.10. *The ring $\operatorname{End}_R(R)$ is isomorphic to R^{op}.*

PROOF. For all $a \in R$, define $\varphi_a \in \operatorname{End}(R)$ by the formula

$$\varphi_a(x) = xa.$$

It is easy to check that
- $\varphi_a \in \operatorname{End}_R(R)$,
- $\varphi_{ba} = \varphi_a \circ \varphi_b$.

In this way we have constructed a homomorphism

$$\varphi : R^{\text{op}} \to \operatorname{End}_R(R).$$

All we have to show that this is an isomorphism.

Injectivity: Assume $\varphi_a = \varphi_b$. Then $\varphi_a(1) = \varphi_b(1)$, hence $a = b$.

Surjectivity: let $\gamma \in \operatorname{End}_R(R)$. One has for all $x \in R$

$$\gamma(x) = \gamma(x1) = x\gamma(1).$$

Therefore $\gamma = \varphi_{\gamma(1)}$. □

LEMMA 1.11. *Let $\rho_i : G \to \mathrm{GL}(V_i)$, $i = 1, \dots, l$, be a finite set of pairwise non-isomorphic irreducible representations of a finite group G over an algebraically closed field k, and let*

$$V = V_1^{\oplus m_1} \oplus \cdots \oplus V_l^{\oplus m_l}.$$

Then

$$\operatorname{End}_G(V) \cong M_{m_1}(k) \times \cdots \times M_{m_l}(k).$$

PROOF. If φ is an element of $\operatorname{End}_G(V)$, then Schur's Lemma implies that φ preserves isotypic components. Therefore we have an isomorphism

$$\operatorname{End}_G(V) \cong \operatorname{End}_G\left(V_1^{\oplus m_1}\right) \times \cdots \times \operatorname{End}_G\left(V_l^{\oplus m_l}\right).$$

Thus it suffices to prove the following

LEMMA 1.12. *Let G be a finite group, k be an algebraically closed field of characteristic zero and W be a simple $k(G)$-module. Then $\operatorname{End}_G(W^{\oplus m})$ is isomorphic to the matrix ring $M_m(k)$.*

PROOF. For all $i, j = 1, \dots, m$ denote by p_j the canonical projection of $W^{\oplus m}$ onto its j-th factor and by q_i the emebedding of W as the i-th factor into $W^{\oplus m}$. Take $\varphi \in \operatorname{End}_G(W^{\oplus m})$. For all $i, j = 1, \dots, m$ denote by φ_{ij} the composition map

$$W \xrightarrow{q_i} W^{\oplus m} \xrightarrow{\varphi} W^{\oplus m} \xrightarrow{p_j} W.$$

Since $\varphi_{ij} \in \operatorname{End}_G(W)$, Schur's Lemma implies

$$\varphi_{ij} = c_{ij} \operatorname{Id}_W$$

for some $c_{ij} \in k$. Thus we obtain a map

$$\Phi : \operatorname{End}\left(W^{\oplus m}\right) \to M_m(k).$$

Moreover, φ can be written uniquely as

$$\varphi = \sum_{i,j=1}^{m} c_{ij} q_i \circ p_j.$$

If ψ is another element in $\operatorname{End}(W^{\oplus m})$ we write

$$\psi = \sum_{i,j=1}^{m} d_{ij} q_i \circ p_j.$$

Then we have, for the composition

$$\varphi \circ \psi = \sum_{i,j,k=1}^{m} c_{ik} d_{kj} q_i \circ p_j.$$

This shows that Φ is a homomorphism of rings. Injectivity and surjectivity of Φ are direct consequences of the definition. □

□

THEOREM 1.13. *Let G be a finite group. Assume k is algebraically closed and* $\operatorname{char} k = 0$. *Then*

$$k\left(G\right) \cong M_{n_1}(k) \times \cdots \times M_{n_r}(k),$$

where $n_1, \ldots, n_r$ are the dimensions of all up to isomorphism irreducible representations.

PROOF. By Lemma 1.10

$$\operatorname{End}_{k(G)}\left(k\left(G\right)\right) \cong k\left(G\right)^{\mathrm{op}}.$$

Moreover, $g \mapsto g^{-1}$ gives an isomorphism

$$k\left(G\right)^{\mathrm{op}} \cong k\left(G\right).$$

On the other hand, by Theorem 4.13 Chapter 1 one has

$$k\left(G\right) = V_1^{\oplus n_1} \oplus \cdots \oplus V_r^{\oplus n_r},$$

where $V_1, \ldots, V_r$ are simple G-modules. Applying Lemma 1.11 we get the theorem. □

2. Finitely generated modules and Noetherian rings.

DEFINITION 2.1. An R-module M is *finitely generated* if there exist finitely many elements $x_1, \ldots, x_n \in M$ such that $M = Rx_1 + \cdots + Rx_n$.

LEMMA 2.2. *Let*

$$0 \to N \xrightarrow{q} M \xrightarrow{p} L \to 0$$

be an exact sequence of R-modules.

(a) If M is finitely generated, then L is finitely generated.

(b) If N and L are finitely generated, then M is finitely generated.

PROOF. The first assertion is obvious. For the second let

$$L = Rx_1 + \cdots + Rx_n, \quad N = Ry_1 + \cdots + Ry_m,$$

then one has $M = Rp^{-1}\left(x_1\right) + \cdots + Rp^{-1}\left(x_n\right) + Rq\left(y_1\right) + \cdots + Rq\left(y_m\right)$. □

LEMMA 2.3. *Let R be a ring. The following conditions are equivalent*

(1) *Every increasing chain of left ideals in R is finite, in other words for any sequence $I_1 \subset I_2 \subset \dots$ of left ideals, there exists n_0 such that for all $n > n_0$, $I_n = I_{n_0}$.*
(2) *Every left ideal is a finitely generated R-module.*

Proof. (1) $\Rightarrow$ (2). Assume that some left ideal I is not finitely generated. Then there exists an infinite sequence of $x_n \in I$ such that

$$x_{n+1} \notin Rx_1 + \cdots + Rx_n.$$

But then $I_n = Rx_1 + \cdots + Rx_n$ form an infinite increasing chain of ideals which does not stabilize.

(2) $\Rightarrow$ (1). Let $I_1 \subset I_2 \subset \dots$ be an increasing chain of ideals. Consider

$$I := \bigcup_n I_n.$$

Then by (2) I is finitely generated. Therefore $I = Rx_1 + \cdots + Rx_s$ for some $x_1, \dots x_s \in I$. Then there exists n_0 such that $x_1, \dots, x_s \in I_{n_0}$. Hence $I = I_{n_0}$ and the chain stabilizes. $\square$

Definition 2.4. A ring satisfying the conditions of Lemma 2.3 is called *(left) Noetherian*.

Lemma 2.5. *Let R be a left Noetherian ring and M be a finitely generated R-module. Then every submodule of M is finitely generated.*

Proof. First, we prove the statement when M is free. Then M is isomorphic to R^n for some n and we use induction on n. For $n = 1$ the statement follows from definition. Consider the exact sequence

$$0 \to R^{n-1} \to R^n \to R \to 0.$$

Let N be a submodule of R^n. Consider the exact sequence obtained by restriction to N

$$0 \to N \cap R^{n-1} \to N \to N' \to 0.$$

By induction assumption $N \cap R^{n-1}$ is finitely generated and $N' \subset R$ is finitely generated. Therefore by Lemma 2.2 (b), N is finitely generated.

In the general case M is a quotient of a free module of finite rank. We use the exact sequence

$$0 \to K \to R^n \xrightarrow{p} M \to 0.$$

If N is a submodule of M, then $p^{-1}(N) \subset R^n$ is finitely generated. Therefore by Lemma 2.2 (a), N is also finitely generated. $\square$

Exercise 2.6. (a) A principal ideal domain is a Noetherian ring. In particular, $\mathbb{Z}$ and the polynomial ring $k[X]$ are Noetherian.

(b) Show that the polynomial ring $k[X_1, \dots, X_n, \dots]$ of infinitely many variables is not Noetherian.

(c) A subring of a Noetherian ring is not automatically Noetherian. For example, let R be a subring of $\mathbb{C}[X,Y]$ consisting of polynomial functions constant on the cross $X^2 - Y^2 = 0$. Show that R is not Noetherian.

Let R be a commutative ring. An element $r \in R$ is called *integral over* $\mathbb{Z}$ if there exists a monic polynomial $p(X) \in \mathbb{Z}[X]$ such that $p(r) = 0$.

Exercise 2.7. Check that r is integral over $\mathbb{Z}$ if and only if $\mathbb{Z}[r] \subset R$ is a finitely generated $\mathbb{Z}$-module.

Remark. The complex numbers which are integral over $\mathbb{Z}$ are usually called algebraic integers. All the rational numbers which are integral over $\mathbb{Z}$ belong to $\mathbb{Z}$.

Lemma 2.8. *Let R be a commutative ring and S be the set of elements integral over $\mathbb{Z}$. Then S is a subring of R.*

Proof. Let $x, y \in S$. By assumption $\mathbb{Z}[x]$ and $\mathbb{Z}[y]$ are finitely generated $\mathbb{Z}$-modules. Then $\mathbb{Z}[x,y]$ is also finitely generated. Since $\mathbb{Z}$ is Noetherian ring, Lemma 2.5 implies that for every $s \in \mathbb{Z}[x,y]$ the $\mathbb{Z}$-submodule $\mathbb{Z}[s]$ is finitely generated. □

3. The center of the group algebra $k(G)$

In this section we assume that k is algebraically closed of characteristic 0 and G is a finite group. In this section we obtain some results about the center $Z(G)$ of the group ring $k(G)$. It is clear that $Z(G)$ can be identified with the subspace of class functions:

$$Z(G) = \left\{ \sum_{s \in G} f(s)\, s \mid f \in \mathcal{C}(G) \right\}.$$

Recall that if $n_1, \dots, n_r$ are the dimensions of isomorphism classes of simple G-modules, then by Theorem 1.13 we have an isomorphism

$$k(G) \simeq M_{n_1}(k) \times \dots \times M_{n_r}(k).$$

If $e_i \in k(G)$ denotes the element corresponding to the identity matrix in $M_{n_i}(K)$, the $e_1, \dots, e_r$ form a basis of $Z(G)$ and one has

$$e_i e_j = \delta_{ij} e_i$$
$$1_G = e_1 + \dots + e_r.$$

If $\rho_j : G \to \mathrm{GL}(V_j)$ is an irreducible representation, then e_j acts on V_j as the identity element and we have

$$\rho_j(e_i) = \delta_{ij}\, \mathrm{Id}_{V_j}. \tag{2.1}$$

Lemma 3.1. *If χ_i is the character of the irreducible representation ρ_i of dimension n_i, then one has*

$$e_i = \frac{n_i}{|G|} \sum_{g \in G} \chi_i\left(g^{-1}\right) g. \tag{2.2}$$

PROOF. We have to check (2.1). Since $\rho_j(e_i)$ belongs to $\mathrm{End}_G(V_j)$, Schur's Lemma implies $\rho_j(e_i) = \lambda \,\mathrm{Id}$ for some λ. Now we use orthogonality relations, Theorem 4.8

$$\operatorname{tr} \rho_j(e_i) = \frac{n_i}{|G|} \sum \chi_i\left(g^{-1}\right) \chi_j(g) = \frac{n_i}{|G|}(\chi_i, \chi_j) = \delta_{ij} n_i.$$

Therefore we have $n_j \lambda = \delta_{ij} n_i$ which implies $\lambda = \delta_{ij}$. □

EXERCISE 3.2. Define $\omega_i : Z(G) \to k$ by the formula

$$\omega_i\left(\sum a_s s\right) = \frac{1}{n_i} \sum a_s \chi_i(s).$$

and $\omega : Z(G) \to k^r$ by

$$\omega = (\omega_1, \dots, \omega_r).$$

Check that ω is an isomorphism of rings. Hint: check that $\omega_i(e_j) = \delta_{ij}$ using again the orthogonality relations.

For any conjugacy class C in G let

$$\eta_C := \sum_{g \in C} g.$$

Clearly, the set η_C for C running the set of conjugacy classes is a basis in $Z(G)$.

LEMMA 3.3. *For any conjugacy class $C \subset G$ we have*

$$\eta_C = |C| \sum_{i=1}^{r} \frac{\chi_i(g)}{n_i} e_i,$$

where g is any element of C.

PROOF. If we extend by linearity $\chi_1, \ldots \chi_r$ to linear functionals on $k(G)$, then (2.1) implies $\chi_j(e_i) = n_i \delta_{i,j}$. Thus, $\chi_1, \dots, \chi_r$ form a basis in the dual space $Z(G)^*$. Therefore it suffices to check that

$$\chi_j(\eta_c) = |C| \sum_{i=1}^{r} \frac{\chi_i(g)}{n_i} \chi_j(e_i) = |C| \chi_j(g).$$

□

LEMMA 3.4. *If $g, h \in G$ lie in the same conjugacy class C, we have*

$$\sum_{i=1}^{r} \chi_i(g) \chi_i(h^{-1}) = \frac{|G|}{|C|}.$$

If g and h are not conjugate we have

$$\sum_{i=1}^{r} \chi_i(g) \chi_i(h^{-1}) = 0.$$

Proof. The statement follows from Lemma 3.1 and Lemma 3.3. Indeed, if g is in the congugacy class C, we have

$$\eta_c = |C| \sum_{i=1}^{r} \frac{\chi_i(g)}{n_i} e_i = \frac{|C|}{|G|} \sum_{i=1}^{r} \sum_{h \in G} \chi_i(g)\chi_i(h^{-1})h.$$

The coefficient of h in the last expression is 1 if $h \in C$ and zero otherwise. This implies the lemma. □

Lemma 3.5. *Let $u = \sum_{g \in G} a_g g \in Z(G)$. If all a_g are algebraic integers, then u is integral over $\mathbb{Z}$.*

Proof. Consider the basis η_C of $Z(G)$. Every η_C is integral over $\mathbb{Z}$ since the subring generated by all η_C is a finitely generated $\mathbb{Z}$-module. Now the statement follows from Lemma 2.8. □

Theorem 3.6. *Let ρ be an n-dimenisonal irreducible representation of G. Then n divides $|G|$.*

Proof. For every $g \in G$, all eigenvalues of $\rho(g)$ are roots of 1. Therefore $\chi_\rho(g)$ is an algebraic integer. By Lemma 3.5 $u = \sum_{g \in G} \chi_\rho(g)\, g$ is integral over $\mathbb{Z}$. Recall the homomorphism ω_i from Exercise 3.2. Since $\omega_i(u)$ is an algebraic integer we have

$$\omega_i(u) = \frac{1}{n_i} \sum \chi_i(s)\, \chi_i\left(s^{-1}\right) = \frac{|G|}{n_i} (\chi_i, \chi_i) = \frac{|G|}{n_i}.$$

Therefore $\frac{|G|}{n_i} \in \mathbb{Z}$. □

Theorem 3.7. *Let Z be the center of G and ρ be an irreducible n-dimensional representation of G. Then n divides $\frac{|G|}{|Z|}$.*

Proof. Let G^m be the direct product of m copies of G and ρ^m be the exterior product of m copies of ρ. The dimension of ρ^m is n^m. Furthermore, ρ^m is irreducible by Exercise 4.12. Consider the normal subgroup N of G^m defined by

$$N = \{(z_1, \dots, z_m) \in Z^m \mid z_1 z_2 \dots z_m = 1\}.$$

We have $|N| = |Z|^{m-1}$. Furthermore, N lies in the kernel of ρ^m. Therefore ρ^m is a representation of the quotient group $H = G/N$. Hence, by Theorem 3.6, n^m divides $\frac{|G|^m}{|Z|^{m-1}}$ for every $m > 0$. It follows from prime factorization that n divides $\frac{|G|}{|Z|}$. □

4. Generalities on induced modules

Let A be a ring, B be a subring of A and M be a B-module. Consider the abelian group $A \otimes_B M$ defined by generators and relations in the following way. The generators are all elements of the Cartesian product $A \times M$ and the relations:

$$(a_1 + a_2) \times m - a_1 \times m - a_2 \times m, \quad a_1, a_2 \in A, m \in M, \tag{2.3}$$

$$a \times (m_1 + m_2) - a \times m_1 - a \times m_2, \quad a \in A, m_1, m_2 \in M, \tag{2.4}$$

$$ab \times m - a \times bm, \quad a \in A, b \in B, m \in M. \tag{2.5}$$

This group has a structure of A-module, A acting on it by left multiplication. For every $a \in A$ and $m \in M$ we denote by $a \otimes m$ the corresponding element in $A \otimes_B M$.

Definition 4.1. The A-module $A \otimes_B M$ is called the induced module.

Exercise 4.2. (a) Show that $A \otimes_B B$ is isomoprhic to A.

(b) Show that if M_1 and M_2 are two B-modules, then there exists a canonical isomorphism of A-modules

$$A \otimes_B (M_1 \oplus M_2) \simeq A \otimes_B M_1 \oplus A \otimes_B M_2.$$

(c) Check that for any $n \in \mathbb{Z}$ one has

$$\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0.$$

Theorem 4.3. *(Frobenius reciprocity.) For every B-module M and for every A-module N, there is an isomorphism of abelian groups*

$$\operatorname{Hom}_B (M, N) \cong \operatorname{Hom}_A (A \otimes_B M, N).$$

Proof. Let M be a B-module and N be an A-module. Consider $j : M \to A \otimes_B M$ defined by

$$j (m) := 1 \otimes m,$$

which is a homomorphism of B-modules.

Lemma 4.4. *For every $\varphi \in \operatorname{Hom}_B (M, N)$ there exists a unique $\psi \in \operatorname{Hom}_A (A \otimes_B M, N)$ such that $\psi \circ j = \varphi$. In other words, the following diagram is commutative*

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[dr, "\varphi"'] & A \otimes_B M \arrow[d, "\psi"] \\ & N \end{tikzcd}$$

Proof. We define ψ by the formula

$$\psi(a \otimes m) := a\varphi(m),$$

for all $a \in A$ and $m \in M$. The reader can check that ψ is well defined, i.e. the relations defining $A \otimes_B M$ are preserved by ψ. That proves the existence of ψ

To check uniqueness we just note that for all $a \in A$ and $m \in M$, ψ must satisfy the relation

$$\psi (a \otimes m) = a\psi (1 \otimes m) = a\varphi (m).$$

$\square$

To prove the theorem we observe that by the above lemma the map $\psi \mapsto \varphi := \psi \circ j$ gives an isomorphism between $\mathrm{Hom}_A(A \otimes_B M, N)$ and $\mathrm{Hom}_B(M, N)$. $\square$

REMARK 4.5. For readers familiar with category theory the former theorem can be reformulated as follows. Since any A-module M is automatically a B-module, we have a natural functor Res from the category of A-modules to the category of B-modules. This functor is usually called the *restriction* functor. The *induction* functor Ind from the category of B-modules to the category of A-modules which sends M to $A \otimes_B M$ is left adjoint of Res.

EXAMPLE 4.6. Let $k \subset F$ be a field extension. For any vector space M over k, $F \otimes_k M$ is a vector space of the same dimension over F. If we have an exact sequence of vector spaces

$$0 \to N \to M \to L \to 0,$$

then the sequence

$$0 \to F \otimes_k N \to F \otimes_k M \to F \otimes_k L \to 0$$

is also exact. In other words the induction in this situation is an exact functor.

EXERCISE 4.7. Let A be a ring and B be a subring of A.

(a) Show that if a sequence of B-modules

$$N \to M \to L \to 0$$

is exact, then the sequence

$$A \otimes_B N \to A \otimes_B M \to A \otimes_B L \to 0$$

of induced modules is also exact. In other words the induction functor is right exact.

(b) Assume that A is a free right B-module, then the induction functor is exact. In other words, if a sequence

$$0 \to N \to M \to L \to 0$$

of B-modules is exact, then the sequence

$$0 \to A \otimes_B N \to A \otimes_B M \to A \otimes_B L \to 0$$

is also exact.

(c) Let $A = \mathbb{Z}[X]/(X^2, 2X)$ and $B = \mathbb{Z}$. Consider the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

where φ is the multiplication by 2. Check that after applying induction we get a sequence of A-modules

$$0 \to A \to A \to A/2A \to 0,$$

which is not exact.

Later we discuss general properties of induction but now we are going to study induction for the case of groups.

5. Induced representations for groups.

Let G be a finite group. Let H be a subgroup of G and $\rho : H \to \mathrm{GL}(V)$ be a representation of H with characater χ. Then the induced representation $\mathrm{Ind}_H^G \rho$ is by definition the $k(G)$-module

$$k(G) \otimes_{k(H)} V.$$

The following lemma has a straightforward proof.

Lemma 5.1. *The dimension of* $\mathrm{Ind}_H^G \rho$ *equals the product of* $\dim \rho$ *by the index* $[G : H]$ *of* H*. More precisely, let* S *be a set of representatives of left cosets in* G/H*, i.e.*

$$G = \coprod_{s \in S} sH,$$

then

$$k(G) \otimes_{k(H)} V = \bigoplus_{s \in S} s \otimes V. \tag{2.6}$$

Moreover, for any $g \in G$*,* $s \in S$ *there exists a unique* $s' \in S$ *such that* $(s')^{-1}gs \in H$*. Then the action of* g *on* $s \otimes v$ *for all* $v \in V$ *is given by*

$$g(s \otimes v) = s' \otimes \rho_{(s')^{-1}gs} v. \tag{2.7}$$

Example 5.2. Let ρ be the trivial representation of H. Then $\mathrm{Ind}_H^G \rho$ is the permutation representation of G obtained from the natural left action of G on the set of left cosets G/H, see Example 3 in Section 4.2 Chapter 1.

Lemma 5.3. *We keep the notations of the previous lemma. Denote by* $\mathrm{Ind}_H^G \chi$ *the character of the induced representation. Then one has for* $g \in G$

$$\mathrm{Ind}_H^G \chi(g) = \sum_{s \in S, s^{-1}gs \in H} \chi\left(s^{-1}gs\right). \tag{2.8}$$

Proof. (2.6) and (2.7) imply

$$\mathrm{Ind}_H^G \chi(g) = \sum_{s \in S} \delta_{s,s'} \operatorname{tr} \rho_{(s')^{-1}gs}.$$

∎

Corollary 5.4. *In the notations of Lemma 5.3 we have*

$$\mathrm{Ind}_H^G \chi(g) = \sum_{u \in G, u^{-1}gu \in H} \chi\left(u^{-1}gu\right).$$

Proof. If $s^{-1}gs \in H$, then for all $u \in sH$ we have $\chi(u^{-1}gu) = \chi(s^{-1}gs)$. Therefore

$$\chi(s^{-1}gs) = \frac{1}{|H|} \sum_{u \in sH} \chi\left(u^{-1}gu\right).$$

Hence the statement follows from (2.8). ∎

COROLLARY 5.5. *Let H be a normal subgroup in G. Then* $\operatorname{Ind}_H^G \chi\,(g) = 0$ *for any* $g \notin H$.

EXERCISE 5.6. (a) Let $G = S_3$ and $H = A_3$ be its normal cyclic subgroup. Consider, a one-dimensional representation of H such that $\rho(123) = \varepsilon$, where ε is a primitive 3-d root of 1. Show that then

$$\operatorname{Ind}_H^G \chi_\rho\,(1) = 2,$$

$$\operatorname{Ind}_H^G \chi_\rho\,(12) = 0,$$

$$\operatorname{Ind}_H^G \chi_\rho\,(123) = -1.$$

Therefore $\operatorname{Ind}_H^G \rho$ is the irreducible 2-dimensional representation of S_3.

(b) Next, consider the 2-element subgroup K of $G = S_3$ generated by the transposition (12), and let σ be the (unique) non-trivial one-dimensional representation of K. Show that

$$\operatorname{Ind}_K^G \chi_\sigma\,(1) = 3,$$

$$\operatorname{Ind}_K^G \chi_\sigma\,(12) = -1,$$

$$\operatorname{Ind}_H^G \chi_\rho\,(123) = 0.$$

Therefore $\operatorname{Ind}_K^G \sigma$ is the direct sum of the sign representation and the 2-dimensional irreducible representation.

Now we assume that k has characteristic zero. Let us recall that, in Section 4.2 Chapter 1, we defined a scalar product on the space $\mathcal{C}(G)$ of class functions by (1.2). When we consider several groups at the same time we specify the group by the a lower index.

THEOREM 5.7. *Consider two representations* $\rho : G \to \mathrm{GL}\,(V)$ *and* $\sigma \colon H \to \mathrm{GL}\,(W)$. *Then we have the identity*

$$\left(\operatorname{Ind}_H^G \chi_\sigma, \chi_\rho\right)_G = \left(\chi_\sigma, \operatorname{Res}_H \chi_\rho\right)_H . \tag{2.9}$$

PROOF. The statement follows from Frobenius reciprocity (Theorem 4.3) and Corollary 4.7 in Chapter 1, since

$$\dim \operatorname{Hom}_G\left(\operatorname{Ind}_H^G W, V\right) = \dim \operatorname{Hom}_H (W, V)\,.$$

$\square$

EXERCISE 5.8. Prove Theorem 5.7 directly from Corollary 5.4. Define two maps

$$\operatorname{Res}_H : \mathcal{C}\,(G) \to \mathcal{C}\,(H)\,,\ \operatorname{Ind}_H^G : \mathcal{C}\,(H) \to \mathcal{C}\,(G)\,,$$

the former is the restriction on a subgroup, the latter is defined by (2.8). Then for any $\varphi \in \mathcal{C}\,(G)\,, \psi \in \mathcal{C}\,(H)$

$$\left(\operatorname{Ind}_H^G \varphi, \psi\right)_G = \left(\varphi, \operatorname{Res}_H \psi\right)_H .$$

6. Double cosets and restriction to a subgroup

If K and H are subgroups of G one can define the equivalence relation on G: $s \sim t$ if and only if $s \in KtH$. The equivalence classes are called *double cosets*. We can choose a set of representative $T \subset G$ such that

$$G = \coprod_{t \in T} K \,\mathrm{tH}\,.$$

We define the set of double cosets by $K\backslash G/H$. One can identify $K\backslash G/H$ with K-orbits on $S = G/H$ in the obvious way and with G-orbits on $G/K \times G/H$ by the formula

$$KtH \to G\,(K, tH)\,.$$

Example 6.1. Let $\mathbb{F}$ be a field. Let $G = \mathrm{GL}_2\,(\mathbb{F})$ be the group of all invertible 2×2 matrices with coefficients in $\mathbb{F}$. Consider the natural action of G on $\mathbb{F}^2$. Let B be the subgroup of upper-triangular matrices in G. We denote by $\mathbb{P}^1$ the projective line which is the set of all one-dimensional linear subspaces of $\mathbb{F}^2$. Clearly, G acts on $\mathbb{P}^1$.

Exercise 6.2. Prove that G acts transitively on $\mathbb{P}^1$ and that the stabilizer of any point in $\mathbb{P}^1$ is isomorphic to B.

By the above exercise one can identify G/B with the set of lines $\mathbb{P}^1$. The set of double cosets $B\backslash G/B$ can be identified with the set of G-orbits in $\mathbb{P}^1 \times \mathbb{P}^1$ or with the set of B-orbits in $\mathbb{P}^1$.

Exercise 6.3. Check that G has only two orbits on $\mathbb{P}^1 \times \mathbb{P}^1$: the diagonal and its complement. Thus, $|B\backslash G/B| = 2$ and

$$G = B \cup BsB,$$

where

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Theorem 6.4. *Let $T \subset G$ such that $G = \coprod_{s \in T} KsH$. Then*

$$\mathrm{Res}_K \mathrm{Ind}_H^G \rho = \oplus_{s \in T} \mathrm{Ind}_{K \cap sHs^{-1}}^K \rho^s,$$

where

$$\rho_h^s \overset{def}{=} \rho_{s^{-1}hs},$$

for any $h \in sHs^{-1}$.

Proof. Let $s \in T$ and $W^s = k\,(K)\,(s \otimes V)$. Then by construction, W^s is K-invariant and

$$k\,(G) \otimes_{k(H)} V = \oplus_{s \in T} W^s.$$

Thus, we need to check that the representation of K in W^s is isomorphic to $\operatorname{Ind}_{K\cap \mathrm{sHs}^{-1}}^K \rho^s$. We define a homomorphism

$$\alpha : \operatorname{Ind}_{K\cap sHs^{-1}}^K V \to W^s$$

by $\alpha(t\otimes v) = ts\otimes v$ for any $t\in K, v\in V$. It is well defined

$$\alpha\left(th\otimes v - t\otimes \rho_h^s v\right) = ths\otimes v - ts\otimes \rho_{s^{-1}hs}v = ts\left(s^{-1}hs\right)\otimes v - ts\otimes \rho_{s^{-1}hs}v = 0$$

and obviously surjective. Injectivity can be proved by counting dimensions. □

EXAMPLE 6.5. Let us go back to our example $B\subset \mathrm{SL}_2(\mathbb{F})$ (see Exercise 6.3). We now assume that $\mathbb{F}=\mathbb{F}_q$ is the finite field with q elements. Theorem 6.4 tells us that for any representation ρ of B

$$\operatorname{Ind}_B^G \rho = \rho \oplus \operatorname{Ind}_H^G \rho',$$

where $H = B\cap sBs^{-1}$ is a subgroup of diagonal matrices and

$$\rho'\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \rho\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$$

COROLLARY 6.6. *If H is a normal subgroup of G, then*

$$\operatorname{Res}_H \operatorname{Ind}_H^G \rho = \oplus_{s\in G/H}\rho^s.$$

7. Mackey's criterion

In order to compute $\left(\operatorname{Ind}_H^G \chi, \operatorname{Ind}_H^G \chi\right)$, we use Frobenius reciprocity and Theorem 6.4. One has:

$$\left(\operatorname{Ind}_H^G \chi, \operatorname{Ind}_H^G \chi\right)_G = \left(\operatorname{Res}_H \operatorname{Ind}_H^G \chi, \chi\right)_H = \sum_{s\in T}\left(\operatorname{Ind}_{H\cap sHs^{-1}}^H \chi^s, \chi\right)_H =$$

$$= \sum_{s\in T}\left(\chi^s, \operatorname{Res}_{H\cap sHs^{-1}}\chi\right)_{H\cap sHs^{-1}} = (\chi,\chi)_H + \sum_{s\in T\setminus\{1\}}\left(\chi^s, \operatorname{Res}_{H\cap sHs^{-1}}\chi\right)_{H\cap sHs^{-1}}.$$

We call two representation *disjoint* if they do not have any irreducible component in common, or in other words if their characters are orthogonal.

THEOREM 7.1. *(Mackey's criterion) The representation $\operatorname{Ind}_H^G \rho$ is irreducible if and only if ρ is irreducible and ρ^s and ρ are disjoint representations of $H\cap sHs^{-1}$ for all $s\in T\setminus\{1\}$.*

PROOF. Write the condition

$$\left(\operatorname{Ind}_H^G \chi, \operatorname{Ind}_H^G \chi\right)_G = 1$$

and use the above formula. □

COROLLARY 7.2. *Let H be a normal subgroup of G. Then $\operatorname{Ind}_H^G \rho$ is irreducible if and only if ρ^s is not isomorphic to ρ for any $s\in G/H$, $s\notin H$.*

REMARK 7.3. Note that if H is normal, then G/H acts on the set of representations of H. In fact, this is a part of the action of the group $\operatorname{Aut} H$ of automorphisms of H on the set of representation of H. Indeed, if $\varphi \in \operatorname{Aut} H$ and $\rho : H \to \mathrm{GL}(V)$ is a representation, then $\rho^{\varphi} : H \to \mathrm{GL}(V)$ defined by

$$\rho^{\varphi}_t = \rho_{\varphi(t)},$$

is a new representation of H.

8. Hecke algebras, a first glimpse

DEFINITION 8.1. Let G be a group, $H \subset G$ a subgroup, consider $\mathcal{H}(G,H) \subset k(G)$ defined by:

$$\mathcal{H}(G,H) := \operatorname{End}_G(\operatorname{Ind}_H^G \operatorname{triv}).$$

This is the *Hecke algebra* associated to the pair (G,H).

Define the projector

$$\Pi_H := \frac{1}{|H|} \sum_{h \in H} h \in k(G).$$

EXERCISE 8.2. Show that

$$\operatorname{Ind}_H^G \operatorname{triv} = k(G)\Pi_H.$$

Applying Frobenius reciprocity, one gets:

$$\operatorname{End} G \operatorname{Ind}_H^G \operatorname{triv} = \operatorname{Hom}_H(\operatorname{triv}, \operatorname{Ind}_H^G \operatorname{triv}).$$

We can identify the Hecke algebra with $\Pi_H k(G) \Pi_H$. Therefore a basis of the Hecke algebra can be enumerated by the double cosets, i.e. elements of $H \backslash G / H$.

Set, for $g \in G$,

$$\eta_g := \Pi_H g \Pi_H.$$

it is clear that those functions are constant on double cosets and give a basis of the Hecke algebra. Then, the multiplication is given by the formula

$$\eta_g \eta_{g'} = \sum_{g'' \in G} \frac{1}{|H|} |gHg' \cap Hg''H| \eta_{g''}. \tag{2.10}$$

EXERCISE 8.3. Consider the pair $G = GL_2(\mathbb{F}_q)$, $H = B$ the subgroup of upper triangular matrices. Then by Exercise 6.3 we know that the Hecke algebra $\mathcal{H}(G,B)$ is 2-dimensional. The identity element η_e corresponds to the double coset B. The second element of the basis is η_s. Let us compute η_s^2 using (2.10). We have

$$\eta_s^2 = a\eta_e + b\eta_s,$$

where

$$a = \frac{|sBs \cap B|}{|B|}, \quad b = \frac{sBs \cap BsB}{|B|}.$$

Since sBs is the subgroup of the lower triangular matrices in G, the intersection subgroup $sBs \cap B$ is the subgroup of diagonal matrices. Therefore we have

$$|B| = (q-1)^2 q, \quad |sBs \cap B| = (q-1)^2, \quad a = \frac{1}{q}, \quad b = 1 - a = \frac{q-1}{q}.$$

Definition 8.4. We say that a G-module V is *multiplicity free* if any simple G-module appears in V with multiplicity either 0 or 1.

Proposition 8.5. *Assume that k is algebraically closed. The following conditions on the pair $H \subset G$ are equivalent*

(1) *The G-module $\operatorname{Ind}_H^G \operatorname{triv}$ is multiplicity free;*
(2) *For any G-module M the dimension of subspace M^H of H-invariants is at most one;*
(3) *The Hecke algebra $\mathcal{H}(G,H)$ is commutative.*

Proof. (1) is equivalent to (2) by Frobenius reciprocity. Equivalence of (1) and (3) follows from Lemma 1.11. $\square$

Lemma 8.6. *Let G be a finite group and $H \subset G$ be a subgroup. Let $\varphi : G \to G$ be antiautomorphism of G such that for any $g \in G$ we have $\varphi(g) \subset HgH$. Then $\mathcal{H}(G,H)$ is commutative.*

Proof. Extend φ to the whole group algebra $k(G)$ by linearity. Then φ is an antiautomorphism of $k(G)$ and for all $g \in G$ we have $\varphi(\eta_g) = \eta_g$. Therefore for any $g, h \in H\backslash G/H$ we have

$$\eta_g \eta_h = \sum c^u_{g,h} \eta_u = \sum_{u \in H\backslash G/H} c^u_{g,h} \varphi(\eta_u) = \varphi(\eta_g \eta_h) = \varphi(\eta_h)\varphi(\eta_g) = \eta_h \eta_g.$$

$\square$

Exercise 8.7. Let G be the symmetric group S_n and $H = S_p \times S_{n-p}$. Prove that $\mathcal{H}(G,H)$ is abelian. Hint: consider $\varphi(g) = g^{-1}$ and apply Lemma 8.6.

9. Some examples

Let H be a subgroup of G of index 2. Then H is normal and $G = H \cup sH$ for some $s \in G\backslash H$. Suppose that ρ is an irreducible representation of H. There are two possibilities

(1) ρ^s is isomorphic to ρ;
(2) ρ^s is not isomorphic to ρ.

Hence there are two possibilities for $\operatorname{Ind}_H^G \rho$:

(1) $\operatorname{Ind}_H^G \rho = \sigma \oplus \sigma'$, where σ and σ' are two non-isomorphic irreducible representations of G;
(2) $\operatorname{Ind}_H^G \rho$ is irreducible.

For instance, let $G = S_5$, $H = A_5$ and $\rho_1, \dots, \rho_5$ be the irreducible representations of H introduced in Example 5.3. 3. Then for $i = 1, 2, 3$

$$\mathrm{Ind}_H^G \rho_i = \sigma_i \oplus (\sigma_i \otimes \mathrm{sgn}),$$

where sgn denotes the sign representation. Furthermore, the induced modules $\mathrm{Ind}_H^G \rho_4$ and $\mathrm{Ind}_H^G \rho_5$ are isomorphic and irreducible. Thus in dimensions $1, 4$ and 5, S_5 has two non-isomorphic irreducible representations and only one in dimension 6.

Now let G be the subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$ consisting of matrices of shape

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

where $a \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q$. Let us classify complex irreducible representations of G. One has $|G| = q^2 - q$. Furtheremore G has q conjugacy classes with the following representatives

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

(in the last case $a \neq 1$). Note that

$$H = \{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{F}_q \}$$

is a normal subgroup of G and the quotient G/H is isomorphic to $\mathbb{F}_q^*$ which is cyclic of order $q - 1$.

Therefore G has $q - 1$ one-dimensional representations which can be lifted from G/H. That leaves one more representation, its dimension must be $q-1$. Let us try to obtain it using induction from H. Let σ be a non-trivial irreducible representation of H, its dimension is automatically 1. Then the dimension the induced representation $\mathrm{Ind}_H^G \sigma$ is equal to $q - 1$ as required. We claim that it is irreducible. Indeed, if ρ is a one-dimensional representation of G, then by Frobenius reciprocity, Theorem 5.7, we have

$$\left(\mathrm{Ind}_H^G \sigma, \rho\right)_G = (\sigma, \mathrm{Res}_H \rho)_H = 0,$$

since $\mathrm{Res}_H \rho$ is trivial. Therefore $\mathrm{Ind}_H^G \sigma$ is irreducible.

Exercise 9.1. Compute the character of this representation.

Exercise 9.2. Let G' denote the commutator of G, namely the subgoup of G generated by $ghg^{-1}h^{-1}$ for all $g, h \in G$. Show that all one-dimensional representations of G are obtained by lifting from one-dimensional representations of G/G'.

10. Some generalities about field extension

Lemma 10.1. *If* $\mathrm{char}\, k = 0$ *and* G *is finite, then a representation* $\rho : G \to \mathrm{GL}(V)$ *is irreducible if and only if* $\mathrm{End}_G(V)$ *is a division ring.*

PROOF. In one direction it is Schur's Lemma. In the opposite direction if V is not irreducible, then $V = V_1 \oplus V_2$ and the projectors p_1 and p_2 are intertwiners such that $p_1 \circ p_2 = 0$. $\square$

For any extension F of k and any representation $\rho : G \to \mathrm{GL}(V)$ over k we denote by ρ_F the representation $G \to \mathrm{GL}(F \otimes_k V)$.

For any representation $\rho : G \to \mathrm{GL}(V)$ we denote by V^G the subspace of G-invariants in V, i.e.

$$V^G = \{v \in V \mid \rho_s v = v, \forall s \in G\}.$$

LEMMA 10.2. *One has* $(F \otimes_k V)^G = F \otimes_k V^G$.

PROOF. The embedding $F \otimes_k V^G \subset (F \otimes_k V)^G$ is trivial. On the other hand, V^G is the image of the operator

$$p = \frac{1}{|G|} \sum_{s \in G} \tau_s,$$

in particular $\dim V^G$ equals the rank of p. Since rank p does not depend on the base field, we have

$$\dim F \otimes_k V^G = \dim (F \otimes_k V)^G.$$

$\square$

COROLLARY 10.3. *Let* $\rho : G \to \mathrm{GL}(V)$ *and* $\sigma : G \to \mathrm{GL}(W)$ *be two representations over* k. *Then*

$$\mathrm{Hom}_G(F \otimes_k V, F \otimes_k W) = F \otimes \mathrm{Hom}_G(V, W).$$

In particular,

$$\dim_k \mathrm{Hom}_G(V, W) = \dim_F \mathrm{Hom}_G(F \otimes_k V, F \otimes_k W).$$

PROOF.

$$\mathrm{Hom}_G(V, W) = (V^* \otimes W)^G.$$

$\square$

COROLLARY 10.4. *The formula*

$$\mathit{dim}\ \mathrm{Hom}_G(V, W) = (\chi_\rho, \chi_\sigma)$$

holds even if the field is not algebraically closed.

A representation $\rho : G \to \mathrm{GL}(V)$ over k is called *absolutely irreducible* if it remains irreducible after any extension of k. This property is equivalent to the egality $(\chi_\rho, \chi_\rho) = 1$.

A field K is called *splitting* for a group G if every irreducible representation of G over K is absolutely irreducible. It is not difficult to see that for a finite group G, there exists a finite extension of $\mathbb{Q}$ which is a splitting field for G.

CHAPTER 3

Representations of compact groups

1. Compact groups

Let G be a group which is also a topological space. We say that G is a *topological group* if both the multiplication from $G \times G$ to G and the inverse from G to G are continuous maps. Naturally, we say that G is compact (respectively, locally compact) if it is a compact (resp., locally compact) topological space.

Examples.

- The circle
$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$
- The torus $T^n = S^1 \times \cdots \times S^1$.
  Note that in general, the direct product of two compact groups is compact.
- The unitary group
$$U_n = \left\{X \in \mathrm{GL}_n(\mathbb{C}) \mid \bar{X}^t X = 1_n\right\}.$$
  To see that U_n is compact, note that a matrix $X = (x_{ij}) \in U_n$ satisfies the equations $\sum_{j=1}^n |x_{ij}|^2 = 1$ for $j = 1, \ldots, n$. Hence U_n is a closed subset of the product of n spheres of dimension $(2n-1)$.
- The special unitary group
$$SU_n = \{X \in U_n \mid \det X = 1\}.$$
- The orthogonal group
$$O_n = \left\{x \in \mathrm{GL}_n(\mathbb{R}) \mid X^t X = 1_n\right\}.$$
- The special orthogonal group
$$SO_n = \{X \in O_n \mid \det X = 1\}.$$

1.1. Haar measure. A measure dg on a locally compact group G is called right-invariant if, for every integrable function f on G and every h in G, one has:
$$\int_G f(gh)\, dg = \int_G f(g)\, dg.$$

Similarly, a measure $d'g$ on G is called left-invariant if for every integrable function f on G and every h in G, one has:

$$\int_G f(hg)\, d'g = \int_G f(g)\, d'g.$$

THEOREM 1.1. *Let G be compact group. There exists a unique right-invariant measure dg on G such that*

$$\int_G dg = 1.$$

In the same way there exists a unique left-invariant measure $d'g$ such that

$$\int_G d'g = 1.$$

Moreover, $dg = d'g$.

DEFINITION 1.2. The measure dg is called the Haar measure on G.

We do not give the proof of this theorem in general. In this sketch of proof, we assume general knowledge of submanifolds and of the notion of vector bundle. All examples we consider here are smooth submanifolds in $\mathrm{GL}_k(\mathbb{R})$ or $\mathrm{GL}_k(\mathbb{C})$.

EXERCISE 1.3. Assume that G is a subgroup of $\mathrm{GL}_k(\mathbb{R})$ or $\mathrm{GL}_k(\mathbb{C})$ and G is the set of zeros of smooth functions $f_1, \dots, f_k$. Then G is a smooth submanifold in GL_k. Hint: consider the map $m_g : G \to G$ given by left multiplication by $g \in G$. Then its differential $(m_g)_* : T_eG \to T_gG$ is an isomorphism between tangent spaces at e and g.

To define the invariant measure we just need to define a volume form on the tangent space at identity T_eG and then use right (left) multiplication to define it on the whole group. More precisely, let $\gamma \in \Lambda^{\mathrm{top}} T_e^* G$. Then the map

$$g \mapsto \gamma_g := m_g^* (\gamma),$$

where $m_g : G \to G$ is the right (left) multiplication by g and m_g^* is the induced differential map $\Lambda^{\mathrm{top}} T_e^* G \to \Lambda^{\mathrm{top}} T_g^* G$, is a section of the bundle $\Lambda^{\mathrm{top}} T^* G$. This section is a right (left) invariant differential form of maximal degree on the group G, i.e. an invariant volume form. One can normalize γ to satisfy $\int_G \gamma = 1$.

REMARK 1.4. If G is locally compact but not compact, there are still left-invariant and right-invariant measures on G, each is unique up to scalar multiplication, but the left-invariant ones are not necessarily proportional to the right-invariant ones. We speak of left-Haar measure or right-Haar measure.

1.2. Continuous representations. Consider a vector space V over $\mathbb{C}$ equipped with a topology such that addition and multiplication by a scalar are continuous. We always assume that a topological vector space satisfies the following conditions

(1) for any $v \in V \setminus 0$ there exists a neighbourhood of 0 which does not contain v;
(2) there is a base of convex neighbourhoods of zero.

Topological vector spaces satisfying the above conditions are called *locally convex.* We do not go into the theory of such spaces. All we need to know is the fact that there is a non-zero continuous linear functional on a locally convex space.

DEFINITION 1.5. A representation $\rho : G \to \mathrm{GL}(V)$ is called *continuous* if the map $G \times V \to V$ given by $(g, v) \mapsto \rho_g v$ is continuous. Two continuous representations are *equivalent* or *isomorphic* if there is a bicontinuous invertible intertwining operator between them. In this chapter we consider only continuous representations.

A representation $\rho : G \to \mathrm{GL}(V)$, $V \neq \{0\}$ is called *topologically irreducible* if the only G-invariant closed subspaces of V are V and 0.

1.3. Unitary representations. Recall that a Hilbert space is a vector space over $\mathbb{C}$ equipped with a positive definite Hermitian form $\langle , \rangle$, which is complete with respect to the topology defined by the norm

$$\|v\| = \langle v, v \rangle^{1/2}.$$

We will use the following facts about Hilbert spaces:

(1) A Hilbert space V has an orthonormal topological basis, i.e. an orthonormal system of vectors $\{e_i\}_{i \in I}$ such that $\bigoplus_{i \in I} \mathbb{C} e_i$ is dense in V. Two Hilbert spaces are isomorphic if and only if their topological orthonormal bases have the same cardinality.
(2) If V^* denotes the space of all continuous linear functionals on V, then we have an isomorphism $V^* \simeq V$ given by $v \mapsto \langle v, \cdot \rangle$.

DEFINITION 1.6. A continuous representation $\rho : G \to \mathrm{GL}(V)$ is called *unitary* if V is a Hilbert space and

$$\langle v, w \rangle = \langle \rho_g v, \rho_g w \rangle$$

for any $v, w \in V$ and $g \in G$. If $U(V)$ denotes the group of all unitary operators in V, then ρ defines a homomorphism $G \to U(V)$.

The following is an important example of a unitary representation.

Regular representation. Let G be a compact group and $L^2(G)$ be the space of all complex valued functions φ on G such that

$$\int |\varphi(g)|^2 dg$$

exists. Then $L^2(G)$ is a Hilbert space with respect to the Hermitian form

$$\langle \varphi, \psi \rangle = \int_G \bar{\varphi}(g)\, \psi(g)\, dg.$$

Moreover, the representation R of G in $L^2(G)$ given by

$$R_g \varphi(h) = \varphi(hg)$$

is continuous and the Hermitian form is G-invariant. This representation is called the *regular reprensentation of* G.

1.4. Linear operators in a Hilbert space. We will recall certain facts about linear operators in a Hilbert space. We only sketch the proofs hiding technical details in exercises. The enthusiastic reader is encouraged to supply those details and the less enthusiastic reader can find those details in textbooks on the subject, for instance,???.

Definition 1.7. A linear operator T in a Hilbert space is called *bounded* if there exists $C > 0$ such that for any $v \in V$ we have $\|Tv\| \leq C\|v\|$.

Exercise 1.8. Let $\mathcal{B}(V)$ denote the set of all bounded operators in a Hilbert space V.

(a) Check that $\mathcal{B}(V)$ is an algebra over $\mathbb{C}$ with multiplication given by composition.

(b) Show that $T \in \mathcal{B}(V)$ if and only if the map $T : V \to V$ is continuous.

(c) Introduce the norm on $\mathcal{B}(V)$ by setting

$$\|T\| = \sup_{\|v\|=1} \|Tv\|.$$

Check that $\|T_1T_2\| \leq \|T_1\|\|T_2\|$ and $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$ for all $T_1, T_2 \in \mathcal{B}(V)$ and that $\mathcal{B}(V)$ is complete in the topology defined by this norm. Thus, $\mathcal{B}(V)$ is a Banach algebra.

Theorem 1.9. *Let $T \in \mathcal{B}(V)$ be invertible. Then T^{-1} is also bounded.*

Proof. Consider the unit ball

$$B := \{x \in V \mid \|x\| < 1\}.$$

For any $k \in \mathbb{N}$ denote by S_k the closure of $T(kB) = kT(B)$ and let $U_k = V \setminus S_k$. Note that

$$V = \bigcup_{k \in \mathbb{N}} kB.$$

Since T is invertible, it is surjective, and therefore

$$\bigcup_{k \in \mathbb{N}} S_k = V.$$

We claim that there exists k such that U_k is not dense. Indeed, otherwise there exists a sequence of embedded balls $B_k \subset U_k$, $B_{k+1} \subset B_k$, which has a common point by completeness of V. This contradicts to the fact that the intersection of all U_k is empty. Then S_k contains a ball $x+\varepsilon B$ for some $x \in V$ and $\varepsilon > 0$. It is not hard to see that for any $r > \frac{k}{\varepsilon} + \|x\|$, S_r contains B.

Now we will prove the inclusion $B \subset T(2rB)$ for r as above. Indeed, let $y \in B \subset S_r$. There exists $x_1 \in rB$ such that $\|y - Tx_1\| < \frac{1}{2}$. Note that $y - Tx_1 \in \frac{1}{2}B \subset \frac{1}{2}S_r$. Then one can find $x_2 \in \frac{r}{2}B$ such that $\|y - Tx_1 - Tx_2\| < \frac{1}{4}$. Proceeding in this way we can construct a sequence $\{x_n \in \frac{1}{2^{n-1}}B\}$ such that $\|y - T(x_1 + \cdots + x_n)\| < \frac{1}{2^n}$. Consider $w = \displaystyle\sum_{i=1}^{\infty} x_i$, which is well defined due to completeness of V. Then $w \in 2rB$ and $Tw = y$. That implies $B \subset T(2rB)$.

Now we have $T^{-1}B \subset 2rB$ and hence $\|T^{-1}\| \leq 2r$. □

Bounded operators have a nice spectral theory, see ???.

Definition 1.10. Let T be bounded. The *spectrum* $\sigma(T)$ of T is the subset of complex numbers λ such that $T - \lambda \,\mathrm{Id}$ is not invertible.

In a finite-dimensional Hilbert space $\sigma(T)$ is the set of eigenvalues of T. In the infinite-dimensional case a point of the spectrum is not necessarily an eigenvalue. We need the following fundamental result.

Theorem 1.11. *If T is bounded, then $\sigma(T)$ is a non-empty closed bounded subset of $\mathbb{C}$.*

Proof. The main idea is to consider the resolvent $\mathcal{R}(\lambda) = (T - \lambda \,\mathrm{Id})^{-1}$ as a function of λ. If T is invertible, then we have the decomposition

$$\mathcal{R}(\lambda) = T^{-1}(\mathrm{Id} + T^{-1}\lambda + T^{-2}\lambda^2 + \dots),$$

which converges for $|\lambda| < \frac{1}{\|T^{-1}\|}$. Thus, $\mathcal{R}(\lambda)$ is analytic in a neighbourhood of 0. Using shift $\mathcal{R}(\lambda) \to \mathcal{R}(\lambda + c)$ we obtain that $\mathcal{R}(\lambda)$ is analytic in its domain which is $\mathbb{C} \setminus \sigma(T)$. The domain of $\mathcal{R}(\lambda)$ is an open set. Hence $\sigma(T)$ is closed.

Furthermore, we can write the series for $\mathcal{R}(\lambda)$ at infinity:

$$\mathcal{R}(\lambda) = -\lambda^{-1}(\mathrm{Id} + \lambda^{-1}T + \lambda^{-2}T^2 + \dots). \tag{3.1}$$

This series converges for $|\lambda| > \|T\|$. Therefore $\sigma(\lambda)$ is a subset of the circle $|\lambda| \leq \|T\|$. Hence $\sigma(T)$ is bounded.

Finally, (3.1) also implies $\lim\limits_{\lambda \to \infty} \mathcal{R}(\lambda) = 0$. Suppose that $\sigma(T) = \emptyset$, then $\mathcal{R}(\lambda)$ is analytic and bounded. By Liouville's theorem $\mathcal{R}(\lambda)$ is constant, which is impossible. □

Definition 1.12. For any linear operator T in a Hilbert space V we denote by T^* the adjoint operator. Since $V^* \simeq V$, we can consider T^* as a linear operator in V such that for any $x, y \in V$

$$\langle x, Ty \rangle = \langle T^*x, y \rangle .$$

An operator T is *self-adjoint* if $T^* = T$. A self-adjoint operator T defines on V a Hermitian form $\langle x, y\rangle_T = \langle x, Ty\rangle$. We call T (semi)positive if this form is (semi)positive definite. For any operator X the operator X^*X is semipositive self-adjoint.

EXERCISE 1.13. (a) If T is bounded, then T^* is bounded and $\sigma(T^*)$ is the complex conjugate of $\sigma(T)$.

(b) If T is bounded self-adjoint, then $\sigma(T) \subset \mathbb{R}$.

LEMMA 1.14. *Let T be a self-adjoint operator in a Hilbert space. Then $\|T^2\| = \|T\|^2$.*

PROOF. For any bounded operator A the Cauchy–Schwartz inequality implies that for all $v \in V$
$$\langle Av, v\rangle \leq \|Av\|\|v\| \leq \|A\|\|v\|^2.$$
For a self-adjoint T we have
$$\left\langle T^2 v, v\right\rangle = \|Tv\|^2.$$
Therefore
$$\|T^2\| \geq \sup_{\|v\|=1} \left\langle T^2 v, v\right\rangle = \sup_{\|v\|=1} \|Tv\|^2 = \|T\|^2.$$
On the other hand $\|T^2\| \leq \|T\|^2$. Hence $\|T^2\| = \|T\|^2$. □

LEMMA 1.15. *Let T be a self-adjoint operator in a Hilbert space V such that $\sigma(T) = \{\mu\}$ is a single point. Then $T = \mu\,\mathrm{Id}$.*

PROOF. Without loss of generality we may assume $\mu = 0$. Then the series (3.1) converges for all $\lambda \neq 0$. Therefore by the root test we have
$$\lim_{n\to\infty} \sup \|T^n\| = 0.$$
By Lemma 1.14 if $n = 2^k$, then $\|T^n\| = \|T\|^n$. This implies $\|T\| = 0$. Hence $T = 0$. □

EXERCISE 1.16. Let X be a self-adjoint bounded operator.

(a) If $f \in \mathbb{R}[x]$ is a polynimial with real coefficients, then $\sigma(f(X)) = f(\sigma(X))$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that one can define $f(X)$ by approximating f by polynomials f_n on the interval $|x| \leq \|X\|$ and setting $f(X) = \lim_{n\to\infty} f_n(X)$ and the result does not depend on the choice of approximation.

(c) For a continuous function f we still have $\sigma(f(X)) = f(\sigma(X))$.

DEFINITION 1.17. An operator T in a Hilbert space V is called *compact* if the closure of the image $T(S)$ of the unit sphere $S = \{x \in V \mid \|x\| = 1\}$ is compact.

Clearly, any compact operator is bounded.

EXERCISE 1.18. Let $\mathcal{C}(V)$ be the subset of all compact operators in a Hilbert space V.

(a) Show that $\mathcal{C}(V)$ is a closed ideal in $\mathcal{B}(V)$.

(b) Let $\mathcal{F}(V)$ be the ideal in $\mathcal{B}(V)$ of all operators with finite-dimensional image. Prove that $\mathcal{C}(V)$ is the closure of $\mathcal{F}(V)$.

LEMMA 1.19. *Let A be a compact self-adjoint operator in V. Then*

$$\lambda := \sup_{u \in S} \langle Au, u \rangle$$

is either zero or an eigenvalue of A.

PROOF. Consider the hermitian form $x \mapsto \lambda \langle x, x\rangle - \langle Ax, x\rangle$ on V, it is positive therefore the Cauchy–Schwarz inequality gives

$$|\lambda \langle x, y\rangle - \langle Ax, y\rangle|^2 \leq (\lambda \langle x, x\rangle - \langle Ax, x\rangle)(\lambda \langle y, y\rangle - \langle Ay, y\rangle) \tag{3.2}$$

Let (x_n) be a sequence in S such that $\langle Ax_n, x_n\rangle$ converges to λ. Since A is a compact operator, after extracting a subsequence we may assume that Ax_n converges to $z \in V$. By the inequality 3.2, we get that $\langle \lambda x_n - Ax_n, y\rangle$ tends to 0 uniformly in $y \in S$. Hence, $\|\lambda x_n - Ax_n\|$ tends to 0. Therefore, (x_n) converges to $\frac{1}{\lambda}z$ and z is a eigenvector for A with eigenvalue λ, if $\lambda > 0$. □

1.5. Schur's lemma for unitary representations.

THEOREM 1.20. *Let $\rho : G \to U(V)$ a topologically irreducible unitary representation of G and $T \in \mathcal{B}(V)$ be a bounded intertwining operator. Then $T = \lambda \operatorname{Id}$ for some $\lambda \in \mathbb{C}$.*

PROOF. First, by Theorem 1.11, the spectrum $\sigma(T)$ is not empty. Therefore by adding a suitable scalar operator we may assume that T is not invertible. Note that T^* is also an interwiner, and therefore $S = TT^*$ is an interwiner as well. Moreover, S is not invertible. If $\sigma(S) = \{0\}$, then $S = 0$ by Lemma 1.15. Then we claim that $\operatorname{Ker} T \neq 0$. Indeed, if T is injective, then $\operatorname{Im} T^* \subset \operatorname{Ker} T = 0$. That implies $T^* = T = 0$. Since $\operatorname{Ker} T$ is a closed G-invariant subspace of V, we obtain $T = 0$.

Now we assume that $\sigma(S)$ consists of more than one point. We will use Exercise 1.16. One can always find two continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $fg(\sigma(S)) = 0$, but $f(\sigma(S)) \neq 0$ and $g(\sigma(S)) \neq 0$. Then Exercise 1.16(3) together with Lemma 1.15 implies $f(X)g(X) = 0$. Both $f(X)$ and $g(X)$ are non-zero intertwiners. At least one of $\operatorname{Ker} f(X)$ and $\operatorname{Ker} g(X)$ is a proper non-zero G-invariant subspace of V. Contradiction. □

COROLLARY 1.21. *Let $\rho : G \to U(V)$ and $\rho' : G \to U(V')$ be two topologically irreducible unitary representations and $T : V \to V'$ be a continuous intertwining operator. Then either $T = 0$ or there exists $c > 0$ such that $cT : V \to V'$ is an isometry of Hilbert spaces.*

PROOF. Let $T \neq 0$. By Theorem 1.20 we have $T^*T = TT^* = \lambda \operatorname{Id}$ for some positive real λ. Set $c = \lambda^{-1/2}$ and $U = cT$. Then $U^* = U^{-1}$, hence U is an isometry. □

COROLLARY 1.22. *Every topologically irreducible unitary representation of an abelian topological group G is one-dimensional.*

1.6. Irreducible unitary representations of compact groups.

PROPOSITION 1.23. *Every non-zero unitary representation of a compact group G contains a non-zero finite dimensional invariant subspace.*

PROOF. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible unitary representation. Choose $v \in V$, $\|v\| = 1$. Define an operator $T : V \to V$ by the formula

$$Tx = \langle v, x \rangle v.$$

One can check easily that T is a semipositive self-adjoint operator of rank 1.
Define the operator

$$Qx = \int_G \rho_g T\left(\rho_g^{-1} x\right) dg.$$

EXERCISE 1.24. Check $Q : V \to V$ is a compact semipositive intertwining operator.

Lemma 1.19 implies that Q has a positive eigenvalue λ. Consider $W = \mathrm{Ker}(Q - \lambda \,\mathrm{Id})$. Then W is an invariant subspace of V. Note that for any orthonormal system of vectors $e_1, \dots, e_n \in W$, one has

$$\sum_{i=1}^{n} \langle e_i, T e_i \rangle \leq 1.$$

Hence

$$\sum_{i=1}^{n} \langle e_i, Q e_i \rangle = \sum_{i=1}^{n} \int_G \langle \rho_g e_i, T \rho_g e_i \rangle \leq 1.$$

That implies $\lambda n \leq 1$. Hence $\dim W \leq \frac{1}{\lambda}$. □

COROLLARY 1.25. *Every irreducible unitary representation of a compact group G is finite-dimensional.*

LEMMA 1.26. *Every topologically irreducible representation of G is isomorphic to a subrepresentation of the regular representation in $L^2(G)$.*

PROOF. Let $\rho : G \to \mathrm{GL}(V)$ be irreducible. Pick a non-zero continuous linear functional φ on V and define the map $\Phi : V \to L^2(G)$ which sends v to the matrix coefficient $f_{v,\varphi}(g) = \langle \varphi, \rho_g v \rangle$. The continuity of ρ and φ implies that $f_{v,\varphi}$ is a continuous function on G, therefore $f_{v,\varphi} \in L^2(G)$. Furthermore

$$R_g f_{v,\varphi}(h) = f_{v,\varphi}(hg) = \langle \varphi, \rho_{hg} v \rangle = \langle \varphi, \rho_h \rho_g v \rangle = f_{\rho_g v, \varphi}(h).$$

Hence Φ is a continuous intertwining operator and the irreducibility of ρ implies $\mathrm{Ker}\, \Phi = 0$. The bicontinuity assertion follows from Corollary 1.25. □

Corollary 1.27. *Every topologically irreducible representation of a compact group G is equivalent to some unitary representation.*

Corollary 1.28. *Every irreducible continuous representation of a compact group G is finite-dimensional.*

Theorem 1.29. *If $\rho : G \to GL(V)$ is a unitary representation, then for any closed invariant subspace $W \subset V$ there exists a closed invariant subspace $U \subset V$ such that $V = U \oplus W$.*

Proof. Take $U = W^{\perp}$. ☐

Let $\widehat{G}$ denotes the set of isomorphism classes of irreducible unitary representations of G. This set is called the *unitary dual* of G.

Lemma 1.30. *Let V be a unitary representation of a compact group G. Then it has a unique dense semi-simple G-submodule, namely $\oplus_{\rho\in\widehat{G}} \operatorname{Hom}_G(V_\rho, V) \otimes V_\rho$.*

Proof. Let $M = \oplus_{\rho\in\widehat{G}} \operatorname{Hom}_G(V_\rho, V) \otimes V_\rho$, and $\bar{M}$ denote the closure of M. We claim that $\bar{M} = V$. Indeed, if $\bar{M}^{\perp}$ is not zero, then it contains an irreducible finite-dimensional subrepresentation by Proposition 1.23, but any such representation is contained in M.

On the other hand, if N is a dense semisimple submodule of V, then N must contain all finite-dimensional irreducible subrepresentations of V. Therefore $N = M$. ☐

2. Orthogonality relations and Peter–Weyl Theorem

2.1. Matrix coefficients. Let $\rho : G \to \mathrm{GL}(V)$ be a unitary representation of a compact group G. The function $G \to \mathbb{C}$ defined by the formula

$$f_{v,w}(g) = \langle w, \rho_g v\rangle .$$

for v, w in V is called a *matrix coefficient* of the representation ρ.

Since ρ is unitary, one has:

$$f_{v,w}\left(g^{-1}\right) = \bar{f}_{w,v}(g) . \tag{3.3}$$

Theorem 2.1. *For every irreducible unitary representation $\rho : G \to \mathrm{GL}(V)$, one has:*

$$\langle f_{v,w}, f_{v',w'}\rangle = \frac{1}{\dim\rho} \langle v, v'\rangle \langle w', w\rangle .$$

Moreover, the matrix coefficients of two non-isomorphic representations of G are orthogonal in $L^2(G)$.

Proof. Take v and v' in V. Define $T \in \operatorname{End}_{\mathbb{C}}(V)$ by

$$Tx := \langle v, x\rangle v'$$

and let

$$Q = \int_G \rho_g T \rho_g^{-1} dg.$$

As follows from Schur's lemma, since ρ is irreducible, Q is a scalar multiplication. Since

$$\operatorname{tr} Q = \operatorname{tr} T = \langle v, v' \rangle ,$$

we obtain

$$Q = \frac{\langle v, v' \rangle}{\dim \rho} \operatorname{Id}.$$

Hence

$$\langle w', Qw \rangle = \frac{1}{\dim \rho} \langle v, v' \rangle \langle w', w \rangle .$$

On the other hand,

$$\langle w', Qw \rangle = \int_G \left\langle w', \left\langle v, \rho_g^{-1} w \right\rangle \rho_g v' \right\rangle dg = \int_G f_{w,v} \left(g^{-1} \right) f_{v',w'} \left(g \right) dg =$$

$$= \int_G \bar{f}_{v,w} \left(g \right) f_{v',w'} \left(g \right) dg = \frac{1}{\dim \rho} \left\langle f_{v,w}, f_{v',w'} \right\rangle .$$

If $f_{v,w}$ and $f_{v',w'}$ are matrix coefficients of two non-isomorphic representations, then $Q = 0$, and the calculation is even simpler. $\square$

Corollary 2.2. *Let $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$ be two irreducible unitary representations, then $\langle \chi_\rho, \chi_\sigma \rangle = 1$ if ρ is isomorphic to σ and $\langle \chi_\rho, \chi_\sigma \rangle = 0$ otherwise.*

Proof. Let $v_1, \dots, v_n$ be an orthonormal basis in V and $w_1, \dots, w_m$ be an orthonormal basis in W. Then

$$\langle \chi_\rho, \chi_\sigma \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \left\langle f_{v_i,v_i}, f_{w_j,w_j} \right\rangle .$$

Therefore the statement follows from Theorem 2.1. $\square$

Lemma 2.3. *Let $\rho : G \to GL(V)$ be an irreducible unitary representation of G. Then the map $V \to \operatorname{Hom}_G(V, L^2(G))$ defined by*

$$w \mapsto \varphi_w, \quad \varphi_w(v) := f_{v,w} \text{ for all } v, w \in V$$

is an isomorphism of vector spaces.

Proof. It is easy to see that $\varphi_w \in \operatorname{Hom}_G(V, L^2(G))$. Moreover, the value of $\varphi_w(w)$ at e equals $\langle w, w \rangle$. Hence $\varphi_w \neq 0$ if $w \neq 0$. Thus, the map is injective. To check surjectivity note that $\operatorname{Hom}_G(V, L^2(G))$ is the subspace of functions $f : G \times V \to \mathbb{C}$ satisfying the condition

$$f(gh, v) = f(g, \rho_h v) \quad \text{for all } v \in V,\ g, h \in G.$$

For any such f there exists $w \in V$ such that $f(e,v) = \langle w, v \rangle$. The above condition implies $f(g,v) = \langle w, \rho_g v \rangle$, i.e. $f = \varphi_w$. $\square$

THEOREM 2.4. *(Peter–Weyl) Matrix coefficients of all irreducible unitary representations span a dense subspace in $L^2(G)$ for a compact group G.*

PROOF. We apply Lemma 1.30 to the regular representation of G. Let $\rho \in \widehat{G}$. Lemma 2.3 implies that $V_\rho \otimes \mathrm{Hom}_G(V_\rho, L^2(G))$ coincides with the space of all matrix coefficients of ρ. Hence the span of matrix coefficients is the unique semisimple G-submodule in $L^2(G)$. $\square$

2.2. Convolution algebra. For a group G we define by $L^1(G)$ the set of all complex valued functions φ on G such that

$$\|f\|_1 := \int_G |\varphi(g)| dg$$

is finite.

DEFINITION 2.5. The convolution product of two continuous complex valued functions φ and ψ on G is defined by the formula:

$$(\varphi * \psi)(g) := \int_G \varphi(h)\psi(h^{-1}g) dh. \tag{3.4}$$

EXERCISE 2.6. The following properties are easily checked:

(1) $\|\varphi * \psi\|_1 \leq \|\varphi\|_1 \|\psi\|_1$
(2) The convolution product extends uniquely as a continuous bilinear map $L^1(G) \times L^1(G) \to L^1(G)$.
(3) The convolution is an associative product.
(4) Let V be a unitary representation of G, show that we can see it as a $L^1(G)$-module by setting $\varphi.v := \int_G \varphi(g) g^{-1} v dg$.

COROLLARY 2.7. *Let G be a compact group and R denote the representation of $G \times G$ in $L^2(G)$ given by the formula*

$$R_{s,t} f(x) = f\left(s^{-1}xt\right).$$

Then

$$L^2(G) \cong \widehat{\bigoplus}_{\rho \in \widehat{G}} V_\rho^* \boxtimes V_\rho,$$

where the direct sum is in the sense of Hilbert spaces.

Moreover, this isomorphism is actually an isomorphism of algebras (without unit) between $L^2(G)$ equipped with the convolution and $\widehat{\bigoplus}_{\rho \in \widehat{G}} End(V_\rho)$.

PROOF. For any $\rho \in \widehat{G}$ consider the map $\Phi_\rho : V_\rho^* \boxtimes V_\rho \to L^2(G)$ defined by

$$\Phi_\rho(v \otimes w)(g) = \langle v, \rho_g w \rangle.$$

It is easy to see that Φ_ρ defines an embedding of the irreducible $G\times G$-representation $\rho^*\boxtimes\rho$ in $L^2(G)$. Moreover, by orthogonality relation $\langle \operatorname{Im}\Phi_\rho, \operatorname{Im}\Phi_\sigma\rangle = 0$ if ρ and σ are not isomorphic. The direct sum $\bigoplus_{\rho\in\widehat{G}}\operatorname{Im}\Phi_\rho$ coincides with the span of all matrix coefficients of all irreducible representations of G. Hence it is dense in $L^2(G)$. That implies the first statement. The final statement is clear by applying item (4) of Exercise 2.6. □

Remark 2.8. A finite group G is a compact group in discrete topology and $L^2(G)$ with convolution product is the group algebra $\mathbb{C}(G)$. Therefore Theorem 1.13 of Chapter II is a particular case of Corollary 2.7 when the ground field is $\mathbb{C}$.

Corollary 2.9. *The characters of irreducible representations form an orthonormal basis in the subspace of class function in $L^2(G)$.*

Proof. Let $\mathcal{C}(G)$ denote the subspace of class functions in $L^2(G)$, it is clearly the center of $L^2(G)$. On the other hand, the center of $End(V_\rho)$ is $\mathbb{C}$ and its image in $L^2(G)$ is $\mathbb{C}\chi_\rho$ (χ_ρ denotes as usual the character of ρ). The assertion is a direct consequence of Corollary 2.7. □

Exercise 2.10. Let $r: G\to U(V)$ be a unitary representation of a compact group G and ρ be an irreducible representation with character χ_ρ. Then the linear operator

$$P_\rho(x) := \frac{1}{\dim\rho}\int_G \chi_\rho(g^{-1})r_g x dg$$

is a projector onto the corresponding isotypic component.

Exercise 2.11. Let E be a faithful finite-dimensional representation of a compact group G. Show that all irreducible representations of G appear in $T(E)\otimes T(E^*)$ as subrepresentations. Hint: Note that G is a subgroup in $GL(E)$. Using Weierstrass theorem prove that matrix coefficient of E and E^* generate a dense subalgebra in $L^2(G)$ (with usual pointwise multiplication).

3. Examples

3.1. The circle. Let $\mathbb{T} = S^1 = \{z\in\mathbb{C} \mid |z| = 1\}$, if $z\in S^1$, one can write $z = e^{i\theta}$ with θ in $\mathbb{R}/2\pi\mathbb{Z}$. The Haar measure on S^1 is equal to $\frac{d\theta}{2\pi}$. All irreducible representations of S^1 are one-dimensional since S^1 is abelian. They are given by the characters $\chi_n: S^1\to\mathbb{C}^*$, $\chi_n(\theta) = e^{in\theta}$, $n\in\mathbb{Z}$. Hence $\widehat{S}^1 = \mathbb{Z}$ and

$$L^2\left(S^1\right) = \oplus_{n\in\mathbb{Z}}\mathbb{C}e^{in\theta}.$$

This is a representation-theoretic explanation of the Parseval theorem, meaning that every square integrable periodic function is the sum (with respect to the L^2 norm) of its Fourier series.

3.2. The group SU_2. Consider the compact group $G = SU_2$. Then G consists of all matrices

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

satisfying the relations $|a|^2 + |b|^2 = 1$. Thus, as a topological space, SU_2 is isomorphic to the 3-dimensional sphere S^3.

EXERCISE 3.1. Check that SU_2 is isomorphic to the multiplicative subgroup of quaternions with norm 1 by identifying the quaternion $a+bi+cj+dk = a+bi+j(c+di)$ with the matrix $\begin{pmatrix} a+bi & c+di \\ -c+di & a-di \end{pmatrix}$.

To find the irreducible representations of SU_2, consider the polynomial ring $\mathbb{C}[x,y]$, with the action of SU_2 given by the formula

$$\rho_g(x) = ax + by, \ \rho_g(y) = -\bar{b}x + \bar{a}y, \text{ if } g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

Let ρ_n be the representation of G in the space $\mathbb{C}_n[x,y]$ of homogeneous polynomials of degree n. The monomials x^n, $x^{n-1}y, \dots, y^n$ form a basis of $\mathbb{C}_n[x,y]$. Therefore $\dim \rho_n = n+1$. We claim that all ρ_n are irreducible and that every irreducible representation of SU_2 is isomorphic to ρ_n for some $n \geq 0$. We will show this by checking that the characters χ_n of ρ_n form an orthonormal basis in the Hilbert space of class functions on G.

Recall that every unitary matrix is diagonal in some orthonormal basis, therefore every conjugacy class of SU_2 intersects the diagonal subgroup. Moreover, $\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$ and $\begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix}$ are conjugate. Hence the set of conjugacy classes can be identified with the quotient of S^1 by the equivalence relation $z \sim \bar{z}$. Let $z = e^{i\theta}$, then

$$\chi_n(z) = z^n + z^{n-2} + \dots + z^{-n} = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{3.5}$$

First, let us compute the Haar measure for G.

EXERCISE 3.2. Let $G = SU_2$.

(a) Using Exercise 3.1 show that the action of $G \times G$ given by the multiplication on the right and on the left coincides with the standard action of $SO(4)$ on S^3. Use it to prove that $SO(4)$ is isomorphic to the quotient of $G \times G$ by the two element subgroup $\{(1,1),(-1,-1)\}$.

(b) Prove that the Haar measure on G is proportional to the standard volume form on S^3 invariant under the action of the orthogonal group SO_4.

More generally: let us compute the volume form on the n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$ which is invariant under the action SO_{n+1}. We use the spherical coordinates in

$\mathbb{R}^{n+1}$,

$$
\begin{aligned}
&x_1 = r\cos\theta, x_2 = r\sin\theta\cos\varphi_1, x_3 = r\sin\theta\sin\varphi_1\cos\varphi_2,\\
&\dots\\
&x_{n-1} = r\sin\theta\sin\varphi_1\sin\varphi_2\dots\sin\varphi_{n-2}\cos\varphi_{n-1},\\
&x_n = r\sin\theta\sin\varphi_1\sin\varphi_2\dots\sin\varphi_{n-2}\sin\varphi_{n-1},
\end{aligned}
$$

where $r > 0$, $\theta, \varphi_1, \dots, \varphi_{n-2}$ vary in $[0, \pi]$ and $\varphi_{n-1} \in [0, 2\pi]$. The Jacobian relating spherical and Euclidean coordinates is equal to

$$r^n \sin^{n-1}\theta \sin^{n-2}\varphi_1 \dots \sin\varphi_{n-2},$$

thus when we restrict to the sphere $r = 1$ we obtain the volume

$$\sin^{n-1}\theta\sin^{n-2}\varphi_1\dots\sin\varphi_{n-2}d\theta d\varphi_1\dots d\varphi_{n-1},$$

which is SO_{n+1}-invariant. It is not normalized.

Let us return to the case $G = SU_2 \simeq S^3$. After normalization the invariant volume form is

$$\frac{1}{2\pi^2}\sin^2\theta\sin\varphi_1 d\theta d\varphi_1 d\varphi_2.$$

The conjugacy class $C(\theta)$ of all matrices with eigenvalues $e^{i\theta}, e^{-i\theta}$ ($\theta \in [0, \pi]$) is the set of points in S^3 with spherical coordinates $(1, \theta, \varphi_1, \varphi_2)$: indeed, the minimal polynomial on $\mathbb{R}$ of the quaternion with those coordinates is

$$t^2 - 2t\cos\theta + 1$$

which is also the characteristic polynomial of the corresponding matrix in SU_2, so it belongs to $C(\theta)$.

Hence, one gets that, for a class function ψ on G

$$\int_G \psi(g)\,dg = \frac{1}{2\pi^2}\int_0^\pi \psi(\theta)\sin^2\theta d\theta \int_0^\pi \sin\varphi_1 d\varphi_1 \int_0^{2\pi} d\varphi_2 = \frac{2}{\pi}\int_0^\pi \psi(\theta)\sin^2\theta d\theta.$$

Exercise 3.3. Prove that the functions χ_n form an orthonormal basis of the space $L^2([0, \pi])$ with the measure $\frac{2}{\pi}\sin^2\theta d\theta$ and hence of the space of class functions on G.

3.3. The orthogonal group $G = SO_3$. Recall that SU_2 can be realized as the set of quaternions with norm 1. Consider the representation γ of SU_2 in the space of quaternions $\mathbb{H}$ defined by the formula $\gamma_g(\alpha) = g\alpha g^{-1}$. One can see that the 3-dimensional space $\mathbb{H}_{\text{im}}$ of pure imaginary quaternions is invariant and $(\alpha, \beta) = \operatorname{Re}\left(\alpha\bar{\beta}\right)$ is an invariant positive definite scalar product on $\mathbb{H}_{\text{im}}$. Therefore ρ defines a homomorphism $\gamma\colon SU_2 \to SO_3$.

Exercise 3.4. Check that $\operatorname{Ker}\gamma = \{1, -1\}$ and that γ is surjective. Hence $SO_3 \cong SU_2/\{1, -1\}$. Thus, we can see that as a topological space SO_3 is a 3-dimensional sphere with opposite points identified, or the real 3-dimensional projective space.

Therefore every representation of SO_3 can be lifted to the representations of SU_2, and a representation of SU_2 factors to the representation of SO_3 if and only if it is trivial on -1. One can check easily that $\rho_n(-1) = 1$ if and only if n is even. Thus, any irreducible representations of SO_3 is isomorphic to ρ_{2m} for some $m > 0$ and $\dim \rho_{2m} = 2m + 1$. Below we give an independent realization of irreducible representation of SO_3.

3.4. Harmonic analysis on a sphere. Consider the sphere S^2 in $\mathbb{R}^3$ defined by the equation

$$x_1^2 + x_2^2 + x_3^2 = 1.$$

The action of SO_3 on S^2 induces the representation of SO_3 in the space $L^2(S^2)$ of complex-valued squre integrable functions on S^2. This representation is unitary. We would like to decompose it into a sum of irreducible representations of SO_3. We note first that the space $\mathbb{C}[S^2]$ obtained by restriction of the polynomial functions $\mathbb{C}[x_1, x_2, x_3]$ to S^2 is the invariant dense subspace in $L^2(S^2)$. Indeed, it is dense in the space of continuous functions on S^2 by the Weierstrass theorem and the latter space is dense in $L^2(S^2)$.

Let us introduce the following differential operators in $\mathbb{R}^3$:

$$e := -\frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right),\ h := x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + \frac{3}{2},\ f := \frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right),$$

note that e, f, and h commute with the action of SO_3 and satisfy the relations

$$[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f,$$

where $[a, b] = ab - ba$.

Let P_n be the space of homogeneous polynomial of degree n and $H_n = \operatorname{Ker} f \cap P_n$. The polynomials of H_n are called *harmonic* polynomials since they are annihilated by the Laplace operator f. For any $\varphi \in P_n$ we have

$$h(\varphi) = \left(n + \frac{3}{2}\right)\varphi.$$

If $\varphi \in H_n$, then

$$fe(\varphi) = ef(\varphi) - h(\varphi) = -\left(n + \frac{3}{2}\right)\varphi,$$

and by induction

$$fe^k(\varphi) = efe^{k-1}(\varphi) - he^{k-1}(\varphi) = -\left(nk + k(k-1) + \frac{3k}{2}\right)e^{k-1}\varphi.$$

In particular, this implies that

$$fe^k(H_n) = e^{k-1}(H_n). \tag{3.6}$$

We will prove now that

$$P_n = H_n \oplus e(H_{n-2}) \oplus e^2(H_{n-4}) + \dots \tag{3.7}$$

by induction on n. Indeed, by the induction assumption

$$P_{n-2} = H_{n-2} \oplus e\left(H_{n-4}\right) + \dots .$$

Furthermore, (3.6) implies $fe\left(P_{n-2}\right) = P_{n-2}$. Hence $H_n \cap eP_{n-2} = 0$. On the other hand, $f\colon P_n \to P_{n-2}$ is surjective, and therefore $\dim H_n + \dim P_{n-2} = \dim P_n$. Therefore

$$P_n = H_n \oplus eP_{n-2}, \tag{3.8}$$

which implies (3.7). Note that after restriction to S^2, the operator e acts as the multiplication on $\frac{-1}{2}$.

Hence (3.7) implies that

$$\mathbb{C}\left[S^2\right] = \bigoplus_{n\geq 0} H_n.$$

To calculate the dimension of H_n use (3.8)

$$\dim H_n = \dim P_n - \dim P_{n-2} = \frac{(n+1)(n+2)}{2} - \frac{n(n-1)}{2} = 2n+1.$$

Finally, we will prove that the representation of SO_3 in H_n is irreducible and isomorphic to ρ_{2n}. Consider the subgroup $D \subset SO_3$ consisting of all rotations about x_3-axis. Then D is the image under $\gamma : SU_2 \to SO_3$ of a diagonal subgroup of SU_2. Let R_θ denotes the rotation by the angle θ.

Exercise 3.5. Let V_{2n} be the space of the representation ρ_{2n}. Check that the set of eigenvalues of R_θ in the representation V_{2n} equals $\{e^{k\theta i} \mid -n \leq k \leq n\}$.

Let $\varphi = (x_1 + ix_2)^n$. It is easy to see that $\varphi_n \in H_n$ and $R_\theta(\varphi_n) = e^{n\theta i}\varphi_n$. By Exercise 3.5 this implies that H_n contains a subrepresentation isomorphic to ρ_{2k} for some $k \geq n$. By comparing dimensions we see that this implies $H_n = V_{2n}$. Thus, we obtain the following decompositions

$$\mathbb{C}[S^2] = \bigoplus_{n\in\mathbb{N}} H_n, \;\; L^2(S^2) = \widehat{\bigoplus_{n\in\mathbb{N}}} H_n.$$

Now, we are able to prove the following geometrical theorem.

Theorem 3.6. *A convex centrally symmetric solid in $\mathbb{R}^3$ is uniquely determined by the areas of the plane cross-sections through the origin.*

Proof. A convex solid B can be defined by an even continuous function on S^2. Indeed, for each unit vector v let

$$\varphi(v) = \sup\left\{t^2 \in \mathbb{R} \mid tv \in B\right\}.$$

Define a linear operator T in the space of all even continuous functions on S^2 by the formula

$$T\varphi(v) = \frac{1}{2}\int_0^{2\pi} \varphi(w)\, d\theta,$$

where w runs the set of unit vectors orthogonal to v, and θ is the angular parameter on the circle $S^2 \cap v^{\perp}$. Check that $T\varphi(v)$ is the area of the cross section by the plane $v^{\perp}$. We have to prove that T is invertible.

Obviously T commutes with the SO_3-action. Therefore T can be diagonalized by using Schur's lemma and the decomposition

$$L^2(G)_{even} = \widehat{\bigoplus_{n\in\mathbb{N}}} H_{2n}.$$

Indeed, T acts on H_{2n} as the scalar operator $\lambda_n Id$. We have to check that $\lambda_n \neq 0$ for all n. Consider again $\varphi_{2n} \in H_{2n}$. Then $\varphi_{2n}(1,0,0) = 1$ and

$$T\varphi_{2n}(1,0,0) = \frac{1}{2}\int_0^{2\pi} (iy)^{2n}\, d\theta = \frac{(-1)^n}{2}\int_0^{2\pi} \sin^{2n}\theta d\theta,$$

here we take the integral over the circle $x_2^2 + x_3^2 = 1$, and assume $x_2 = \sin\theta$, $x_3 = \cos\theta$. Since $T\varphi = \lambda_n\varphi$, we obtain

$$\lambda_n = \frac{(-1)^n}{2}\int_0^{2\pi} \sin^{2n}\theta d\theta \neq 0.$$

$\square$

CHAPTER 4

Some results about unitary representations

In this chapter, we consider unitary representations of groups which are locally compact but no longer compact. We do not intend to go very far in this deep subject, but we want to give three examples in order to show how the structure of the dual of the group changes.

1. Unitary representations of $\mathbb{R}^n$ and Fourier transform

1.1. Unitary dual of an abelian group. Let G be an abelian topological group. Then by Corollary 1.22 of Chapter III, every unitary representation of G is one-dimensional. Therefore the unitary dual $\widehat{G}$ is the set of continuous homomorphisms $\rho : G \to S^1$. Moreover, $\widehat{G}$ is an abelian group with multiplication defined by the tensor product.

For example, as we have seen Section 3.1 chapter III, if $G = S^1$ is a circle, then $\widehat{G}$ is isomorphic to $\mathbb{Z}$. In general, if G is compact, then $\widehat{G}$ is discrete. If G is not compact, one can define a topology on $\widehat{G}$ in such a way that the natural homomorphim $s : G \to \widehat{\widehat{G}}$, defined by $s(g)(\rho) = \rho(g)$, is an isomorphism. This fact is usually called the Pontryagin duality.

Let us concentrate on the case when $G = V$ is a real vector space of finite dimension n. Let us fix an invariant volume form dx on V. The unitary dual of V is isomorphic to the usual dual V^* via identification

$$\rho_\xi(x) = e^{2i\pi<\xi,x>} \text{ for all } x \in V, \xi \in V^*,$$

where $< \xi, x >$ is the duality evaluation.

We immediately see that, in contrast with the compact case, $\rho_\xi \notin L^2(V)$. We still can try to write down the formula for the projector P_ξ from $L^2(V)$ onto the irreducible representation ρ_ξ as in Exercise 2.10 Chapter III. For $f \in L^2(V)$, $y \in V$ and $\xi \in V^*$ we set

$$P_\xi(f)(y) := \int_V f(x+y)e^{-2i\pi<\xi,x>}dx = (\int_V f(z)e^{-2i\pi<\xi,z>}dz)\rho_\xi(y).$$

The coefficient $\int_V f(z)e^{-2i\pi<\xi,z>}dz$ is nothing else but the value $\hat{f}(\xi)$ of the Fourier transform $\hat{f}$. However, the integral defining $\hat{f}$ is in general divergent for $f \in L^2(V)$. In this section we explain how to overcome this difficulty, see Plancherel Theorem 1.12.

We also would like to claim that every $f \in L^2(V)$ is "a sum" of its projections, which leads to the formula

$$f(x) = \int_V \hat{f}(y) e^{2i\pi <\xi, x>} d\xi.$$

This is involutivity of the Fourier transfrom, see Theorem 1.7 below.

1.2. Fourier transform: generalities. Let $L^1(V)$ be the set of integrable complex-valued functions on V.

DEFINITION 1.1. Let $f \in L^1(V)$, the Fourier transform of f is the function on V^*

$$\hat{f}(\xi) := \int_V f(x) e^{-2i\pi <\xi, x>} dx,$$

REMARK 1.2. (1) One checks that $\lim_{\xi \to \infty} \hat{f}(\xi) = 0$ and that $\hat{f}$ is continuous on V^*.

(2) Nevertheless, there is no reason for $\hat{f}$ to belong to $L^1(V^*)$ (check on the characteristic function of an interval in $\mathbb{R}$).
(3) The Fourier transform of the convolution (cf Definition 2.5 Chapter III) of two functions is the product of the Fourier transforms of the two factors.
(4) (Adjunction formula for Fourier transforms), let $f \in L^1(V)$ and $\varphi \in L^1(V^*)$, then

$$\int_V f(x) \hat{\varphi}(x) dx = \int_{V^*} \hat{f}(\xi) \varphi(\xi) d\xi.$$

EXERCISE 1.3. Let $\gamma \in GL(V)$, show that the Fourier transform of the function $\gamma.f$ defined by $(\gamma.f)(x) = f(\gamma^{-1}(x))$ is $\det(\gamma)^t\gamma^{-1}.\hat{f}$.

Let us consider the *generalized Wiener algebra* $\mathcal{W}(V)$ consisting of integrable functions on V whose Fourier transform is integrable on V^*.

PROPOSITION 1.4. *The subspace $\mathcal{W}(V) \subset L^1(V)$ is a dense subset (for the L^1-norm).*

PROOF. Let Q be a positive definite quadratic form on V, denote by B its polarization and by Q^{-1} the quadratic form on V^* whose polarization is B^{-1}. Let $Disc(Q)$ denote the discriminant of Q in a basis of V of volume 1.

LEMMA 1.5. *The Fourier transform of the function $\phi : x \mapsto e^{-\pi Q(x)}$ on V is the function $\xi \mapsto Disc(Q)^{-1/2} e^{-\pi Q^{-1}(\xi)}$ on V^*.*

PROOF. (of the lemma) One can reduce this lemma to the case $n = 1$ by using an orthogonal basis for Q and Fubini's theorem. We just need to compute the Fourier transform of the function $\epsilon(x) := x \mapsto e^{-\pi x^2}$ on the line $\mathbb{R}$.

One has

$$\hat{\epsilon}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2 - 2i\pi \xi x} dx = e^{-\pi \xi^2} \int_{\mathbb{R}} e^{-\pi (x + i\xi)^2} dx.$$

By complex integration, the integral factor in the far-right-hand side does not depend on ξ and its value for $\xi = 0$ is the Gauss integral $\int_{\mathbb{R}} e^{-\pi x^2} dx = 1$. Hence the lemma. $\square$

To finish the proof of Proposition let us take Q such that $Disc(Q) = 1$. The lemma implies that ϕ belongs to $\mathcal{W}(V)$. For every $\lambda \in \mathbb{R}_{>0}$, we set $\phi_\lambda(x) := \lambda^n \phi(\lambda x)$.

EXERCISE 1.6. Check that $\phi_\lambda(x)$ is a positive-valued function and $\int_V \phi_\lambda(x)dx = 1$. Prove that when λ tends to infinity $\phi_\lambda(x)$ converges uniformally to 0 in the complement of any neighbourhood of $0 \in V$.

Now take any function $f \in L^1(V)$. By Remark 1.2 the convolution product $f_\lambda := f * \phi_\lambda$ belongs to $\mathcal{W}(V)$. By the exercise f_λ converges to f for the L^1-norm. $\square$

THEOREM 1.7. *(Fourier reciprocity) Let $f \in \mathcal{W}(V)$, one has, for all $x \in V$:*

$$\hat{\hat{f}}(x) = f(-x).$$

PROOF. By Proposition 1.4 the set of continuous bounded functions is dense in $\mathcal{W}(V)$. Hence it suffices to prove the statement for continuous bounded f. We use a slight extension of the adjunction formula (Remark 1.2, (4)): let $\lambda \in \mathbb{R}_{>0}$, one has, for all $f \in L^1(V)$ and $\varphi \in L^1(V^*)$,

$$\int_V f(\lambda x)\hat{\varphi}(x)dx = \int_{V^*} \hat{f}(\xi)\varphi(\lambda\xi)d\xi = \iint_{V\times V^*} f(x)\varphi(\xi)e^{-2i\pi\lambda<\xi,x>}dxd\xi. \tag{4.1}$$

If λ goes to 0, the function $x \mapsto f(\lambda x)$ tends to $f(0)$ and remains bounded by $\sup \mid f \mid$. By dominated convergence, we obtain the equality

$$f(0)\hat{\hat{\varphi}}(0) = \hat{\hat{f}}(0)\varphi(0). \tag{4.2}$$

We know that, if $\varphi(\xi) = \phi(\xi)$ (see Lemma 1.5) $\hat{\hat{\varphi}} = \varphi$, thus $\hat{\hat{f}}(0) = f(0)$.

We use the actions of the additive group V on $\mathcal{W}(V)$ given by

$$\tau_y(f) : (x \mapsto f(x-y))$$

and on $\mathcal{W}(V^*)$ given by

$$\mu_y(\varphi) : \xi \mapsto e^{-2i\pi<\xi,y>}\varphi(\xi)$$

for all $y \in V$.

EXERCISE 1.8. Check that

(1) $\widehat{\tau_y(f)} = \mu_y(\hat{f})$ for all $f \in L^1(V)$,
(2) $\widehat{\mu_y(\varphi)} = \tau_{-y}(\hat{\varphi})$ for all $\varphi \in L^1(V^*)$.

We apply τ_y to f, Exercise 1.8 shows that $\widehat{\widehat{\tau_y(f)}} = \tau_{-y}\hat{\hat{f}}$, hence the result. $\square$

REMARK 1.9. Fourier reciprocity is equivalent to the following statement

$$\bar{\hat{\bar{\hat{f}}}} = f \tag{4.3}$$

where $\bar{f}$ denotes the complex conjugate of f.

COROLLARY 1.10. *The space $\mathcal{W}(V)$ is a dense subspace in $L^2(V)$.*

PROOF. By Theorem 1.7 and Remark 1.2, $\mathcal{W}(V)$ is a subset of the set $\mathcal{C}^0(V)$ of continuous functions on V which tend to 0 at infinity. Here we need a reference, Rudin? Therefore $\mathcal{W}(V)$ is included in $L^2(V)$. The proof of Proposition 1.4 can be adapted to prove the density of $\mathcal{W}(V)$ in $L^2(V)$ (using that $\phi \in L^2(V)$). □

COROLLARY 1.11. *The Fourier transform is an injective map from $L^1(V)$ to $\mathcal{C}^0(V^*)$.*

PROOF. We first notice that $\mathcal{W}(V)$ is dense in $\mathcal{C}^0(V)$ by the same argument as in the proof of the Proposition 1.4.

Hence, if $f \in L^1(V)$ is such that $\hat{f} = 0$, to show that $f = 0$ it suffices to prove that

$$\int_V f(x)g(x)dx = 0$$

for any $g \in \mathcal{W}(V)$. By Theorem 1.7, g is the Fourier transform of $\xi \mapsto \hat{g}(-\xi)$ and by Remark 1.2(4), one has

$$\int_V f(x)g(x)dx = \int_{V^*} \hat{f}(\xi)\hat{g}(-\xi)d\xi = 0$$

. □

THEOREM 1.12. *(Plancherel) The Fourier transform extends to an isometry from $L^2(V)$ to $L^2(V^*)$.*

PROOF. Since $\mathcal{W}(V)$ is dense in $L^2(V)$, all we have to show is that for f and g in $\mathcal{W}(V)$, one has

$$\int_V \bar{f}(x)g(x)dx = \int_{V^*} \bar{\hat{f}}(\xi)\hat{g}(\xi)d\xi. \tag{4.4}$$

By Remark 1.2 (4), the right-hand side is equal to

$$\int_V \hat{\bar{\hat{f}}}(x)g(x)dx.$$

But, by Remark 1.9 (4.3), $\hat{\bar{\hat{f}}} = \bar{f}$, QED. □

REMARK 1.13. This result amounts to saying that the Fourier transform in generalized Wiener algebras changes the usual product into the convolution product.

1.3. The link between Fourier series and Fourier transform on $\mathbb{R}$. Let f be a function over the interval $[-\frac{1}{2}, \frac{1}{2}]$. The Fourier series of f is

$$\sum_{n\in\mathbb{Z}} c_n(f)e^{2i\pi nx} \tag{4.5}$$

where

$$c_n := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)e^{-2i\pi nt}dt.$$

Now, for $\lambda \in \mathbb{R}_{>0}$, if g is a function defined over the interval $[-\frac{\lambda}{2}, \frac{\lambda}{2}]$, changing the variable by $y := \lambda x$, the corresponding Fourier series is written

$$\sum_{n\in\mathbb{Z}} \left(\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \frac{1}{\lambda} g(u)e^{-2i\pi n\frac{u}{\lambda}}du \right) e^{2i\pi n\frac{y}{\lambda}}. \tag{4.6}$$

We consider that, formally, g is the sum of its Fourier series on $[-\frac{\lambda}{2}, \frac{\lambda}{2}]$.

Now if we consider g as a function defined on $\mathbb{R}$ with compact support by extending by 0 outside the interval $[-\frac{\lambda}{2}, \frac{\lambda}{2}]$, we may interpret the n-th Fourier coefficient as $\frac{1}{\lambda}\hat{g}(\frac{n}{\lambda})$ and the Fourier series as the sum

$$\frac{1}{\lambda}\sum_{n\in\mathbb{Z}} \hat{g}(\frac{n}{\lambda})e^{2i\pi\frac{ny}{\lambda}}. \tag{4.7}$$

Formally, this series is exactly the Riemann sum, corresponding to the partition of $\mathbb{R}$ associated to the intervals $[\frac{n}{\lambda}, \frac{n+1}{\lambda}]$, of the infinite integral $\int_{\mathbb{R}} \hat{f}(t)e^{2i\pi nt}dt$.

If now g is compactly supported and λ tends to $+\infty$, this formal expression of the sum suggests the equality

$$g(t) = \int_{\mathbb{R}} \hat{g}(u)e^{2i\pi tu}du.$$

2. Heisenberg groups and the Stone-von Neumann theorem

2.1. The Heisenberg group and some examples of its unitary representation. Let V be a vector space over $\mathbb{R}$ of finite even dimension $n = 2g$ together with a non-degenerate symplectic form $\omega : (x, y) \mapsto (x|y)$. Let $\mathbb{T} = S^1$ be the group of complex numbers of modulus 1. We define the *Heisenberg group* H as the set theoretical product $\mathbb{T} \times V$ with the composition law

$$(t, x)(t', x') := (tt'e^{i\pi(x|x')}, x + x').$$

The centre of H is $\mathbb{T}$, imbedded in H by $t \mapsto (t, 0)$. There is non split exact sequence

$$1 \to \mathbb{T} \to H \to V \to 0.$$

The commutator of elements of H naturally factorises as the map $V \times V \to \mathbb{T}$

$$(x, y) \mapsto e^{2i\pi(x|y)}.$$

EXERCISE 2.1. Define a representation r of the group H in the space $L^2(V)$ by the formula

$$r_{(t,x)}f(y) := tf(y-x)e^{i\pi(x|y)} \text{ for all } t \in \mathbb{T},\ x, y \in V,\ f \in L^2(V).$$

Check that r is a unitary representation of H.

EXERCISE 2.2. Consider in V two maximal isotropic subspaces with zero intersection. If we denote one of them by W, then the second one can be identified by ω with the dual space W^*. Since the restriction of ω to both W and W^* is zero, the map $x \mapsto (1, x)$ from V to H induces groups homomorphisms on both W and W^*.

The *Schrödinger representation* σ of H in the Hilbert space $L^2(W)$ is defined by

$$\sigma_{(t,w+\eta)}f(x) = tf(x-w)e^{2i\pi(\eta|x)} \text{ for all } t \in \mathbb{T},\ x, w \in W,\ \eta \in W^*,\ \ f \in L^2(W).$$

Prove that σ is an irreducible unitary representation of H. To show irreducibility it suffices to check that any bounded operator T in $L^2(W)$ commuting with the action of H is a scalar multiplication. First, since T commutes with the action of W^*, it commutes also with multiplication by any continuous function with compact support. Making use of partitions of unity, show that this implies that T is the multiplication by some bounded measurable function g on W. Moreover, since T commutes with the action of W, the function g is invariant under translations, hence is a constant function.

2.2. The Stone–von Neumann theorem. The aim of this subection is to show

THEOREM 2.3. *(Stone–von Neumann) Let ρ be a unitary representation of H such that $\rho_t = t\,\mathrm{Id}$ for all $t \in \mathbb{T}$. Then ρ is isomorphic to the Schrödinger representation.*

Let $\mathcal{H}$ be a Hilbert space together with an action ρ of the Heisenberg group H: we assume that the hypotheses of the theorem are satisfied by $(\rho, \mathcal{H})$. To simplify the notations, we identify $x \in V$ with $(1, x) \in H$, although it is not a group homomorphism. We set $\rho(x) := \rho_{(1,x)}$ for all $x \in V$. Then the condition that ρ is a representation is equivalent to

$$\rho(x)\rho(y) = e^{i\pi(x|y)}\rho(x+y). \tag{4.8}$$

We denote by $\mathcal{A}$ the minimal closed subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$, which contains the image ρ_H. Let $\mathcal{C}_c^0(V)$ be the space of compactly supported continuous complex valued functions on V. For every $\varphi \in \mathcal{C}_c^0(V)$, set

$$T_\varphi := \int_V \varphi(x)\rho(x)dx,$$

where dx is the Lebesgue measure on V. It is easy to see that $T_\varphi \in \mathcal{A}$. We have

$$T_\varphi T_\psi = \iint_{V\times V} \varphi(x)\psi(y)\rho(x)\rho(y)dxdy$$

$$T_\varphi T_\psi = \iint_{V\times V} \varphi(x)\psi(y)e^{i\pi(x|y)}\rho(x+y)dxdy$$
$$T_\varphi T_\psi = \iint_{V\times V} \varphi(x)\psi(u-x)e^{i\pi(x|u-x)}\rho(u)dxdu$$
$$T_\varphi T_\psi = T_{\varphi \circledast \psi},$$

where $\varphi \circledast \psi$ is defined by the formula

$$\varphi \circledast \psi(u) = \int_V \varphi(x)\psi(u-x)e^{i\pi(x|u-x)}dx. \tag{4.9}$$

Since clearly $||T_\varphi|| \leq \|\varphi\|_1 (= \int_V |\varphi(x)|dx)$, we get the following statements:

- The map $\varphi \mapsto T_\varphi$ extends by continuity to $L^1(V)$, the space of integrable complex valued functions on V.
- The product $(\varphi, \psi) \mapsto \varphi \circledast \psi$ extends to a product $L^1(V) \times L^1(V) \to L^1(V)$
- The formula (4.9) remains valid for φ and ψ in $L^1(V)$ for almost every $u \in V$.

Lemma 2.4. *The map $\varphi \mapsto T_\varphi$ is injective on $L^1(V)$.*

Proof. Denote by N the kernel of this map. We notice the equality

$$\rho(y)T_\varphi\rho(-y) = \int_V \varphi(x)\rho(y)\rho(x)\rho(-y)dx = \int_V \varphi(x)e^{2i\pi(y|x)}\rho(x)dx.$$

It shows that if $\varphi(x)$ is in N then $\varphi(x)e^{2i\pi(y|x)}$ lies in N for every $y \in V$. For a, b in $\mathcal{H}$, consider the matrix coefficient function

$$\chi_{a,b}(x) =< \rho(x)a, b >,$$

where $<,>$ is the scalar product on $\mathcal{H}$. It is a continuous bounded function of $x \in V$. Moreover, for any x, there exists at least one coefficient function which doesn't vanish at x.

If φ belongs to N, we have

$$\int_V \varphi(x)\chi_{a,b}(x)dx = 0,$$

and therefore

$$\int_V \varphi(x)\chi_{a,b}(x)e^{2i\pi(x|y)}dx = 0$$

for all $y \in V$. This means that the Fourier transform of the function $\varphi\chi_{a,b} \in L^1(V)$ is identically zero, hence $\varphi\chi_{a,b} = 0$ for all a, b, therefore $\varphi = 0$. $\square$

We will also use the following equality:

$$T_\varphi^* = T_{\varphi^*}, \tag{4.10}$$

with $\varphi^*(x) := \overline{\varphi}(-x)$.

Our ultimate goal is to construct a continuous intertwiner $\tau : L^2(V) \to \mathcal{H}$. The following observation is crucial for this construction.

LEMMA 2.5. *(a) For all $f \in \mathcal{C}_c^0(V)$ and $h \in H$ we have $T_{r_h f} = \rho_h T_f$.*

(b) For any $u \in \mathcal{H}$ the map $\pi_u := \mathcal{C}_c^0(V) \to \mathcal{H}$ defined by $f \mapsto T_f u$ is H-equivariant.

PROOF. It suffices to check (a) for $h = (1, y)$ with $y \in V$. Then using (4.8) and making the substitution $z = x - y$ we have

$$T_{r_h f} = \int_V f(x-y)\rho(x)e^{i\pi(y|x)}dx = \int_V f(x-y)\rho(y)\rho(x-y)dx = \rho(y)\int_V f(z)\rho(z)dz = \rho(y)T_f.$$

(b) follows immediately from (a). □

Thus we have an equivariant map $\pi_u : \mathcal{C}_c^0(V) \to \mathcal{H}$. It remains to show that for a suitable choice of $u \in \mathcal{H}$ we are able to extend π_u to a continuous map $L^2(V) \to \mathcal{H}$.

LEMMA 2.6. *Let φ be a continuous bounded function on V which lies in the intersection $L^1(V) \cap L^2(V)$. Assume that T_φ is an orthogonal projection onto a line $\mathbb{C}\varepsilon_\phi$ for some vector ε_φ in $\mathcal{H}$ of norm 1. Then the map $\pi_{\varepsilon_\varphi} : \mathcal{C}_c^0(V) \to \mathcal{H}$ extends to a continuous linear H-equivariant map $\tau : L^2(V) \to \mathcal{H}$.*

PROOF. Observe that for any $f \in L^2(V)$ the convolution $f \circledast \varphi$ lies in $L^1(V)$. Hence we can use

$$T_f\varepsilon_\varphi = T_f T_\varphi \varepsilon_\varphi = T_{f\circledast\varphi}\varepsilon_\varphi.$$

□

The next step is to look for a function φ such that T_φ is an orthogonal projector of rank 1.

LEMMA 2.7. *Let $P \in \mathcal{B}(\mathcal{H})$ be a self-adjoint bounded operator and $P \neq 0$. Then P is a scalar multiple of an orthogonal projector of rank 1 if and only if for any $x \in V$ we have*

$$P\rho(x)P \in \mathbb{C}P \tag{4.11}$$

PROOF. Note that if P is a multiple of an orthogonal projector of rank 1, then clearly $P\rho(x)P \in \mathbb{C}P$ for all $x \in V$.

Assume now that P satisfies the latter condition. First, we have $P^2 = \lambda P$ for some non-zero λ. Hence after normalization we can assume $P^2 = P$. Hence P is a projector. It is an orthogonal projector since P is self-adjoint.

It remains to prove that P has rank 1. Let u be a non-zero vector in $P(\mathcal{H})$ and M be the span of $\rho(x)u$ for all $x \in V$. The assumption on P implies that M is included in $\mathbb{C}u \oplus \operatorname{Ker} P$. Irreducibility of $\mathcal{H}$ implies that M is dense in $\mathcal{H}$. Hence we have $\mathcal{H} = \mathbb{C}u \oplus \operatorname{Ker} P$. Hence P has rank 1. □

LEMMA 2.8. *Let φ be an element in $L^1(V)$ and $\varphi = \varphi^*$. Then T_φ is a multiple of an orthogonal projection on a line if and only if for all $u \in V$, the function*

$$x \mapsto \varphi(u + x)\varphi(u - x)$$

is its own Fourier transform.

REMARK 2.9. Note that the Fourier transform is defined on the dual V^* of V, and those spaces are identitified through the symplectic form ω. We will refer to this characterisation of φ as the *functional equation*.

PROOF. We use Lemma 2.7. Let $v \in V$. We compute

$$T_\varphi \rho(v) T_\varphi = \iint_{V\times V} \varphi(x)\varphi(y)\rho(x)\rho(v)\rho(y)dxdy$$
$$= \iint_{V\times V} \varphi(x)\varphi(y)e^{i\pi((x|v)+(x|y)+(v|y))}\rho(x+v+y)dxdy$$
$$= \iint_{V\times V} \varphi(x)\varphi(z-v-x)e^{i\pi((x|v)+(x|z)+(v|z))}\rho(z)dxdz.$$

For almost every value of z, this operator is T_ψ for

$$\psi(v,z) = \int_V \varphi(x)\varphi(z-v-x)e^{i\pi((x|v)+(x|z)+(v|z))}dx$$

by Fubini's theorem. The relation (4.11) is equivalent to the fact that for every v, $\psi = C(v)\varphi$. So (4.11) is equivalent to:

$$\int_V \varphi(x)\varphi(z-v-x)e^{i\pi((x|v)+(x|z)+(v|z))}dx = C(v)\varphi(z).$$

In the left hand side, we set $x = -y$ and use $\varphi^* = \varphi$. Then we obtain

$$\int_V \overline{\varphi}(y)\overline{\varphi}(v-z-y)e^{-i\pi((y|v)+(y|z)+(z|v))}dy = C(v)\varphi(z) = \overline{C}(z)\overline{\varphi}(v).$$

Hence

$$\frac{\varphi(z)}{\overline{C}(z)} = \frac{\overline{\varphi}(v)}{C(v)} \tag{4.12}$$

so that $\frac{\varphi(z)}{\overline{C}(z)}$ does not depend on z, moreover it is equal to its complex conjugate hence belongs to $\mathbb{R}$. We set $C = \frac{\varphi(z)}{\overline{C}(z)}$, and get $C(z) = C\varphi(-z)$.

Finally,

$$\int_V \varphi(x)\varphi(z-v-x)e^{i\pi((x|v)+(x|z)+(v|z))}dx = C\varphi(-v)\varphi(z).$$

Now we set $t := x - \frac{1}{2}(z-v)$ and we get

$$\int_V \varphi(\frac{z-v}{2}+t)\varphi(\frac{z-v}{2}-t)e^{i\pi(t|z+v)}dt = C\varphi(-v)\varphi(z).$$

The left hand side is precisely the value at $\frac{z+v}{2}$ of the Fourier transform of $t \mapsto \varphi(\frac{z-v}{2}+t)\varphi(\frac{z-v}{2}-t)$, now Fourier reciprocity implies $C^2 = 1$ and C is a positive real number as can be seen by setting $z = v$ in (4.12), hence the Lemma. $\square$

In order to find a non-trivial solution of the functional equation, we choose a positive definite quadratic form Q on V, denote by $B : V \to V^*$ the morphism induced by the polarization of Q. We recall (Lemma 1.5) that the Fourier transform of the function $z \mapsto e^{-\pi Q(z)}$ on V is the function $w \mapsto Disc(Q)^{-\frac{1}{2}} e^{-\pi Q^{-1}(w)}$ on V^*. Let $\Omega : V \to V^*$ be the isomorphism induced by the symplectic form ω.

LEMMA 2.10. *The function $x \mapsto \psi(x) = e^{-\pi Q(x)}$ is its own Fourier transform if and only if one has*

$$(\Omega^{-1}B)^2 = -Id_V.$$

PROOF. Straightforward computation. □

LEMMA 2.11. *The function*

$$\varphi(x) := e^{-\pi \frac{Q(x)}{2}}$$

satisfies the functional equation of Lemma 2.8.

PROOF. This is easily shown using the fact that $\varphi(u+x)\varphi(u-x) = \varphi^2(u)\varphi^2(x)$. □

Now by application of Lemma 2.6 we obtain a bounded H-invariant linear operator $\tau : L^2(V) \to \mathcal{H}$. Consider the dual operator $\tau^* : \mathcal{H} \to L^2(V)$. The composition $\tau\tau^*$ is a bounded intertwiner in $\mathcal{H}$. Hence Theorem 1.20 Chapter III implies that $\tau\tau^* = \lambda\,\mathrm{Id}_{\mathcal{H}}$ for some positive real λ (since $\tau\tau^*$ is self-adjoint positive). Next we will show that $\lambda = 1$.

LEMMA 2.12. *We have $\tau^*(\varepsilon_\varphi) = \varphi$ and $\tau\tau^* = \mathrm{Id}_{\mathcal{H}}$.*

PROOF. Consider the operator $Y : L^2(V) \to L^2(V)$ defined by $Y(f) := \varphi \circledast f \circledast \varphi$. Lemma 2.4 and relations $\varphi \circledast \varphi = \varphi$, $T_\varphi T_f T_\varphi \in \mathbb{C}T_\varphi$ imply that Y is an orthogonal projection on the line $\mathbb{C}\varphi$. Hence $Y(f) = \langle \varphi, f\rangle_{L^2(V)}\,\varphi$. If f is orthogonal to $\tau^*(\varepsilon_\varphi)$, then

$$\langle \tau^*(\varepsilon_\varphi), f\rangle_{L^2(V)} = \langle \varepsilon_\varphi, \tau(f)\rangle_{\mathcal{H}} = \langle \varepsilon_\varphi, T_f(\varepsilon_\varphi)\rangle_{\mathcal{H}} = 0.$$

This is equivalent to $T_\varphi T_f T_\varphi = T_{Y(f)} = 0$. Hence f is orthogonal to φ. We obtain that $\tau^*(\varepsilon_\varphi) = c\varphi$ for some $c \in \mathbb{C}$. But

$$c = \langle c\varphi, \varphi\rangle_{L^2(V)} = \langle \tau^*(\varepsilon_\varphi), \varphi\rangle_{L^2(V)} = \langle \varepsilon_\varphi, \tau(\varphi)\rangle_{\mathcal{H}} = \langle \varepsilon_\varphi, \varepsilon_\varphi\rangle_{\mathcal{H}} = 1.$$

The first assertion is proved.

Now

$$\langle \tau\tau^*(\varepsilon_\varphi), \varepsilon_\varphi\rangle_{\mathcal{H}} = \langle \tau^*(\varepsilon_\varphi), \tau^*(\varepsilon_\varphi)\rangle_{L^2(V)} = \langle \varphi, \varphi\rangle_{L^2(V)} = 1.$$

Hence the second assertion. □

Thus, we have shown that an arbitrary irreducible unitary representation $\mathcal{H}$ is equivalent to the subrepresentation of $L^2(V)$ generated by $\varphi(x) = e^{-\pi \frac{Q(x)}{2}}$. Hence the Stone-von Neumann theorem is proved.

2.3. Fock representation. Let us continue with a lovely avatar of this representation, the *Fock representation*. We would like to characterize the image $\tau^*(\mathcal{H})$ inside $L^2(V)$.

Just before Lemma 2.10, we chose a quadratic form Q on V such that $(\Omega^{-1}B)^2 = -Id_V$, and this equips V with a structure of complex vector space of dimension g for which $\Omega^{-1}B$ is the scalar multiplication by the imaginary unit i. We denote by J this complex structure and by V_J the corresponding complex space.

Furthermore $B + i\Omega : V_J \to V_J^*$ is a sesiquilinear isomorphism, we denote by A the corresponding Hermitian form on V_J.

In this context, for a given $x \in V$ we have:

$$r_{(1,x)}\varphi(y) = \varphi(y-x)e^{i\pi(x|y)} = e^{-\pi\frac{Q(y-x)}{2}+i\pi(x|y)} = e^{-\pi(\frac{Q(x)+Q(y)}{2}-A(x,y))} \tag{4.13}$$

which is the product of $\varphi(y)$ with a holomorphic function of $f(y) = e^{-\pi(\frac{Q(x)}{2}-A(x,y))}$.

The *Fock representation* associated to the complex structure J is the subspace $\mathcal{F}_J \subset L^2(V)$ consisting of the products $f\varphi$ where φ was defined before and f is a holomorphic function on V_J. We have just proven that this space is stable under the H-action. Moreover, it is closed in $L^2(V)$ since holomorphy is preserved under uniform convergence on compact sets. Let us choose complex coordinates $z = (z_1, \dots, z_g)$ in V_J so that the Hermitiam product has the form $A(w,z) = \sum \bar{w}_i z_i$. The scalar product $(\cdot,\cdot)_F$ in $\mathcal{F}_J$ is given by

$$(f\varphi, g\varphi)_F = \int_V \bar{f}(z)g(z)e^{-\pi|z|^2}d\bar{z}dz,$$

where $|z|^2 = \sum_{i=1}^g |z_i|^2$. If $\mathbf{m} = (m_1, \dots, m_g) \in \mathbb{Z}^g$ we denote by $z^{\mathbf{m}}$ the monomial function $z_1^{m_1} \dots z_g^{m_g}$. Any analytic function $f(z)$ can be represented by a convergent series

$$f(z) = \sum_{\mathbf{m}\in\mathbb{Z}^g} a_{\mathbf{m}} z^{\mathbf{m}}. \tag{4.14}$$

Exercise 2.13. Check that if $f(z)\varphi \in \mathcal{F}_J$ then the series

$$f(z)\varphi = \sum_{\mathbf{m}\in\mathbb{Z}^g} a_{\mathbf{m}} z^{\mathbf{m}}\varphi$$

is convergent in the topology defined by the norm in $\mathcal{F}_J$. Furthermore, prove that $\{z^{\mathbf{m}} \mid \mathbf{m} \in \mathbb{Z}^g\}$ is an orthogonal topological basis of $\mathcal{F}_J$.

Lemma 2.14. *The image $\tau^*(\mathcal{H})$ is equal to $\mathcal{F}_J$. Hence the representation of H in $\mathcal{F}_J$ is irreducible.*

Proof. Recall that $\tau^*(\varepsilon_\varphi) = \varphi$. Therefore taking into account (4.13) it is sufficient to show that the set $\{e^{-\pi A(x,y)}\varphi(y) \mid x \in V\}$ is dense in $\mathcal{F}_J$. Let $f\varphi \in \mathcal{F}_J$. Assume that

$$(f(y)\varphi(y), e^{-\pi A(x,y)})_F = 0 \quad \text{for all } x \in V.$$

In the z-coordinates it amounts to saying that

$$F(w) = \int_V \bar{f}(z) e^{\sum_{i=1}^g w_i z_i} e^{-\pi|z|^2} d\bar{z}dz$$

is identically zero. Note that then the partial derivative

$$\frac{\partial F}{\partial w_i} = \int_V z_i \bar{f}(z) e^{\sum_{i=1}^g w_i z_i} e^{-\pi|z|^2} d\bar{z}dz,$$

is also zero. Hence for every monomial $z^{\mathbf{m}}$ and $w \in V_J$ we have

$$\int_V z^{\mathbf{m}} \bar{f}(z) e^{\sum_{i=1}^g w_i z_i} e^{-\pi|z|^2} d\bar{z}dz = 0.$$

Consider the Taylor series (4.14). By Exercise 2.13 we have for any $w \in V_J$

$$\int_V f(z)\bar{f}(z) e^{\sum_{i=1}^g w_i z_i} e^{-\pi|z|^2} d\bar{z}dz = 0,$$

in particular,

$$\int_V f(z)\bar{f}(z) d\bar{z}dz = 0,$$

which implies $f(z) = 0$. Hence the set $\{e^{-\pi A(x,y)}\varphi(y) \,|\, x \in V\}$ is dense in $\mathcal{F}_J$. $\square$

EXERCISE 2.15. Check that $f \circledast \varphi \in \mathcal{F}_J$ for any $f \in L^2(V)$. Therefore the map $f \mapsto f \circledast \varphi$ from $L^2(V)$ to $L^2(V)$ is an orthogonal projection onto $\mathcal{F}_J$.

2.4. Unitary dual of H. Now it is not hard to classify unitary irreducible representations of the Heisenberg group H. If ρ is an irreducible representation of H in a Hilbert space $\mathcal{H}$, then by Theorem 1.20 Chapter III, for every $t \in \mathbb{T}$ we have $\rho_t = \chi_t \operatorname{Id}_{\mathcal{H}}$ for some character $\chi \in \widehat{\mathbb{T}}$. In other words, using the description of $\widehat{\mathbb{T}}$, $\rho_t = t^n \operatorname{Id}_{\mathcal{H}}$ for some $n \in \mathbb{Z}$. Hence we have defined the map $\Phi : \widehat{H} \to \mathbb{Z} = \widehat{\mathbb{T}}$.

We know that the fiber $\Phi^{-1}(1) = \{\sigma\}$ is a single point due to the Stone–von Neumann theorem. We claim that for any $n \neq 0$ the fiber $\Phi^{-1}(n)$ is also a single point. Indeed, consider a linear transformation γ of V such that $\langle\gamma(x)|\gamma(y)\rangle = n\,\langle x|y\rangle$. Then we can define a homomorphism $\tilde{\gamma} : H \to H$ by setting $\tilde{\gamma}(t,x) = (t^n, \gamma(x))$. We have the exact sequence of groups

$$1 \to \mathbb{Z}/n\mathbb{Z} \to H \xrightarrow{\tilde{\gamma}} H \to 1.$$

If ρ lies in the fiber over n, then $\operatorname{Ker}\rho \subset \operatorname{Ker}\tilde{\gamma}$. Hence $\rho = \tilde{\gamma} \circ \rho'$, where ρ' lies in $\Phi^{-1}(1)$. Thus, $\rho \simeq \tilde{\gamma} \circ \sigma$.

Finally, $\Phi^{-1}(0)$ consists of all representations which are trivial on $\mathbb{T}$. Therefore $\Phi^{-1}(0)$ coincides with the unitary dual of $V = H/\mathbb{T}$ and hence isomorphic to V^*.

3. Representations of $\mathrm{SL}_2(\mathbb{R})$

In this section we give a construction of all up to isomorphism unitary irreducible representations of the group $SL_2(\mathbb{R})$. We do non provide a proof that our list is complete and refer to ?? for this.

3.1. Geometry of $SL_2(\mathbb{R})$. In this section we use the notation

$$G = \mathrm{SL}_2(\mathbb{R}) = \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det g = 1\}.$$

EXERCISE 3.1. (a) Since $G = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1\}$, topologically G can be described as a non-compact 3-dimensional quadric in $\mathbb{R}^4$.

(b) Conjugacy classes in G are given by the equations $\operatorname{tr} g = c$, where $c \in \mathbb{R}$, with exception of the case $\operatorname{tr} g = \pm 2$.

(c) The only proper non-trivial normal closed subgroup of G is the center $\{1, -1\}$.

Let us start with the following observation.

LEMMA 3.2. *Let $\rho : G \to GL(V)$ be a unitary finite-dimensiona representation of G. Then ρ is trivial.*

PROOF. Let $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then g^k is congugate to g for every non-zero integer k. Hence $\operatorname{tr} \rho_g = \operatorname{tr} \rho_{g^k}$. Note that ρ_g is unitary and hence diagonalizable in V. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of ρ_g (taken with muliplicities). Then for any $k \neq 0$ we have

$$\lambda_1 + \cdots + \lambda_n = \lambda_1^k + \cdots + \lambda_n^k.$$

Hence $\lambda_1 = \cdots = \lambda_n = 1$. Then $g \in \operatorname{Ker} \rho$. By Exercise 3.1 we have $G = \operatorname{Ker} \rho$. □

Let K be the subgroup of matrices

$$g_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The group K is a maximal compact subgroup of G, clearly K is isomorphic to $\mathbb{T} = S^1$. If $\rho \colon G \to \mathrm{GL}(V)$ is a unitary representation of G in a Hilbert space then the restricted K-representation $\mathrm{Res}_K \rho$ splits into the sum of 1-dimensional representations of K. In particular, one can find $v \in V$ such that, for some n, $\rho_{g_\theta}(v) = e^{in\theta} v$. We define the matrix coefficient function $f : G \to \mathbb{C}$ by the fomula

$$f(g) = \langle v, \rho_g v \rangle.$$

Then f satisfies the condition

$$f(gg_\theta) = e^{in\theta} f(g).$$

Thus, one can consider f as a section of a line bundle on the space G/K (if $n = 0$, then f is a function). Thus, it is clear that the space G/K is an important geometric object, on which the representations of G are "realized". To be a trifle more precise, consider the quotient $(G \times \mathbb{C})/K$ where K acts on G by right multiplication and on $\mathbb{C}$ by $e^{in\theta}$. It is a topological line bundle on G/K, and one can see f as a section of this bundle.

Consider the Lobachevsky plane

$$H = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

equipped with the Riemannian metric defined by the formula $\frac{dx^2+dy^2}{y^2}$ and the corresponding volume form $\frac{dxdy}{y^2}$. Then G coincides with the group of rigid motions of H preserving orientation. The action of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ on H is given by the formula

$$z \mapsto \frac{az+b}{cz+d}.$$

EXERCISE 3.3. Check that G acts transitively on H, preserves the metric and the volume. Moreover, the stabilizer of $i \in H$ coincides with K. Thus, we identify H with G/K.

3.2. Discrete series. Those are the representations with matrix coefficients in $L^2(G)$. For $n \in \mathbb{Z}_{>1}$, let $\mathcal{H}_n^+$ be the space of holomorphic densities on H, i.e. the set of formal expressions $\varphi(z)(dz)^{n/2}$, where $\varphi(z)$ is a holomorphic function on H satisfying the condition that the integral

$$\int_H |\varphi|^2 y^{n-2} dz d\bar{z}$$

is finite. Define a representation of G in $\mathcal{H}_n^+$ by the formula

$$\rho_g \left(\varphi(z)(dz)^{n/2}\right) = \varphi\left(\frac{az+b}{cz+d}\right) \frac{1}{(cz+d)^n} (dz)^{n/2},$$

and a Hermitian product on $\mathcal{H}_n$ by the formula

$$\left\langle \varphi(dz)^{n/2}, \psi(dz)^{n/2} \right\rangle = \int_H \bar{\varphi}\psi y^{n-2} dz d\bar{z}, \tag{4.15}$$

for $n > 1$. For $n = 1$ the product is defined by

$$\left\langle \varphi(dz)^{n/2}, \psi(dz)^{n/2} \right\rangle = \int_{-\infty}^{\infty} \bar{\varphi}\psi dx, \tag{4.16}$$

in this case $\mathcal{H}_1^+$ consists of all densities which converge to L^2-functions on the boundary (real line).

EXERCISE 3.4. Check that this Hermitian product is invariant.

To show that $\mathcal{H}_n^+$ is irreducible it is convenient to consider the Poincaré model of the Lobachevsky plane using the conformal map

$$w = \frac{z-i}{z+i},$$

that maps H to the unit disk $|w| < 1$. Then the group G acts on the unit disk by linear-fractional maps $w \mapsto \frac{aw+b}{\bar{b}w+\bar{a}}$ for all complex a, b satisfying $|a|^2 - |b|^2 = 1$, and K is defined by the condition $b = 0$. If $a = e^{i\theta}$, then $\rho_{g_\theta}(w) = e^{2i\theta}w$. The invariant volume form is $\frac{dw d\bar{w}}{1-\bar{w}w}$.

It is clear that $w^k (dw)^{n/2}$ for all $k \geq 0$ form an orthogonal basis in $\mathcal{H}_n^+$, each vector $w^k (dw)^{n/2}$ is an eigen vector with respect to K, namely

$$\rho_{g_\theta}\left(w^k (dw)^{n/2}\right) = e^{(2k+n)i\theta} w^k (dw)^{n/2}.$$

It is easy to check now that $\mathcal{H}_n^+$ is irreducible. Indeed, every invariant closed subspace M in $\mathcal{H}_n^+$ has a topological basis consisting of eigenvectors of K, in other words $w^k (dw)^{n/2}$ for some positive k must form a topological basis of M. Without loss of generality assume that M contains $(dw)^{n/2}$, then by applying ρ_g one can get that $\frac{1}{(bw+a)^n} (dw)^{n/2}$, and in Taylor series for $\frac{1}{(bw+a)^n}$ all elements of the basis appear with non-zero coefficients. That implies $w^k (dw)^{n/2} \in M$ for all $k \geq 0$, hence $M = \mathcal{H}_n^+$.

One can construct another series $\mathcal{H}_n^-$ by considering holomorphic densities in the lower half-plane $\operatorname{Re} z < 0$.

Exercise 3.5. Check that all representations in the discrete series $\mathcal{H}_n^\pm$ are pairwise non-isomorphic.

3.3. Principal series. These representations are parameterized by a continuous parameter $s \in \mathbb{R}i\,(s \neq 0)$. Consider now the action of G on the real line by linear fractional transformations $x \mapsto \frac{ax+b}{cx+d}$. Let $\mathcal{P}_s^+$ denotes the space of densities $\varphi(x)(dx)^{\frac{1+s}{2}}$ with G-action given by

$$\rho_g\left(\varphi(x)(dx)^{\frac{1+s}{2}}\right) = \varphi\left(\frac{ax+b}{cx+d}\right)|cx+d|^{-s-1}.$$

The Hermitian product given by

$$\langle \varphi, \psi \rangle = \int_{-\infty}^{\infty} \bar{\varphi}\psi dx \tag{4.17}$$

is invariant. The property of invariance justify the choice of weight for the density as $(dx)^{\frac{1+s}{2}} (dx)^{\frac{1+\bar{s}}{2}} = dx$, thus the integration is invariant. To check that the representation is irreducible one can move the real line to the unit circle as in the example of discrete series and then use $e^{ik\theta}(d\theta)^{\frac{1+s}{2}}$ as an orthonormal basis in $\mathcal{P}_s^+$. Note that the eigen values of ρ_{g_θ} in this case are $e^{2ki\theta}$ for all integer k.

The second principal series $\mathcal{P}_s^-$ can be obtained if instead of densities we consider the pseudo densities which are transformed by the law

$$\rho_g\left(\varphi\left(x\right)\left(dx\right)^{\frac{1+s}{2}}\right)=\varphi\left(\frac{ax+b}{cx+d}\right)\left|cx+d\right|^{-s-1}\operatorname{sgn}\left(cx+d\right)dx^{\frac{1+s}{2}}.$$

3.4. Complementary series. Those are representations which do not appear in the regular representation $L^2\left(G\right)$. They can be realized as the representations in $\mathcal{C}_s$ of all densities $\varphi\left(x\right)\left(dx\right)^{\frac{1+s}{2}}$ for real $0<s<1$ and have an invariant Hermitian product

$$\left\langle\varphi,\psi\right\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\varphi}\left(x\right)\psi\left(y\right)\left|x-y\right|^{s-1}dxdy. \tag{4.18}$$

CHAPTER 5

On algebraic methods

1. Introduction

Say a few words about infinite direct sums and products, talk about Zorn's lemma. Emphasize that we are now in full generality.

2. Semisimple modules and density theorem

2.1. Semisimplicity. Let R be a unital ring. We will use indifferently the terms R-module and module whenever the context is clear.

DEFINITION 2.1. An R-module M is *semisimple* if for any submodule $N \subset M$ there exists a submodule N' of M such that $M = N \oplus N'$.

Recall that an R-module M is *simple* if any submodule of M is either M or 0. Clearly, a simple module is semi-simple.

EXERCISE 2.2. Show that if M a semisimple R-module and if N is a quotient of M, then N is isomorphic to some submodule of M.

LEMMA 2.3. *Every submodule, every quotient of a semisimple R-module is semisimple.*

PROOF. Let N be a submodule of a semisimple module M, and let P be a submodule of N. By semisimplicity of M, there exists a submodule $P' \subset M$ such that $M = P \oplus P'$, then there exists an R-invariant projector $p : M \to P$ with kernel P'. The restriction of p to N defines the projector $N \to P$ and the kernel of this projector is the complement of P in N. Apply Exercise 2.2 to complete the proof. □

For what comes next, it is essential that the ring R is unital. Indeed it is necessary to have this property to ensure that R has a maximal left ideal and this can be proved using Zorn's Lemma.

LEMMA 2.4. *Any semisimple R-module contains a simple submodule.*

PROOF. Let M be semisimple, $m \in M$. Let I be a maximal left ideal in R. Then Rm is semisimple by Lemma 2.3 and $Rm = Im \oplus N$. We claim that N is simple. Indeed, every submodule of Rm is of the form Jm for some left ideal $J \subset R$. If N' is a submodule of N, then $N' \oplus Im = Jm$ and hence $I \subset J$. But, by maximality of I, $J = I$ or R, therefore $N' = 0$ or N. □

LEMMA 2.5. *Let M be an R-module. The following conditions are equivalent:*

(1) *M is semisimple;*

(2) $M = \sum_{i \in I} M_i$ *for a family of simple submodules M_i of M indexed by a set I;*

(3) $M = \bigoplus_{j \in J_0} M_j$ *for a family of simple submodules M_j of M indexed by a set J_0.*

PROOF. (1) $\Rightarrow$ (2) Let $\{M_i\}_{i \in I}$ be the collection of all simple submodules of M. We want to show that $N = \sum_{i \in I} M_i$. Let $N = \sum_{i \in I} M_i$ and assume that N is a proper submodule of M. Then $M = N \oplus N'$ by the semisimplicity of M. By Lemma 2.4, N' contains a simple submodule which can not be contained in the family $\{M_i\}_{i \in I}$. Contradiction.

Let us prove (2) $\Rightarrow$ (3). We consider all possible families $\{M_j\}_{j \in J}$ of simple submodules of M such that $\sum_{j \in J} M_j = \oplus_{j \in J} M_j$. First, we note the set of such families satisfies the conditions of Zorn's lemma, namely that any totally ordered subset of such families has a maximal element, where the order is the inclusion order. (To check this just take the union of all sets in the totally ordered subset.) Hence there is a maximal subset J_0 and the corresponding maximal family $\{M_j\}_{j \in J_0}$ such that $\sum_{j \in J_0} M_j$ is direct. We claim that $M = \oplus_{j \in J_0} M_j$. Indeed, if this is not true, there exists a simple submodule M_i which is not contained in $\oplus_{j \in J_0} M_j$. Since M_i simple, that means $M_i \cap \oplus_{j \in J_0} M_j = 0$. Hence

$$M_i + \bigoplus_{j \in J_0} M_j = M_i \oplus \bigoplus_{j \in J_0} M_j.$$

This contradicts maximality of J_0.

Finally, let us prove (3) $\Rightarrow$ (1). Let $N \subset M$ be a submodule and $S \subset J_0$ be a maximal subset such that $N \cap (\oplus_{j \in S} M_j) = 0$ (Zorn's lemma once more). Let $M' = N \oplus (\oplus_{j \in S} M_j)$. We claim that $M' = M$. Indeed, otherwise there exists $k \in J_0$ such that M_k does not belong to M'. Then $M_k \cap M' = 0$ by simplicity of M_k, and therefore $N \cap (\oplus_{j \in S \cup k} M_j) = 0$. Contradiction. □

EXERCISE 2.6. If R is a field, after noticing that an R-module is a vector space, show that every simple R-module is one-dimensional, and therefore, through the existence of bases, show that every module is semisimple.

EXERCISE 2.7. If $R = \mathbb{Z}$, then some R-modules which are not semisimple, for instance $\mathbb{Z}$ itself.

LEMMA 2.8. *Let M be a semisimple module. Then M is simple if and only if* $\mathrm{End}_R(M)$ *is a division ring.*

PROOF. In one direction this is Schur's lemma. In the opposite direction let $M = M_1 \oplus M_2$ for some proper submodules M_1 and M_2 of M. Let p_1, p_2 be the canonical projections onto M_1 and M_2 respectively. Then $p_1 \circ p_2 = 0$ and therefore p_1, p_2 can not be invertible. $\square$

2.2. Jacobson density theorem. Let M be any R-module. Set $K := \mathrm{End}_R(M)$, then set $S := \mathrm{End}_K(M)$. There exists a natural homomorphism $R \to S$. In general it is neither surjective nor injective. In the case when M is semisimple it is very close to being surjective.

THEOREM 2.9. *(Jacobson density theorem). Assume that M is semisimple. Then for any $m_1, \dots, m_n \in M$ and $s \in S$ there exists $r \in R$ such that $rm_i = sm_i$ for all $i = 1, \dots, n$.*

PROOF. First let us prove the statement for $n = 1$. We just have to show that $Rm_1 = Sm_1$. The inclusion $Rm_1 \subset Sm_1$ is obvious. We will prove the inverse inclusion. The semisimplicity of M implies $M = Rm_1 \oplus N$ for some submodule N of M. Let p be the projector $M \to N$ with kernel Rm_1. Then $p \in K$ and therefore $p \circ s = s \circ p$ for every $s \in S$. Hence $\mathrm{Ker}\, p$ is S-invariant. So $Sm_1 \subset Rm_1$.

For arbitrary n we use the following lemma.

LEMMA 2.10. *Let $\widehat{K} := \mathrm{End}_R(M^{\oplus n})$ and $\widehat{S} := \mathrm{End}_{\widehat{K}}(M^{\oplus n}) \cong S$. Then $\widehat{K}$ is isomorphic to the matrix ring $\mathrm{Mat}_n(K)$ and $\widehat{S}$ is isomorphic to S. The latter isomorphism is given by the diagonal action*

$$s(m_1, \dots, m_n) = (sm_1, \dots, sm_n).$$

EXERCISE 2.11. Adapt the proof of Lemma 1.12 to check the above lemma.

$\square$

COROLLARY 2.12. *Let M be a semisimple R-module, which is finitely generated over K. Then the natural map $R \to \mathrm{End}_K(M)$ is surjective.*

PROOF. Let $m_1, \dots, m_n$ be generators of M over K, apply Theorem 2.9. $\square$

COROLLARY 2.13. *Let R be an algebra over a field k, and $\rho\colon R \to \mathrm{End}_k(V)$ be an irreducible finite-dimensional representation of R. Then*

- *There exists a division ring D containing k such that $\rho(R) \cong \mathrm{End}_D(V)$.*
- *If k is algebraically closed, then $D = k$ and therefore ρ is surjective.*

PROOF. Apply Schur's lemma. $\square$

EXERCISE 2.14. Let V be an infinite-dimensional vector space over $\mathbb{C}$ and R be the span of Id and all linear operators with finite-dimensional image. Check that R is a ring and V is a simple R-module. Then $K = \mathbb{C}$, S is the ring of all linear operators in V and R is dense in S but R does not coincide with S.

3. Wedderburn–Artin theorem

A ring R is called *semisimple* if every R-module is semisimple. For example, a group algebra $k(G)$, for a finite group G such that $\operatorname{char} k$ does not divide $|G|$, is semisimple by Maschke's Theorem 3.3.

LEMMA 3.1. *Let R be a semisimple ring. Then as a module over itself R is isomorphic to a finite direct sum of minimal left ideals.*

PROOF. Consider R as an R-module. By definition the simple submodules of R are exactly the minimal left ideals of R. Hence since R is semisimple we can write R as a direct sum $\oplus_{i\in I} L_i$ of minimal left ideals L_i. It remains to show that this direct sum is finite. Indeed, let $l_i \in L_i$ be the image of the identity element 1 under the projection $R \to L_i$. But R as a module is generated by 1. Therefore $l_i \neq 0$ for all $i \in I$. Hence I is finite. $\square$

COROLLARY 3.2. *A direct product of finitely many semisimple rings is semisimple.*

EXERCISE 3.3. Let D be a division ring, and $R = \operatorname{Mat}_n(D)$ be a matrix ring over D.

(a) Let L_i be the subset of R consisting of all matrices which have zeros everywhere outside the i-th column. Check that L_i is a minimal left ideal of R and that $R = L_1 \oplus \cdots \oplus L_n$. Therefore R is semisimple.

(b) Show that L_i and L_j are isomorphic R-modules and that any simple R-module is isomorphic to L_i.

(c) Using Corollary 2.12 show that $F := \operatorname{End}_R(L_i)$ is isomorphic to D^{op}, and that R is isomorphic to $\operatorname{End}_F(L_i)$.

By the above exercise and Corollary 3.2 a direct product $\operatorname{Mat}_{n_1}(D_1) \times \cdots \times \operatorname{Mat}_{n_k}(D_k)$ of finitely many matrix rings is semisimple. In fact any semisismple ring is of this form.

THEOREM 3.4. *(Wedderburn-Artin) Let R be a semisimple ring. Then there exist division rings $D_1, \dots, D_k$ such that R is isomorphic to a finite product of matrix rings*

$$\operatorname{Mat}_{n_1}(D_1) \times \cdots \times \operatorname{Mat}_{n_k}(D_k).$$

Furthermore, $D_1, \dots, D_k$ are unique up to isomorphism and this presentation of R is unique up to permutation of the factors.

PROOF. Take the decomposition of Lemma 3.1 and combine isomorphic factors together. Then the following decomposition holds

$$R = L_1^{\oplus n_1} \oplus \cdots \oplus L_k^{\oplus n_k},$$

where L_i is not isomorphic to L_j if $i \neq j$. Set $J_i = L_i^{\oplus n_i}$. We claim that J_i is actually a two-sided ideal. Indeed Lemma 1.10 and simplicity of L_i imply that $L_i r$ is isomorphic to L_i for any $r \in R$ such that $L_i r \neq 0$. Thus, $L_i r \subset J_i$.

Now we will show that each J_i is isomorphic to a matrix ring. Let $F_i := \mathrm{End}_{J_i}(L_i)$. The natural homomorphism $J_i \to \mathrm{End}_{F_i}(L_i)$ is surjective by Corollary 2.12. This homomorphism is also injective since $rL_i = 0$ implies $rJ_i = 0$ for any $r \in R$. Then, since J_i is a unital ring $r = 0$. On the other hand, F_i is a division ring by Schur's lemma. Threfore we have an isomorphism $J_i \simeq \mathrm{End}_{F_i}(L_i)$. By Exercise 1.7 L_i is a free F_i-module. Moreover, L_i is finitely generated over F_i as J_i is a sum of finitely many left ideals. Thus, by Exercise 3.3 (c), J_i is isomorphic to $\mathrm{Mat}_{n_i}(D_i)$ where $D_i = F_i^{op}$.

The uniqueness of presentation follows easily from Krull-Schmidt theorem (Theorem 4.19) which we prove in the next section. Indeed, let $S_1, \dots, S_k$ be a complete list of non-isomorphic simple R-modules. Then both D_i and n_i are defined intrinsically, since $D_i^{op} \simeq \mathrm{End}_R(S_i)$ and n_i is the multiplicity of the indecomposable module S_i in R. □

4. Jordan-Hölder theorem and indecomposable modules

Let R be a unital ring.

4.1. Artinian and Noetherian modules.

Definition 4.1. We say that an R-module M is Noetherian or satisfies the ascending chain condition (ACC for short) if every increasing sequence

$$M_1 \subset M_2 \subset \dots$$

of submodules of M stabilizes.

Similarly, we say that M is Artinian or satisfies the descending chain condition (DCC) if every decreasing sequence

$$M_1 \supset M_2 \supset \dots$$

of submodules of M stabilizes.

Exercise 4.2. Consider $\mathbb{Z}$ as a module over itself. Show that it is Noetherian but not Artinian.

Exercise 4.3. (a) A submodule or a quotient of a Noetherian (respectively, Artinian) module is always Noetherian (resp. Artinian).

(b) Let

$$0 \to N \to M \to L \to 0$$

be an exact sequence of R-modules. Assume that both N and L are Noetherian (respectively, Artinian), then M is also Noetherian (respectively, Artinian).

Exercise 4.4. Let M be a semisimple module. Prove that M is Noetherian if and only if it is Artinian.

4.2. Jordan-Hölder theorem.

DEFINITION 4.5. Let M be an R-module. A finite sequence of submodules of M

$$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$$

such that M_i/M_{i+1} is a simple module for all $i = 0, \ldots, k-1$ is called a Jordan-Hölder series of M.

LEMMA 4.6. *An R-module M has a Jordan-Hölder series if and only if M is both Artinian and Noetherian.*

PROOF. Let M be an R-module which is both Artinian and Noetherian. Then it is easy to see that there exists a finite sequence of properly included submodules of M

$$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$$

which can not be refined. Then M_i/M_{i+1} is a simple module for all $i = 0, \ldots, k-1$.

Conversely, assume that M has a Jordan-Hölder series

$$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0.$$

We prove that M is both Noetherian and Artinian by induction on k. If $k = 1$, then M is simple and hence both Noetherian and Artinian. For $k > 1$ consider the exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0$$

and use Exercise 4.3 (b). □

We say that two Jordan-Hölder series of M

$$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$$

and

$$M = N_0 \supset N_1 \supset \cdots \supset N_l = 0$$

are *equivalent* if $k = l$ and for some permutation s of indices $1, \ldots, k-1$ we have $M_i/M_{i+1} \cong M_{s(i)}/M_{s(i)+1}$.

THEOREM 4.7. *Let M be an R-module which is both Noetherian and Artinian. Let*

$$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$$

and

$$M = N_0 \supset N_1 \supset \cdots \supset N_l = 0$$

be two Jordan-Hölder series of M. Then they are equivalent.

PROOF. First note that if M is simple, then the statement is trivial. We will prove that if the statement holds for any proper submodule of M then it is also true for M. If $M_1 = N_1$, then the statement is obvious. Otherwise, $M_1 + N_1 = M$, hence we have two isomorphisms $M/M_1 \cong N_1/(M_1 \cap N_1)$ and $M/N_1 \cong M_1/(M_1 \cap N_1)$. Like the second isomorphism theorem for groups. Now let

$$M_1 \cap N_1 \supset K_1 \supset \cdots \supset K_s = 0$$

be a Jordan-Hölder series for $M_1 \cap N_1$. This gives us two new Jordan-Hölder series of M

$$M = M_0 \supset M_1 \supset M_1 \cap N_1 \supset K_1 \supset \cdots \supset K_s = 0$$

and

$$M = N_0 \supset N_1 \supset N_1 \cap M_1 \supset K_1 \supset \cdots \supset K_s = 0.$$

These series are obviously equivalent. By our assumption on M_1 and N_1 the first series is equivalent to $M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$, and the second one is equivalent to $M = N_0 \supset N_1 \supset \cdots \supset N_l = \{0\}$. Hence the original series are also equivalent. □

Thus, we can now give two definitions:

DEFINITION 4.8. First, we define the length $l(M)$ of an R-module M which satsfies ACC and DCC as the length of any Jordan-Hölder series of M. Note that we can easily see that if N is a proper submodule of M, then $l(N) < l(M)$.

Furthermore, this gives rise to a notion of *finite length* R-module.

REMARK 4.9. Note that in the case of *infinite series* with simple quotients, we may have many non-equivalent series. For example, consider $\mathbb{Z}$ as a $\mathbb{Z}$-module. Then the series

$$\mathbb{Z} \supset 2\mathbb{Z} \supset 4\mathbb{Z} \supset \dots$$

is not equivalent to

$$\mathbb{Z} \supset 3\mathbb{Z} \supset 9\mathbb{Z} \supset \dots.$$

4.3. Indecomposable modules and Krull–Schmidt theorem. A module M is *indecomposable* if $M = M_1 \oplus M_2$ implies $M_1 = 0$ or $M_2 = 0$.

EXAMPLE 4.10. Every simple module is indecomposable. Furthermore, if a semisimple module M is indecomposable then M is simple.

DEFINITION 4.11. An element $e \in R$ is called an *idempotent* if $e^2 = e$.

LEMMA 4.12. *An R-module M is indecomposable if and only if every idempotent in $\operatorname{End}_R(M)$ is either 1 or 0.*

PROOF. If M is decomposable, then $M = M_1 \oplus M_2$ for some proper submodules M_1 and M_2. Then the projection $e : M \to M_1$ with kernel M_2 is an idempotent in $\operatorname{End}_R M$, which is neither 0 nor 1. Conversely, any non-trivial idempotent $e \in \operatorname{End}_R M$ gives rise to a decomposition $M = \operatorname{Ker} e \oplus \operatorname{Im} e$. □

EXERCISE 4.13. Show that $\mathbb{Z}$ is an indecomposable module over itself, although it is not simple.

LEMMA 4.14. *Let M and N be indecomposable R-modules, $\alpha \in \operatorname{Hom}_R(M,N)$, $\beta \in \operatorname{Hom}_R(N,M)$ be such that $\beta \circ \alpha$ is an isomorphism. Then α and β are isomorphisms.*

PROOF. We claim that $N = \operatorname{Im}\alpha \oplus \operatorname{Ker}\beta$. Indeed, since $\operatorname{Im}\alpha \cap \operatorname{Ker}\beta \subset \operatorname{Ker}\beta \circ \alpha$, we have $\operatorname{Im}\alpha \cap \operatorname{Ker}\beta = 0$. Furthermore, for any $x \in N$ set $y := \alpha \circ (\beta \circ \alpha)^{-1} \circ \beta(x)$ and $z = x - y$. Then $\beta(y) = \beta(x)$. One can write $x = y + z$, where $z \in \operatorname{Ker}\beta$ and $y \in \operatorname{Im}\alpha$.

Since N is indecomposable, $\operatorname{Im}\alpha = N$, $\operatorname{Ker}\beta = 0$, hence N is isomorphic to M. □

LEMMA 4.15. *Let M be an indecomposable R-module of finite length and $\varphi \in \operatorname{End}_R(M)$, then either φ is an isomorphism or φ is nilpotent.*

PROOF. Since M is of finite length and $\operatorname{Ker}\varphi^n, \operatorname{Im}\varphi^n$ are submodules, there exists $n > 0$ such that $\operatorname{Ker}\varphi^n = \operatorname{Ker}\varphi^{n+1}$, $\operatorname{Im}\varphi^n = \operatorname{Im}\varphi^{n+1}$. Then $\operatorname{Ker}\varphi^n \cap \operatorname{Im}\varphi^n = 0$. The latter implies that the exact sequence

$$0 \to \operatorname{Ker}\varphi^n \to M \to \operatorname{Im}\varphi^n \to 0$$

splits. Thus, $M = \operatorname{Ker}\varphi^n \oplus \operatorname{Im}\varphi^n$. Since M is indecomposable, either $\operatorname{Im}\varphi^n = 0$, $\operatorname{Ker}\varphi^n = M$ or $\operatorname{Ker}\varphi^n = 0$, $\operatorname{Im}\varphi^n = M$. In the former case φ is nilpotent. In the latter case φ^n is an isomorphism and hence φ is also an isomorphism. □

LEMMA 4.16. *Let M be as in Lemma 4.15 and $\varphi, \varphi_1, \varphi_2 \in \operatorname{End}_R(M)$ such that $\varphi = \varphi_1 + \varphi_2$. Then if φ is an isomorphism, at least one of φ_1 and φ_2 is also an isomorphism.*

PROOF. Without loss of generality we may assume that $\varphi = \mathrm{id}$ (otherwise multiply by φ^{-1}). In this case φ_1 and φ_2 commute. If both φ_1 and φ_2 are nilpotent, then $\varphi_1 + \varphi_2$ is nilpotent, but this is impossible as $\varphi_1 + \varphi_2 = \mathrm{id}$. □

COROLLARY 4.17. *Let M be as in Lemma 4.15. Let $\varphi = \varphi_1 + \cdots + \varphi_k \in \operatorname{End}_R(M)$. If φ is an isomorphism then φ_i is an isomorphism at least for one i.*

EXERCISE 4.18. Let M be of finite length. Show that M has a decomposition

$$M = M_1 \oplus \cdots \oplus M_k,$$

where all M_i are indecomposable.

THEOREM 4.19. *(Krull-Schmidt) Let M be an R-module of finite length. Consider two decompositions*

$$M = M_1 \oplus \cdots \oplus M_k \quad \text{and} \quad M = N_1 \oplus \cdots \oplus N_l$$

such that all M_i and N_j are indecomposable. Then $k = l$ and there exists a permutation s of indices $1, \dots, k$ such that M_i is isomorphic to $N_{s(i)}$.

PROOF. We prove the statement by induction on k. The case $k = 1$ is clear since in this case M is indecomposable.

Let

$$p_i^{(1)} : M \to M_i, \quad p_j^{(2)} : M \to N_j$$

denote the natural projections, and

$$q_i^{(1)} : M_i \to M, \quad q_j^{(2)} : N_j \to N$$

denote the injections. We have

$$\sum_{j=1}^{l} q_j^{(2)} \circ p_j^{(2)} = \mathrm{id}_M,$$

hence

$$\sum_{j=1}^{l} p_1^{(1)} \circ q_j^{(2)} \circ p_j^{(2)} \circ q_1^{(1)} = \mathrm{id}_{M_1}.$$

By Corollary 4.17 there exists j such that $p_1^{(1)} \circ q_j^{(2)} \circ p_j^{(2)} \circ q_1^{(1)}$ is an isomorphism. After permuting indices we may assume that $j = 1$. Then Lemma 4.14 implies that $p_1^{(2)} \circ q_1^{(1)}$ is an isomorphism between M_1 and N_1. Set

$$M' := M_2 \oplus \cdots \oplus M_k, \quad N' := N_2 \oplus \cdots \oplus N_l.$$

Since M_1 intersects trivially $N' = \operatorname{Ker} p_1^{(2)}$ we have $M = M_1 \oplus N'$. But we also $M = M_1 \oplus M'$. Therefore M' is isomorphic to N'. By induction assumption the statement holds for $M' \simeq N'$. Hence the statement holds for M. □

Come up with examples of modules for which Krull–Schmidt does not hold.

5. A bit of homological algebra

Let R be a unital ring.

5.1. Complexes. Let $C_\bullet = \oplus_{i \geq 0} C_i$ be a graded R-module. An R-morphism f from $C_\bullet$ to $D_\bullet$ is of *degree* k ($k \in \mathbb{Z}$) if f maps to C_i to D_{i+k} for all i. An R-*differential* on $C_\bullet$ is an R-morphism d from $C_\bullet$ to $C_\bullet$ of degree -1 such that $d^2 = 0$.

An R-module $C_\bullet$ together with a differential d is called a *complex*.

We usually represent $C_\bullet$ the following way:

$$\ldots \xrightarrow{d} C_i \xrightarrow{d} \ldots \xrightarrow{d} C_1 \xrightarrow{d} C_0 \to 0.$$

REMARK 5.1. It will be convenient to look at similar situations for an R-morphism δ of degree $+1$ on a graded R-module such that $\delta^2 = 0$. In this case, we will use upper indices C^i (instead of C_i) and represent the complex the following way:

$$0 \to C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} \ldots \xrightarrow{\delta} C^i \xrightarrow{\delta} \ldots$$

Exercise 5.2. (**Koszul complex**) The following example is very important.

Let V be a finite-dimensional vector space over a field k and denote by V^* its dual space. By $S(V) = \bigoplus S^i(V)$ and $\Lambda(V) = \bigoplus \Lambda^i(V)$ we denote the symmetric and the exterior algebras of V respectively.

Choose a basis $e_1, \ldots, e_n$ of V and let $f_1, \ldots, f_n$ be the dual basis in V^*, i.e. $f_i(e_j) = \delta_{ij}$. For any $x \in V^*$ we define the linear derivation $\partial_x : S(V) \to S(V)$ given by $\partial_x(v) := x(v)$ for $v \in V$ and extend it to the whole $S(V)$ via the Leibniz relation

$$\partial_x(u_1u_2) = \partial_x(u_1)u_2 + u_1\partial_x(u_2) \quad \text{for all} \quad u_1, u_2 \in S(V).$$

Now set $C^k := S(V) \otimes \Lambda^k(V)$ and $C^\bullet := S(V) \otimes \Lambda(V)$. Define $\delta : C^\bullet \to C^\bullet$ by

$$\delta(u \otimes w) := \sum_{j=1}^{n} d_{f_j}(u) \otimes (e_j \wedge w) \quad \text{for all} \quad u \in S(V), w \in \Lambda(V).$$

(a) Show that δ does not depend on the choice of basis in V.

(b) Prove that $\delta^2 = 0$, and therefore $(C^\bullet, \delta)$ is a complex. It is called the *Koszul complex*.

(c) Let $p(w)$ denote the parity of the degree of w if w is homogeneous in $\Lambda(V)$. For any $x \in V^*$ define the linear map $\partial_x : \Lambda(V) \to \Lambda(V)$ by setting $\partial_x(v) := x(v)$ for all $v \in V$ and extend it to the whole $\Lambda(V)$ by the $\mathbb{Z}_2$-graded version of the Leibniz relation

$$\partial_x(w_1 \wedge w_2) = \partial_x(w_1) \wedge w_2 + (-1)^{p(w_1)} w_1 \wedge \partial_x(w_2) \quad \text{for all} \quad w_1, w_2 \in \Lambda(V).$$

Check that one can construct a differential d of degree -1 on the Koszul complex by

$$d(u \otimes w) := \sum_{j=1}^{n} (ue_j) \otimes \partial_{f_j}(w) \quad \text{for all} \quad u \in S(V), w \in \Lambda(V).$$

5.2. Homology and Cohomology. Since in any complex $d^2 = 0$, we have $\operatorname{Im} d \subset \operatorname{Ker} d$ (in every degree). The complex $(C_\bullet, d)$ is *exact* if $\operatorname{Im} d = \operatorname{Ker} d$. The key notion of homological algebra is defined below. This notion expresses how far a given complex is from being exact.

Definition 5.3. Let $(C_\bullet, d)$ be a complex of R-modules (with d of degree -1). Its i-th *homology*, $H_i(C_\bullet)$, is the quotient

$$H_i(C_\bullet) = (\operatorname{Ker} d \cap C_i) / (\operatorname{Im} d \cap C_i).$$

A complex $(C_\bullet, d)$ is exact if and only if $H_i(C_\bullet) = 0$ for all $i \geq 0$.

If $(C^\bullet, \delta)$ is a complex with a differential δ of degree $+1$ we use the term *cohomology* instead of homology and we consistently use upper indices in the notation:

$$H^i(C^\bullet) = \left(\operatorname{Ker} \delta \cap C^i\right) / \left(\operatorname{Im} \delta \cap C^i\right).$$

Definition 5.4. Given two complexes $(C_\bullet, d)$ and $(C'_\bullet, d')$, a homomorphism $f : C_\bullet \to C'_\bullet$ of R-modules of degree 0 which satisfies the relation $f \circ d = d' \circ f$ is called a *morphism of complexes*.

EXERCISE 5.5. Let $f : C_\bullet \to C'_\bullet$ be a morphism of complexes. Check that $f(\operatorname{Ker} d) \subset \operatorname{Ker} d'$ and $f(\operatorname{Im} d) \subset \operatorname{Im} d'$. Therefore f induces a homomorphism

$$f_*\colon H_i\left(C_\bullet\right) \to H_i\left(C'_\bullet\right)$$

between homology groups of the complexes.

Let $(C_\bullet, d)$, $(C'_\bullet, d')$, $(C''_\bullet, d'')$ be complexes and $f : C'_\bullet \to C''_\bullet$ and $g : C_\bullet \to C'_\bullet$ be morphisms such that the sequence

$$0 \to C_i \xrightarrow{g} C'_i \xrightarrow{f} C''_i \to 0$$

is exact for all $i \geq 0$.

EXERCISE 5.6. (**Snake Lemma**) One can define a homomorphism $\delta : H_i(C''_\bullet) \to H_{i-1}(C_\bullet)$ as follows. Let $x \in \operatorname{Ker} d'' \cap C''_i$ and y be an arbitrary element in the preimage $f^{-1}(x) \subset C'_i$. Check that $d'(y)$ lies in the image of g. Pick up $z \in g^{-1}(d'(y)) \subset C_{i-1}$. Show that $z \in \operatorname{Ker} d$. Moreover, show that for a different choice of $y' \in f^{-1}(x) \subset C'_i$ and of $z' \in g^{-1}(d'(y')) \subset C_{i-1}$ the difference $z - z'$ lies in the image of $d : C_i \to C_{i-1}$. Thus, $x \mapsto z$ gives a well-defined map $\delta : H_i(C''_\bullet) \to H_{i-1}(C_\bullet)$.

Why is it called "snake lemma"? Look at the following diagram

$$\begin{array}{ccccc} C_i & \xrightarrow{g} & C'_i & \xrightarrow{f} & C''_i \\ {\scriptstyle d}\downarrow & & {\scriptstyle d'}\downarrow & & {\scriptstyle d''}\downarrow \\ C_{i-1} & \xrightarrow{g} & C'_{i-1} & \xrightarrow{f} & C''_{i-1} \end{array}$$

In this diagram $\delta = g^{-1} \circ d' \circ f^{-1}$ goes from the upper right to the lower left corners.

THEOREM 5.7. *(Long exact sequence). The following sequence*

$$\xrightarrow{\delta} H_i\left(C_\bullet\right) \xrightarrow{g_*} H_i\left(C'_\bullet\right) \xrightarrow{f_*} H_i\left(C''_\bullet\right) \xrightarrow{\delta} H_{i-1}\left(C_\bullet\right) \xrightarrow{g_*} \ldots$$

is actually an exact complex.

We skip the proof of this theorem. The enthusiastic reader might verify it as an exercise or read the proof in Weibel, MacLane.

5.3. Homotopy.

DEFINITION 5.8. Consider complexes $(C_\bullet, d)$, $(C'_\bullet, d')$ of R-modules and let $f, g : C_\bullet \to C'_\bullet$ be morphisms. We say that f and g are *homotopically equivalent* if there exists a map $h : C_\bullet \to C'_\bullet$ of degree 1 such that

$$f - g = h \circ d + d' \circ h.$$

LEMMA 5.9. *If f and g are homotopically equivalent then $f_* = g_*$.*

Proof. Let $\phi := f - g$ and $x \in C_i$ such that $dx = 0$. Then

$$\phi\,(x) = h\,(dx) + d'\,(hx) = d'\,(hx) \in \operatorname{Im} d'.$$

Hence $f_* - g_* = 0$. □

We say that complexes $C_\bullet$ and $C'_\bullet$ are *homotopically equivalent* if there exist $f : C_\bullet \to C'_\bullet$ and $g : C'_\bullet \to C_\bullet$ such that $f \circ g$ is homotopically equivalent to $\mathrm{id}_{C'}$ and $g \circ f$ is homotopically equivalent to id_C. Lemma 5.9 implies that homotopically equivalent complexes have isomorphic homology.

Let $(C_\bullet, d)$ be a complex of R-modules and M be an R-module. Then we have a complex of abelian groups

$$0 \to \operatorname{Hom}_R(C_0, M) \xrightarrow{\delta} \operatorname{Hom}_R(C_1, M) \xrightarrow{\delta} \dots \xrightarrow{\delta} \operatorname{Hom}_R(C_i, M) \xrightarrow{\delta} \dots,$$

where $\delta : \operatorname{Hom}_R(C_i, M) \to \operatorname{Hom}_R(C_{i+1}, M)$ is defined by

$$\delta(\varphi)(x) := \varphi(dx) \quad \text{for all} \quad \varphi \in \operatorname{Hom}_R(C_i, M) \quad \text{and} \quad x \in C_{i+1}. \tag{5.1}$$

Note that the differential δ on $\operatorname{Hom}_R(C_\bullet, M)$ has degree 1. The following Lemma will be used in the next section. The proof is straightforward and we leave it to the reader.

Lemma 5.10. *Let $(C_\bullet, d)$ and $(C'_\bullet, d')$ be homotopically equivalent complexes and M be an arbitrary R-module. Then the complexes $(\operatorname{Hom}_R(C_\bullet, M), \delta)$ and $(\operatorname{Hom}_R(C'_\bullet, M), \delta')$ are also homotopically equivalent.*

The following lemma is useful for calculating cohomology.

Lemma 5.11. *Let $(C_\bullet, d)$ be a complex of R-modules and $h : C_\bullet \to C_\bullet$ be a map of degree 1. Set $f := d \circ h + h \circ d$. Then f is a morphism of complexes. Furthermore, if $f : C_i \to C_i$ is an isomorphism for all $i \geq 0$, then $C_\bullet$ is exact.*

Proof. First, f has degree 0 and since $d^2 = 0$ we have

$$d \circ f = d \circ h \circ d = f \circ d.$$

Thus, f is a morphism of complexes.

Now let f be an isomorphism. Then $f_* : H_i(C_\bullet) \to H_i(C_\bullet)$ is also an isomorphism for all i. On the other hand, f is homotopically equivalent to 0. Hence, by Lemma 5.9, $f_* = 0$. Therefore $H_i(C_\bullet) = 0$ for all i. □

Exercise 5.12. Recall the Koszul complex $(C^\bullet, \delta)$ from Exercise 5.2. Assume the field k has characteristic zero. Show that $H^i(C^\bullet) = 0$ for $i \geq 0$ and $H^0(C^\bullet) = k$.

Hint. For every $m \geq 0$ consider the subcomplex $C^\bullet_m$ with graded components

$$C^i_m := S^{m-i}(V) \otimes \Lambda^i(V).$$

Check that $d(C^i_m) \subset C^{i-1}_m$, $\delta(C^i_m) \subset C^{i+1}_m$ and that the relation

$$d \circ \delta + \delta \circ d = m\,\mathrm{id}$$

holds on $C^\bullet_m$. Then use Lemma 5.11 and the decomposition $C^\bullet = \bigoplus_{m \geq 0} C^\bullet_m$.

6. Projective modules

Let R be a unital ring.

6.1. Projective modules. An R-module P is *projective* if for any surjective morphism $\varphi : M \to N$ of R-modules and any morphism $\psi : P \to N$ there exists a morphism $f : P \to M$ such that $\psi = \varphi \circ f$.

$$\begin{tikzcd} P \arrow[r, dashed, "f"] \arrow[dr, "\psi"'] & M \arrow[d, two heads, "\varphi"] \\ & N \end{tikzcd}$$

Example 6.1. A free R-module F is projective. Indeed, let $\{e_i\}_{i \in I}$ be a set of generators of F. Define $f : F \to M$ by $f(e_i) = \varphi^{-1}(\psi(e_i))$.

Lemma 6.2. *Let P be an R-module, the following conditions are equivalent*

(1) *P is projective;*
(2) *There exists a free module F such that F is isomorphic to $P \oplus P'$;*
(3) *Any exact sequence of R-modules*
$$0 \to N \to M \to P \to 0$$
splits.

Proof. (1) $\Rightarrow$ (3)
Consider the exact sequence
$$0 \to N \to M \xrightarrow{\varphi} P \to 0.$$
Set $\psi = \mathrm{id}_P$. Since φ is surjective and P is projective, there exists $f : P \to M$ such that $\psi = \mathrm{id}_P = \varphi \circ f$.

(3) $\Rightarrow$ (2) Every module is a quotient of a free module. Therefore we just have to apply (3) to the exact sequence
$$0 \to N \to F \to P \to 0$$
for a free module F.

(2) $\Rightarrow$ (1) Choose a free module F such that $F = P \oplus P'$. Let $\varphi : M \to N$ be a surjective morphism of R-modules and ψ a morphism $\psi : P \to N$. Now extend ψ to $\tilde{\psi} : F \to N$ such that the restriction of $\tilde{\psi}$ to P (respectively, P') is ψ (respectively, zero). There exists $f : F \to M$ such that $\varphi \circ f = \tilde{\psi}$. After restriction to P we get
$$\varphi \circ f_{|P} = \tilde{\psi}_{|P} = \psi.$$
□

Exercise 6.3. Recall that a ring A is called a principal ring if A is commutative, has no zero divisors and every ideal of A is principal, i.e. generated by a single element.

(a) Let F be a free A-module. Show that every submodule of F is free. For finitely generated F this can be done by induction on the rank of F. In the infinite case one has to use transfinite induction, see Rotman(651).

(b) Let P be a projective A-module. Show that P is free.

6.2. Projective cover.

Definition 6.4. Let M be an R-module. A submodule N of M is *small* if for any submodule $L \subset M$ such that $L + N = M$, we have $L = M$.

Exercise 6.5. Let $f : P \to M$ be a surjective morphism of modules such that $\operatorname{Ker} f$ is a small submodule of P. Assume that $f = f \circ \gamma$ for some homomorphism $\gamma : P \to P$. Show that γ is surjective.

Definition 6.6. Let M be an R-module. A *projective cover* of M is a projective R-module P equipped with a surjective morphism $f : P \to M$ such that $\operatorname{Ker} f \subset P$ is small.

Lemma 6.7. *Let $f : P \to M$ and $g : Q \to M$ be two projective covers of M. Then there exists an isomorphism $\varphi : P \to Q$ such that $g \circ \varphi = f$.*

Proof. The existence of φ such that $g \circ \varphi = f$ follows immediately from projectivity of P. Similarly, we obtain the existence of a homomorphism $\psi : Q \to P$ such that $f \circ \psi = g$. Therefore we have $g \circ \varphi \circ \psi = g$.. By Exercise 6.5 $\varphi \circ \psi$ is surjective. This implies surjectivity of $\varphi : P \to Q$. Since Q is projective we have an isomorphism $P \simeq Q \oplus \operatorname{Ker} \varphi$. Since $\operatorname{Ker} \varphi \subset \operatorname{Ker} f$, we have $P = Q + \operatorname{Ker} f$. Recall that $\operatorname{Ker} f \subset P$ is a small. Hence $P = Q$ and $\operatorname{Ker} \varphi = 0$. Thus φ is an isomorphism. $\square$

6.3. Projective resolutions.

Definition 6.8. Let M be an R-module. A complex $(P_\bullet, d)$ of R-modules

$$\dots \xrightarrow{d} P_i \xrightarrow{d} \dots \xrightarrow{d} P_1 \xrightarrow{d} P_0 \to 0$$

such that P_i is projective for all $i \geq 0$, $H_0(P_\bullet) = M$ and $H_i(P_\bullet) = 0$ for all $i \geq 1$, is called a *projective resolution* of M.

It is sometimes useful to see a projective resolution as the exact complex

$$\dots \to P_i \to \dots \to P_1 \to P_0 \xrightarrow{p} M \to 0,$$

where $p : P_0 \to M$ is the lift of the identity map between $H_0(P_\bullet)$ and M.

Exercise 6.9. Show that for every R-module M, there exists a resolution of M

$$\dots \to F_i \to \dots \to F_1 \to F_0 \to 0$$

such that all F_i are free. Such a resolution is called a free resolution.

This exercise immediately implies:

PROPOSITION 6.10. *Every R-module has a projective resolution.*

EXAMPLE 6.11. Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field k. Consider the simple R-module $M := R/(x_1, \dots, x_n)$. One can use the Koszul complex, introduced in Exercise 5.2, to construct a projective resolution of M. First, we identify R with the symmetric algebra $S(V)$ of the vector space $V = k^n$. Let P_i denote the free R-module $R \otimes \Lambda^i(V)$ and recall $d : P_i \to P_{i-1}$ from Exercise 5.2 (c). Then $H_0(P_\bullet) = M$ and $H_i(P_\bullet) = 0$ for $i \geq 1$. Hence $(P_\bullet, d)$ is a free resolution of M.

LEMMA 6.12. *Let $(P_\bullet, d)$ and $(P_\bullet, d')$ be two projective resolutions of an R-module M. Then there exists a morphism of complexes $f : P_\bullet \to P'_\bullet$ such that $f_* : H_0(P_\bullet) \to H_0(P'_\bullet)$ induces the identity id_M. Moreover, f is unique up to homotopy equivalence.*

PROOF. We use an induction procedure to construct a morphism $f_i : P_i \to P'_i$. For $i = 0$, we denote by $p : P_0 \to M$ and $p' : P'_0 \to M$ the natural projections. Since P_0 is projective there exists a morphism $f_0 : P_0 \to P'_0$ such that $p' \circ f_0 = p$:

$$\begin{tikzcd}
{} \arrow[r] & P_1 \arrow[r, "d"] & P_0 \arrow[r, "p"] \arrow[d, dashed, "f_0"] & M \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
{} \arrow[r] & P'_1 \arrow[r, "d'"] & P'_0 \arrow[r, "p'"] & M \arrow[r] & 0
\end{tikzcd}$$

then we have $f(\operatorname{Ker} p) \subset \operatorname{Ker} p'$. We construct $f_1 : P_1 \to P'_1$ using the following commutative diagram:

$$\begin{tikzcd}
{} \arrow[r] & P_1 \arrow[r, "d"] \arrow[d, dashed, "f_1"] & \operatorname{Ker} p \arrow[r] \arrow[d, "f_0"] & 0 \\
{} \arrow[r] & P'_1 \arrow[r, "d'"] & \operatorname{Ker} p' \arrow[r] & 0.
\end{tikzcd}$$

The existence of f_1 follows from projectivity of P_1 and surjectivity of d'.

We repeat the procedure to construct $f_i : P_i \to P_i$ for all i.

Suppose now that f and g are two morphisms satisfying the assumptions of the lemma. Let us prove that f and g are homotopically equivalent. Let $\varphi = f - g$. We have to prove the existence of maps $h_i : P_i \to P_{i+1}$ such that $h_i \circ d = d' \circ h_{i+1}$. Let us explain how to construct h_0 and h_1 using the following diagram

$$\begin{tikzcd}
{} \arrow[r] & P_2 \arrow[r, "d"] \arrow[d, "\varphi_2"] & P_1 \arrow[r, "d"] \arrow[d, "\varphi_1"] \arrow[dl, dashed, "h_1"] & P_0 \arrow[r, "p"] \arrow[d, "\varphi_0"] \arrow[dl, dashed, "h_0"] & M \arrow[r] \arrow[d, "0"] & 0 \\
{} \arrow[r] & P'_2 \arrow[r, "d'"] & P'_1 \arrow[r, "d'"] & P'_0 \arrow[r, "p'"] & M \arrow[r] & 0.
\end{tikzcd}$$

Since the morphism $\varphi_* : H_0(P_\bullet) \to H_0(P'_\bullet)$ is zero, we get $p' \circ \varphi_0 = 0$, and hence $\operatorname{Im} \varphi_0 \subset \operatorname{Im} d'$. Recall that P_0 is projective, therefore there exists $h_0 : P_0 \to P'_1$ such that $d' \circ h_0 = \varphi_0$.

To construct the map h_1, set $\psi := \varphi_1 - h_0 \circ d$. The relation

$$d' \circ h_0 \circ d = \varphi_0 \circ d = d' \circ \varphi_1$$

implies $d' \circ \psi = 0$. Since $H_1(P'_\bullet) = 0$, the image of ψ belongs to $d'(P'_2)$, and by projectivity of P_1 there exists a morphism $h_1 : P_1 \to P'_2$ such that

$$d' \circ h_1 = \psi = \varphi_1 - h_0 \circ d.$$

The construction of h_i for $i > 1$ is similar to the one for $i = 1$. The collection of the maps h_i gives the homotopy equivalence. $\square$

The following proposition expresses in what sense a projective resolution is unique.

PROPOSITION 6.13. *Let M be an R-module, and $(P_\bullet, d)$, $(P'_\bullet, d')$ be two projective resolutions of M. Then $(P_\bullet, d)$ and $(P'_\bullet, d')$ are homotopically equivalent.*

PROOF. By Lemma 6.12 there exist $f : P_\bullet \to P'_\bullet$ and $g : P'_\bullet \to P_\bullet$ such that $g \circ f$ is homotopically equivalent to $\mathrm{id}_{P_\bullet}$ and $f \circ g$ is homotopically equivalent to $\mathrm{id}_{P'_\bullet}$. $\square$

6.4. Extensions.

DEFINITION 6.14. Let M and N be two R-modules and $P_\bullet$ be a projective resolution of M. Consider the complex of abelian groups

$$0 \to \mathrm{Hom}_R\,(P_0, N) \xrightarrow{\delta} \mathrm{Hom}_R\,(P_1, N) \xrightarrow{\delta} \dots,$$

where δ is defined by (5.1). We define the i-th *extension group* $\mathrm{Ext}^i_R\,(M, N)$ as the i-th cohomology group of this complex. Lemma 5.10 ensures that $\mathrm{Ext}^i_R\,(M, N)$ does not depend on the choice of a projective resolution of M.

EXERCISE 6.15. Check that $\mathrm{Ext}^0_R(M, N) = \mathrm{Hom}_R(M, N)$.

Let us give an interpretation of $\mathrm{Ext}^1_R(M, N)$. Consider an exact sequence of R-modules

$$0 \to N \xrightarrow{\alpha} Q \xrightarrow{\beta} M \to 0 \tag{5.2}$$

and a projective resolution

$$\dots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\varphi} M \to 0 \tag{5.3}$$

of M. Then by projectivity of $P_\bullet$ there exist $\psi \in \mathrm{Hom}_R(P_0, Q)$ and $\gamma \in \mathrm{Hom}_R(P_1, N)$ which make the following diagram

$$\begin{tikzcd}
{} \arrow[r] & P_2 \arrow[r, "d"] \arrow[d, "0"] & P_1 \arrow[r, "d"] \arrow[d, "\gamma"] & P_0 \arrow[d, "\psi"] \arrow[dr, "\varphi"] & & \\
& 0 \arrow[r] & N \arrow[r, "\alpha"] & Q \arrow[r, "\beta"] & M \arrow[r] & 0
\end{tikzcd}$$

commutative. Let δ be the differential of degree $+1$ in Definition 6.14. The commutativity of this diagram implies that $\gamma \circ d = 0$ and hence $\delta(\gamma) = 0$. The choice of ψ and

γ is not unique. If we choose another pair $\psi' \in \operatorname{Hom}_R(P_0, Q)$ and $\gamma' \in \operatorname{Hom}_R(P_1, N)$, then there exists $\theta \in \operatorname{Hom}_R(P_0, N)$ such that $\psi' - \psi = \alpha \circ \theta$ as in the diagram below

$$\begin{tikzcd}
{} \arrow[r] & P_2 \arrow[r, "d"] \arrow[d, "0"] & P_1 \arrow[r, "d"] \arrow[d] & P_0 \arrow[dl, "\theta"'] \arrow[d] \arrow[dr, "\varphi"] & & \\
& 0 \arrow[r] & N \arrow[r, "\alpha"] & Q \arrow[r, "\beta"] & M \arrow[r] & 0.
\end{tikzcd}$$

Furthermore, $\gamma' - \gamma = \theta \circ d$ or, equivalently, $\gamma' - \gamma = \delta(\theta)$. Thus, we can associate the class $[\gamma] \in \operatorname{Ext}^1_R(M, N)$ to the exact sequence (5.2).

Conversely, if we start with resolution (5.3) and a class $[\gamma] \in \operatorname{Ext}^1_R(M, N)$, we may consider some lift $\gamma \in \operatorname{Hom}_R(P_1, N)$. Then we can associated the following short exact sequence to $[\gamma]$

$$0 \to P_1 / \operatorname{Ker}\gamma \to P_0 / d(\operatorname{Ker}\gamma) \to M \to 0.$$

The reader may check that this exact sequence splits if and only if $[\gamma] = 0$.

Example 6.16. Let R be $\mathbb{C}[x]$. Since $\mathbb{C}$ is algebraically closed, every simple R-module is one-dimensional over $\mathbb{C}$ and isomorphic to $\mathbb{C}_\lambda := \mathbb{C}[x] / (x - \lambda)$. It is easy to check

$$0 \to \mathbb{C}[x] \xrightarrow{d} \mathbb{C}[x] \to 0,$$

where $d(1) = x - \lambda$ is a projective resolution of $\mathbb{C}_\lambda$. We can compute $\operatorname{Ext}^\bullet(\mathbb{C}_\lambda, \mathbb{C}_\mu)$. It amounts to calculating the cohomology of the complex

$$0 \to \mathbb{C} \xrightarrow{\delta} \mathbb{C} \to 0$$

where δ is the multiplication by $\lambda - \mu$. Hence

$$\operatorname{Ext}^0_R(\mathbb{C}_\lambda, \mathbb{C}_\mu) = \operatorname{Ext}^1_R(\mathbb{C}_\lambda, \mathbb{C}_\mu) = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \mathbb{C} & \text{if } \lambda = \mu \end{cases}$$

Example 6.17. Let $R = \mathbb{C}[x] / (x^2)$. Then R has only one (up to isomorphism) simple module, which we denote $\mathbb{C}_0$. Then

$$\dots \xrightarrow{d} R \xrightarrow{d} R \to \ 0,$$

where $d(1) = x$ is a projective resolution for $\mathbb{C}_0$ and $\operatorname{Ext}^i(\mathbb{C}_0, \mathbb{C}_0) = \mathbb{C}$ for all $i \geq 0$.

7. Representations of artinian rings

7.1. Idempotents, nilpotent ideals and Jacobson radical. A (left or right) ideal N of a ring R is called *nilpotent* if there exists $p > 0$ such that $N^p = 0$. The smallest such p is called the *degree of nilpotency* of N. The following lemma is sometimes called "lifting of an idempotent".

Lemma 7.1. *Let N be a left (or right) nilpotent ideal of R and take $r \in R$ such that $r^2 \equiv r \mod N$. Then there exists an idempotent $e \in R$ such that $e \equiv r \mod N$.*

Proof. Let N be a left ideal. We prove the statement by induction on the degree of nilpotency $d(N)$. The case $d(N)=1$ is trivial. Let $d(N)>1$. Set $n=r^2-r$, then n belongs to N and $rn=nr$. Therefore we have

$$(r+n-2rn)^2 \equiv r^2+2rn-4r^2n \mod N^2.$$

We set $s=r+n-2rn$. Then we have

$$s^2 \equiv s \mod N^2, \quad s \equiv r \mod N.$$

Since $d(N^2)<d(N)$, the induction assumption ensures that there exists an idempotent $e \in R$ such that $e \equiv s \mod N^2$, hence $e \equiv r \mod N$. □

For an R-module M let

$$\operatorname{Ann} M = \{x \in R \mid xM = 0\}.$$

Definition 7.2. The *Jacobson radical* $\operatorname{rad} R$ of a ring R is the intersection of $\operatorname{Ann} M$ for all simple R-modules M.

Exercise 7.3. (a) Prove that $\operatorname{rad} R$ is the intersection of all maximal left ideals of R as well as the intersection of all maximal right ideals.

(b) Show that x belongs $\operatorname{rad} R$ if and only if $1+rx$ is invertible for any $r \in R$.

(c) Show that if N is a nilpotent left ideal of R, then N is contained in $\operatorname{rad} R$.

Lemma 7.4. *Let $e \in \operatorname{rad} R$ such that $e^2=e$. Then $e=0$.*

Proof. By Exercise 7.3 (b) we have that $1-e$ is invertible. But $e(1-e)=0$ and therefore $e=0$. □

7.2. The Jacobson radical of an Artinian ring.

Definition 7.5. A ring R is *artinian* if it satisfying the descending chain condition for left ideals.

A typical example of artinian ring is a finite-dimensional algebra over a field. It follows from the definition that any left ideal in an Artinian ring contains a minimal (non-zero) ideal.

Lemma 7.6. *Let R be an artinian ring, $I \subset R$ be a left ideal. If I is not nilpotent, then I contains a non-zero idempotent.*

Proof. Since R is Artinian, one can can find a minimal left ideal $J \subset I$ among all non-nilpotent ideals of I. Then $J^2=J$. We will prove that J contains a non-zero idempotent.

Let $L \subset J$ be some minimal left ideal such that $JL \neq 0$. Then there exists $x \in L$ such that $Jx \neq 0$. By minimality of L we have $Jx = L$. Therefore there exists $r \in J$ such that $rx=x$. Hence $(r^2-r)\,x=0$. Let $N=\{y \in J \mid yx=0\}$. Note that N is a proper left ideal of J and therefore N is nilpotent. Thus, we have $r^2 \equiv r \mod N$. By Lemma 7.1 there exists an idempotent $e \in R$ such that $e \equiv r \mod N$, and we are done. □

Theorem 7.7. *If R is artinian then* $\operatorname{rad} R$ *is the unique maximal nilpotent ideal of R.*

Proof. By Exercise 7.3 every nilpotent ideal of R lies in $\operatorname{rad} R$. It remains to show that $\operatorname{rad} R$ is nilpotent. Indeed, otherwise by Lemma 7.6, $\operatorname{rad} R$ contains a non-zero idempotent. This contradicts Lemma 7.4. $\square$

Lemma 7.8. *An Artinian ring R is semisimple if and only if* $\operatorname{rad} R = 0$.

Proof. If R is semisimple and Artinian, then by Wedderburn-Artin theorem it is a direct product of matrix rings, which does not have non-trivial nilpotent ideals.

If R is Artinian with trivial radical, then by Lemma 7.6 every minimal left ideal L of R contains an idempotent e such that $L = Re$. Hence R is isomorphic to $L \oplus R(1-e)$. Therefore R is a direct sum of its minimal left ideals. $\square$

Corollary 7.9. *If R is Artinian, then $R/\operatorname{rad} R$ is semisimple.*

Proof. By Theorem 7.7 the quotient ring $R/\operatorname{rad} R$ does not have non-zero nilpotent ideals. Hence it is semisimple by Lemma 7.8. $\square$

7.3. Modules over Artinian rings.

Lemma 7.10. *Let R be an Artinian ring and M be an R-module. Then $M/(\operatorname{rad} R)M$ is the maximal semisimple quotient of M.*

Proof. Since $R/\operatorname{rad} R$ is a semisimple ring and $M/(\operatorname{rad} R)M$ is an $R/\operatorname{rad} R$-module, we obtain that $M/(\operatorname{rad} R)M$ is semisimple. To prove maximality, observe that $\operatorname{rad} R$ acts by zero on any semisimple quotient of M. $\square$

Corollary 7.11. *Assume that R is Artinian and M is an R-module. Consider the filtration*

$$M \supset (\operatorname{rad} R)\, M \supset (\operatorname{rad} R)^2\, M \supset \cdots \supset (\operatorname{rad} R)^k\, M = 0,$$

where k is the degree of nilpotency of $\operatorname{rad} R$. *Then all quotients* $(\operatorname{rad} R)^i M/(\operatorname{rad} R)^{i+1} M$ *are semisimple R-modules. In particular, M always has a simple quotient.*

Proposition 7.12. *Let R be Artinian. Consider it as a module over itself. Then R is a finite length module. Hence R is a Noetherian ring.*

Proof. Apply Corollary 7.11 to $M = R$. Then every quotient $(\operatorname{rad} R)^i/(\operatorname{rad} R)^{i+1}$ is a semisimple Artinian R-module. By Exercise 4.4 $(\operatorname{rad} R)^i/(\operatorname{rad} R)^{i+1}$ is a Noetherian R-module. Hence R is a Noetherian module over itself. $\square$

Let us apply the Krull–Schmidt theorem to an Artinian ring R considered as a left module over itself. Then R has a decomposition into a direct sum of indecomposable submodules

$$R = L_1 \oplus \cdots \oplus L_n.$$

Recall that $\mathrm{End}_R(R) = R^{\mathrm{op}}$. Therefore the canonical projection on each component L_i is given by multiplication (on the right) by some idempotent element $e_i \in L_i$.

In other words R has a decomposition

$$R = Re_1 \oplus \cdots \oplus Re_n. \tag{5.4}$$

Moreover, $e_i e_j = \delta_{ij} e_i$. Once more by Krull-Schmidt theorem this decomposition is unique up to multiplication by some invertible element on the right.

Definition 7.13. An idempotent $e \in R$ is called *primitive* if it can not be written $e = e' + e''$ for some non-zero idempotents e', e'' such that $e'e'' = e''e' = 0$.

Exercise 7.14. Prove that the idempotent $e \in R$ is primitive if and only if Re is an indecomposable R-module.

In the decomposition (5.4) the idempotents $e_1, \dots, e_n$ are primitive.

Lemma 7.15. *Assume R is Artinian, $N = \operatorname{rad} R$ and $e \in R$ is a primitive idempotent. Then Ne is a unique maximal submodule of Re.*

Proof. Due to Corollary 7.11 it is sufficient to show that Re/Ne is a simple R-module. Since e is primitive, the left ideal Re is an indecomposable R-module. Assume that Re/Ne is not simple. Then $Re/Ne = Ree_1 \oplus Ree_2$ for some non-zero idempotent elements e_1 and e_2 in the quotient ring R/N. By Lemma 7.1 there exist idempotents $f_1, f_2 \in R$ such that $f_i \equiv e_i \mod N$. Then $Re = Rf_1 \oplus Rf_2$ which contradicts indcomposability of Re. $\square$

Theorem 7.16. *Assume R is Artinian.*

1. *Every simple R-module S has a projective cover which is isomorphic to Re for some primitive idempotent $e \in R$.*
2. *Let P be an indecomposable projective R-module. There exists a primitive idempotent $e \in R$ such that P is isomorphic to Re. Furthermore, P has a unique simple quotient.*

Proof. Let S be a simple R-module. There exists a surjective homomorphism $f : R \to S$. Consider the decomposition (5.4). There exists $i \leq n$ such that the restriction of f on Re_i is non-zero. By the simplicity of S the restriction $f : Re_i \to S$ is surjective. It follows from Lemma 7.15 that Re_i is a projective cover of S.

Let P be an indecomposable projective module. By Lemma 7.10 the quotient $P/(\operatorname{rad} RP)$ is semisimple. Let S be a simple submodule of $P/(\operatorname{rad} RP)$. Then we have a surjection $f : P \to S$. Let $g : Q \to S$ be a projective cover of S. There exists a morphism $\varphi : P \to Q$ such that $f = g \circ \varphi$. Since Q has a unique simple quotient, the morphism φ is surjective. Then P is isomorphic to $Q \oplus \operatorname{Ker} \varphi$. The indecomposability of P implies that P is isomorphic to Q. $\square$

Example 7.17. Consider the group algebra $R = \mathbb{F}_3(S_3)$. First let us classify simple and indecomposable projective R-modules.

Let r be a 3-cycle and s be a transposition. Since s and r generate S_3, one can see easily that the elements $r-1$, r^2-1, $sr-s$ and sr^2-s span a nilpotent ideal N, which turns out to be maximal. The quotient R/N is a semisimple R-module with two simple components L_1 and L_2, where L_1 (resp. L_2) is the trivial (resp. the sign) representation of S_3. Set $e_1 = -s-1$ and $e_2 = s-1$. Then e_1, e_2 are primitive idempotents of R such that $1 = e_1 + e_2$ and $e_1 e_2 = 0$. Hence R has exactly two indecomposable projective modules, namely $P_1 = Re_1$ and $P_2 = Re_2$. Those modules can be seen as induced modules

$$Re_1 \cong \mathrm{Ind}_{S_2}^{S_3}(\mathrm{triv}), \quad Re_2 \cong \mathrm{Ind}_{S_2}^{S_3}(\mathrm{sgn}).$$

Thus P_1 is the 3-dimensional permutation representation of S_3, and $P_2 = P_1 \otimes \mathrm{sgn}$.

EXERCISE 7.18. Compute explicitly the radical filtration of P_1 and P_2. Show that

$$P_1/(\mathrm{rad}\, R)P_1 \simeq L_1, \quad (\mathrm{rad}\, R)P_1/(\mathrm{rad}\, R)^2 P_1 \simeq L_2, \quad (\mathrm{rad}\, R)^2 P_1 \simeq L_1$$

and

$$P_2/(\mathrm{rad}\, R)P_2 \simeq L_2, \quad (\mathrm{rad}\, R)P_2/(\mathrm{rad}\, R)^2 P_2 \simeq L_1, \quad (\mathrm{rad}\, R)^2 P_2 \simeq L_2.$$

Now we will calculate the extension groups between the simple modules. The above exercise implies the following exact sequences

$$0 \to L_2 \to P_2 \to P_1 \to L_1 \to 0, \quad 0 \to L_1 \to P_1 \to P_2 \to L_2 \to 0.$$

By gluing these sequences together we obtain a projective resolution for L_1

$$\cdots \to P_1 \to P_2 \to P_2 \to P_1 \to P_1 \to P_2 \to P_2 \to P_1 \to 0$$

and for L_2

$$\cdots \to P_2 \to P_1 \to P_1 \to P_2 \to P_2 \to P_1 \to P_1 \to P_2 \to 0.$$

Using the following obvious relation

$$\mathrm{Hom}(P_i, P_j) = \begin{cases} \mathbb{F}_3, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

we obtain

$$\mathrm{Ext}^p(L_i, L_i) = \begin{cases} 0, & \text{if } p \equiv 1,2 \mod 4,\ i = j \\ \mathbb{F}_3, & \text{if } p \equiv 0,3 \mod 4,\ i = j \\ 0, & \text{if } p \equiv 0,3 \mod 4,\ i \neq j \\ \mathbb{F}_3, & \text{if } p \equiv 1,2 \mod 4, i \neq j \end{cases}$$

EXERCISE 7.19. Let B_n denote the algebra of upper triangular $n \times n$ matrices over a field $\mathbb{F}$. Denote by E_{ij} the elementary matrix. Show that E_{ii} for $i = 1, \ldots, n$, are primitive idempotents of B_n. Furthermore, show that B_n has n up to isomorphism simple modules $L_1, \dots, L_n$ associated with those idempotents and that the dimension of every L_i over $\mathbb{F}$ is 1. Finally check that

$$\operatorname{Ext}^p(L_i, L_j) = \begin{cases} \mathbb{F}, & \text{if } \ i = j,\ p = 0 \text{ or } i = j+1,\ p = 1 \\ 0, & \text{otherwise} \end{cases} .$$

8. Abelian categories

An abelian category is a generalization of categories of modules over a ring.

Let us start with definition of an additive category.

Definition 8.1. A category $\mathcal{C}$ is called *additive* if for any two objects A and B,

(1) The set of morphisms $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is an abelian group.
(2) There exist an object $A \oplus B$, called a direct sum, and a pair of morphisms $i_A : A \to A \oplus B$ and $i_B : B \to A \oplus B$ such that for any morphisms $\varphi : A \to M$ and $\psi : B \to M$ there exists a unique morphism $\tau : A \oplus B \to M$ such that $\tau \circ i_A = \varphi$ and $\tau \circ i_B = \psi$.
(3) There exist an object $A \times B$ called a direct product and a pair of morphisms $p_A : A \times B \to A$ and $i_B : A \times B \to B$ such that for any morphisms $\alpha : M \to A$ and $\beta : M \to M$ there exists a unique morphism $\theta : M \to A \times B$ such that $p_A \circ \theta = \alpha$ and $p_B \circ \theta = \beta$.
(4) The induced morphism $A \oplus B \to A \times B$ is an isomorphism.

Definition 8.2. An *abelian* category is an additive category $\mathcal{C}$ such that, for every morphism $\varphi \in \operatorname{Hom}_{\mathcal{C}}(A, B)$

(1) There exist an object and a morphism $\operatorname{Ker} \varphi \xrightarrow{i} A$ such that for any morphism $\gamma : M \to A$ such that, $\varphi \circ \gamma = 0$, there exists a unique morphism $\delta : M \to \operatorname{Ker} \varphi$ such that $\gamma = i \circ \delta$.
(2) There exist an object and morphism $B \xrightarrow{p} \operatorname{Coker} \varphi$ such that for any morphism $\tau : B \to M$ such that, $\tau \circ \varphi = 0$, there exists a unique morphism $\sigma : \operatorname{Coker} \varphi \to M$ such that $\tau = \sigma \circ p$.
(3) There is an isomorphism $\operatorname{Coker} i \to \operatorname{Ker} p$.

Exercise 8.3. Let R be a ring, show that the category of finitely generated R-modules is abelian. Show that the category of projective R-modules is additive but not abelian in general. Finally show that the category of projective R-modules is abelian if and only if R is a semisimple ring.

In an abelian category we can define the image of a morphism, a quotient object, exacts sequences, projective and injective objects. All the results of Sections 4, 5 and 6 can be generalized for abelian categories. If we want to define extension groups we have to assume the existence of projective covers.

DEFINITION 8.4. Let $\mathcal{C}$ be an abelian category. Its *Grothendieck group* $\mathcal{K}_{\mathcal{C}}$ is the abelian group defined by generators and relations in the following way. For every object M of $\mathcal{C}$ there is one generator $[M]$. For every exact sequence

$$0 \to N \to M \to K \to 0$$

in $\mathcal{C}$ we have the relation $[M] = [K] + [N]$.

EXERCISE 8.5. Let $\mathcal{C}$ be the category of finite-dimensional vector spaces. Show that $\mathcal{K}_{\mathcal{C}}$ is isomorphic to $\mathbb{Z}$.

EXERCISE 8.6. Let G be a finite group and k be a field of characteristic 0. Let $\mathcal{C}$ be the category of finite-dimensional $k(G)$-modules. Then $\mathcal{K}_{\mathcal{C}}$ is isomorphic to the abelian subgroup of $\mathcal{C}(G)$ generated by the characters of irreducible representations. Furthermore, the tensor product equips $\mathcal{K}_{\mathcal{C}}$ with a structure of commutative ring.

CHAPTER 6

Symmetric groups, Schur–Weyl duality and PSH algebras

This chapter was written with Laurent Gruson

Though this be madness, yet there is method in it (Hamlet, Act II scene 2)

In which we revisit the province of representations of symmetric groups with a vision enriched by our journeys, encounter Schur-Weyl duality and PSH algebras, and put a bit of order in this mess. Not to mention the partitions, Young tableaux and related combinatorics.

In this chapter (from section 3), we will rely on a book by Andrei Zelevinsky, *Representations of finite classical groups, a Hopf algebra approach* (LNM 869, Springer 1981), which gives a very efficient axiomatisation of the essential properties of the representations of symmetric groups and general linear groups over finite fields. In this book lies the first appearance of the notion of *categorification* which has become an ubiquitous tool in representation theory.

1. Representations of symmetric groups

Consider the symmetric group S_n. In this section we classify irreducible representations of S_n over $\mathbb{Q}$. We will see that any irreducible representation over $\mathbb{Q}$ is absolutely irreducible, in other words $\mathbb{Q}$ is a splitting field for S_n. We will realize the irreducible representations of S_n as minimal left ideals in the group algebra $\mathbb{Q}(S_n)$.

Definition 1.1. A *partition* λ of n is a sequence of positive integers $(\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \cdots + \lambda_k = n$. We use the notation $\lambda \vdash n$ when λ is a partition of n. Moreover, the integer k is called the length of the partition λ.

Remark 1.2. Recall that two permutations lie in the same conjugacy class of S_n if and only if there is a bijection between their sets of cycles which preserves the lengths. Therefore we can parametrize the conjugacy classes in S_n by the partitions of n.

Definition 1.3. To every partition $\lambda = (\lambda_1, \dots, \lambda_k)$ we associate a table, also denoted λ, consisting of n boxes with rows of length $\lambda_1, \dots, \lambda_k$, it is called a *Young diagram. A Young tableau* $t(\lambda)$ is a Young diagram λ with entries $1, \dots, n$ in its boxes such that every number occurs in exactly one box. We say that two Young tableaux have the same shape if they are obtained from the same Young diagram. The number of tableaux of shape λ equals $n!$.

EXAMPLE 1.4. Let $n = 7$, $\lambda = (3, 2, 1, 1)$. The corresponding Young diagram is

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1} \\ \cline{1-1} \end{array}$$

and a possible example of tableau $t(\lambda)$ is

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}\ .$$

Given a Young tableau $t(\lambda)$, we denote by $P_{t(\lambda)}$ the subgroup of S_n preserving the rows of $t(\lambda)$ and by $Q_{t(\lambda)}$ the subgroup of permutations preserving the columns.

EXAMPLE 1.5. Consider the tableau $t(\lambda)$ from Example 1.4. Then $P_{t(\lambda)}$ is isomorphic to $S_3 \times S_2$, which is the subgroup of S_7 permuting $\{1, 2, 3\}$ and $\{4, 5\}$, and $Q_{t(\lambda)}$ is isomorphic to $S_4 \times S_2$ which permutes $\{1, 4, 6, 7\}$ and $\{2, 5\}$.

EXERCISE 1.6. Check that $P_{t(\lambda)} \cap Q_{t(\lambda)} = \{1\}$ for any tableau $t(\lambda)$.

Introduce the following elements in $\mathbb{Q}(S_n)$:

$$a_{t(\lambda)} = \sum_{p \in P_t(\lambda)} p,\ b_{t(\lambda)} = \sum_{q \in Q_{t(\lambda)}} (-1)^q q,\ c_{t(\lambda)} = a_{t(\lambda)} b_{t(\lambda)},$$

where $(-1)^q$ stands for $\epsilon(q)$.

The element $c_{t(\lambda)}$ is called a *Young symmetrizer.*

THEOREM 1.7. *Let $t(\lambda)$ be a Young tableau.*

(1) *The left ideal $\mathbb{Q}(S_n)c_{t(\lambda)}$ is minimal, therefore it is a simple $\mathbb{Q}(S_n)$-module.*
(2) *Two $\mathbb{Q}(S_n)$-modules $\mathbb{Q}(S_n)c_{t(\lambda)}$ and $\mathbb{Q}(S_n)c_{t'(\mu)}$ are isomorphic if and only if $\mu = \lambda$.*
(3) *Every simple $\mathbb{Q}(S_n)$-module is isomorphic to $V_{t(\lambda)} := \mathbb{Q}(S_n)c_{t(\lambda)}$ for some Young tableau $t(\lambda)$.*

REMARK 1.8. Note that assertion (3) of the Theorem follows from the first two, since the number of Young diagrams is equal to the number of conjugacy classes (see Remark 1.2).

EXAMPLE 1.9. Consider the partition (of length 1) $\lambda = (n)$. Then the corresponding Young diagram consists of one row with n boxes. For any tableau $t(\lambda)$ we have $P_{t(\lambda)} = S_n$, $Q_{t(\lambda)}$ is trivial and therefore

$$c_{t(\lambda)} = a_{t(\lambda)} = \sum_{s \in S_n} s.$$

The corresponding representation of S_n is trivial.

EXAMPLE 1.10. Consider the partition $\lambda = (1, \ldots, 1)$ whose Young diagram consists of one column with n boxes. Then $Q_{t(\lambda)} = S_n$, $P_{t(\lambda)}$ is trivial and

$$c_{t(\lambda)} = b_{t(\lambda)} = \sum_{s \in S_n} (-1)^s \, s.$$

Therefore the corresponding representation of S_n is the sign representation.

EXAMPLE 1.11. Let us consider the partition $\lambda = (n-1, 1)$ and the Young tableau $t(\lambda)$ which has entries $1, \ldots, n-1$ in the first row and n in the second row. Then $P_{t(\lambda)}$ is isomorphic to S_{n-1} and consists of all permutations which fix n, and $Q_{t(\lambda)}$ is generated by the transposition $(1n)$. We have

$$c_{t(\lambda)} = \left(\sum_{s \in S_{n-1}} s \right) (1 - (1n)) \, .$$

Let E denote the permutation representation of S_n. Let us show that $\mathbb{Q}(S_n)c_{t(\lambda)}$ is the $n-1$ dimensional simple submodule of E. Indeed, $a_{t(\lambda)}c_{t(\lambda)} = c_{t(\lambda)}$, therefore the restriction of $V_{t(\lambda)}$ to $P_{t(\lambda)}$ contains the trivial representation of $P_{t(\lambda)}$. Recall that the permutation representation can be obtained by induction from the trivial representation of S_{n-1}:

$$E = \mathrm{Ind}_{P_\lambda}^{S_n} \mathrm{triv} \, .$$

By Frobenius reciprocity $\mathbb{Q}(S_n)c_{t(\lambda)}$ is a non-trivial submodule of E. .

In the rest of this Section we prove Theorem 1.7.

First, let us note that S_n acts simply transitively on the set of Young tableaux of the same shape by permuting the entries, and for any $s \in S_n$ we have

$$a_{st(\lambda)} = s a_{t(\lambda)} s^{-1}, \; b_{st(\lambda)} = s b_{t(\lambda)} s^{-1}, \; c_{st(\lambda)} = s c_{t(\lambda)} s^{-1}.$$

Therefore if we have two tableaux $t(\lambda)$ and $t'(\lambda)$ of the same shape, then

$$\mathbb{Q}(S_n)c_{t(\lambda)} = \mathbb{Q}(S_n)c_{t'(\lambda)}s^{-1}$$

for some $s \in S_n$. Hence $\mathbb{Q}(S_n)c_{t(\lambda)}$ and $\mathbb{Q}(S_n)c_{t'(\lambda)}$ are isomorphic $\mathbb{Q}(S_n)$-modules.

In what follows we denote by V_λ a fixed representative of the isomorphism class of $\mathbb{Q}(S_n)c_{t(\lambda)}$ for some tableau $t(\lambda)$. As we have seen this does not depend on the tableau but only on its shape.

EXERCISE 1.12. Let $t(\lambda)$ be a Young tableau and $s \in S_n$. Show that if s does not belong to the set $P_{t(\lambda)}Q_{t(\lambda)}$, then there exist two entries i, j which lie in the same row of $t(\lambda)$ and in the same column of $st(\lambda)$. In other words, the transposition (ij) lies in the intersection $P_{t(\lambda)} \cap Q_{st(\lambda)}$. *Hint*: Assume the opposite, and check that one can find $s' \in P_{t(\lambda)}$ and $s'' \in Q_{t(\lambda)}$ such that $s't(\lambda) = s''st(\lambda)$.

Next, observe that for any $p \in P_\lambda$ and $q \in Q_\lambda$ we have

$$pc_{t(\lambda)}q = (-1)^q \, c_{t(\lambda)}.$$

LEMMA 1.13. *Let $t(\lambda)$ be a Young tableau and $y \in \mathbb{Q}(S_n)$. Assume that for all $p \in P_{t(\lambda)}$ and $q \in Q_{t(\lambda)}$ we have*

$$pyq = (-1)^q \, y.$$

Then $y = ac_{t(\lambda)}$ for some $a \in \mathbb{Q}$.

PROOF. Let T be a set of representatives of the double cosets $P_{t(\lambda)} \backslash S_n / Q_{t(\lambda)}$. Then S_n is the disjoint union $\bigsqcup_{s \in T} P_{t(\lambda)} s Q_{t(\lambda)}$ and we can write y in the form

$$\sum_{s \in T} d_s \sum_{p \in P_{t(\lambda)}, q \in Q_{t(\lambda)}} (-1)^q \, psq = \sum_{s \in T} d_s a_{t(\lambda)} s b_{t(\lambda)}.$$

It suffices to show that if $s \notin P_{t(\lambda)} Q_{t(\lambda)}$ then $a_{t(\lambda)} s b_{t(\lambda)} = 0$. This follows from Exercise 1.12. Indeed, there exists a transposition τ in the intersection $P_{t(\lambda)} \cap Q_{st(\lambda)}$. Therefore

$$a_{t(\lambda)} s b_{t(\lambda)} s^{-1} = a_{t(\lambda)} b_{st(\lambda)} = (a_{t(\lambda)} \tau)(\tau b_{st(\lambda)}) = -a_{t(\lambda)} b_{st(\lambda)} = 0.$$

□

This lemma implies

COROLLARY 1.14. *We have $c_{t(\lambda)} \mathbb{Q}(S_n) c_{t(\lambda)} \subset \mathbb{Q} c_{t(\lambda)}$.*

Now we are ready to prove the first assertion of Theorem 1.7.

LEMMA 1.15. *The ideal $\mathbb{Q}(S_n) c_{t(\lambda)}$ is a minimal left ideal of $\mathbb{Q}(S_n)$.*

PROOF. Consider a left ideal $W \subset \mathbb{Q}(S_n) c_{t(\lambda)}$. Then by Corollary 1.14 either $c_{t(\lambda)} W = \mathbb{Q} c_{t(\lambda)}$ or $c_{t(\lambda)} W = 0$.

If $c_{t(\lambda)} W = \mathbb{Q} c_{t(\lambda)}$, then $\mathbb{Q}(S_n) c_{t(\lambda)} W = \mathbb{Q}(S_n) c_{t(\lambda)}$. Hence $W = \mathbb{Q}(S_n) c_{t(\lambda)}$. If $c_{t(\lambda)} W = 0$, then $W^2 = 0$. But $\mathbb{Q}(S_n)$ is a semisimple ring, hence $W = 0$. □

Note that Corollary 1.14 also implies that V_λ is absolutely irreducible because

$$\mathrm{End}_{S_n}(\mathbb{Q}(S_n) c_{t(\lambda)}) = c_{t(\lambda)} \mathbb{Q}(S_n) c_{t(\lambda)} \simeq \mathbb{Q}.$$

COROLLARY 1.16. *For every Young tableau $t(\lambda)$ we have $c_{t(\lambda)}^2 = n_\lambda c_{t(\lambda)}$, where $n_\lambda = \frac{n!}{\dim V_\lambda}$.*

PROOF. By Corollary 1.14 we know that $c_{t(\lambda)} = n_\lambda c_{t(\lambda)}$ for some $n_\lambda \in \mathbb{Q}$. Moreover, there exists a primitive idempotent $e \in \mathbb{Q}(S_n)$ such that $c_{t(\lambda)} = n_\lambda e$. To find n_λ note that the trace of e in the regular representation equals $\dim V_\lambda$, and the trace of $c_{t(\lambda)}$ in the regular representation equals $n!$. □

EXERCISE 1.17. Introduce the lexicographical order on partitions by setting $\lambda > \mu$ if there exists i such that $\lambda_j = \mu_j$ for all $j < i$ and $\lambda_i > \mu_i$. Show that if $\lambda > \mu$, then for any two Young tableaux $t(\lambda)$ and $t'(\mu)$ there exist entries i and j which lie in the same row of $t(\lambda)$ and in the same column of $t'(\mu)$.

LEMMA 1.18. *Let $t(\lambda)$ and $t'(\mu)$ be two Young tableaux such that $\lambda < \mu$. Then $c_{t(\lambda)} \mathbb{Q}(S_n) c_{t'(\mu)} = 0$.*

PROOF. We have to check that $c_{t(\lambda)} s c_{t'(\mu)} s = 0$ for any $s \in S_n$, which is equivalent to $c_{t(\lambda)} c_{st'(\mu)} = 0$. Therefore it suffices to prove that $c_{t(\lambda)} c_{t'(\mu)} = 0$. By Exercise 1.17 there exists a transposition τ which belongs to the intersection $Q_{t(\lambda)} \cap P_{t'(\mu)}$. Then, repeating the argument from the proof of Lemma 1.13, we obtain

$$c_{t(\lambda)} c_{t'(\mu)} = c_{t(\lambda)} \tau^2 c_{t'(\mu)} = -c_{t(\lambda)} c_{t'(\mu)}.$$

$\square$

Now we show the second statement of Theorem 1.7.

LEMMA 1.19. *Two irreducible representations V_λ and V_μ are isomorphic if and only if $\lambda = \mu$.*

PROOF. It suffices to show that if $\lambda \neq \mu$, then V_λ and V_μ are not isomorphic. Without loss of generality we may assume $\lambda < \mu$ and take some Young tableaux $t(\lambda)$ and $t'(\mu)$. By Lemma 1.18 we obtain that $c_{t(\lambda)}$ acts by zero on V_μ. On the other hand, by Corollary 1.16, $c_{t(\lambda)}$ does not annihilate V_λ. Hence the statement. $\square$

By Remark 1.8 the proof of Theorem 1.7 is complete.

REMARK 1.20. Note that in fact we have proved that if $\lambda \neq \mu$ then $c_{t(\lambda)} \mathbb{Q}(S_n) c_{t'(\mu)} = 0$ for any pair of tableaux $t(\lambda), t'(\mu)$. Indeed, if $c_{t(\lambda)} \mathbb{Q}(S_n) c_{t'(\mu)} \neq 0$, then

$$\mathbb{Q}(S_n) c_{t(\lambda)} \mathbb{Q}(S_n) c_{t'(\mu)} = \mathbb{Q}(S_n) c_{t'(\mu)}.$$

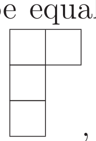
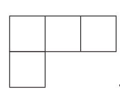
But this is impossible since $\mathbb{Q}(S_n) c_{t(\lambda)} \mathbb{Q}(S_n)$ has only components isomorphic to V_λ.

LEMMA 1.21. *Let $\rho : S_n \to \mathrm{GL}(V)$ be a finite-dimensional representation of S_n. Then the multiplicity of V_λ in V equals the rank of $\rho\left(c_{t(\lambda)}\right)$.*

PROOF. The rank of $c_{t(\lambda)}$ in V_λ is 1 and $c_{t(\lambda)} V_\mu = 0$ for all $\mu \neq \lambda$. Hence the statement. $\square$

EXERCISE 1.22. Let λ be a partition and χ_λ denote the character of V_λ.

(1) Prove that $\chi_\lambda(s) \in \mathbb{Z}$ for all $s \in S_n$.
(2) Prove that $\chi_\lambda(s) = \chi_\lambda(s^{-1})$ for all $s \in S_n$ and hence V_λ is self-dual.
(3) For a tableau $t(\lambda)$ let $\bar{c}_{t(\lambda)} = b_{t(\lambda)} a_{t(\lambda)}$. Prove that $\mathbb{Q}(S_n) c_{t(\lambda)}$ and $\mathbb{Q}(S_n) \bar{c}_{t(\lambda)}$ are isomorphic $\mathbb{Q}(S_n)$-modules.

EXERCISE 1.23. Let λ be a partition. We define the *conjugate* partition $\lambda^\perp$ by setting $\lambda_i^\perp$ to be equal to the length of the i-th row in the Young diagram λ. For example, if $\lambda =$  , then $\lambda^\perp =$  .

Prove that for any partition λ, the representation $V_{\lambda^\perp}$ is isomorphic to the tensor product of V_λ with the sign representation.

Since $\mathbb{Q}$ is a splitting field for S_n, Theorem 1.7 provides classification of irreducible representations of S_n over any field of characteristic zero.

2. Schur–Weyl duality.

2.1. Dual pairs. We will start the following general statement.

THEOREM 2.1. *Let G and H be two groups and $\rho : G \times H \to \mathrm{GL}(V)$ be a representation in a vector space V. Assume that V has a decomposition*

$$V = \bigoplus_{i=1}^{m} V_i \otimes \mathrm{Hom}_G(V_i, V)$$

for some aboslutely irreducible representations $V_1, \ldots, V_m$ of G, and the subalgebra generated by $\rho(H)$ equals $\mathrm{End}_G(V)$. Then every $W_i := \mathrm{Hom}_G(V_i, V)$ is an absolutely irreduicble representation of H and W_i is not isomorphic to W_j if $i \neq j$.

PROOF. Since every V_i is an absolutely irreducible representation of G, we have

$$\mathrm{End}_G(V) = \prod_{i=1}^{m} \mathrm{End}_k(W_i).$$

By our assumption the homomorphism

$$\rho : k(H) \to \prod_{i=1}^{m} \mathrm{End}_k(W_i)$$

is surjective. Hence the statement. □

REMARK 2.2. In general, we say that G and H satisfying the conditions of Theorem 2.1 form a *dual pair*.

EXAMPLE 2.3. Let k be an algebraically closed field, G be a finite group. Let ρ be the regular representation of G in $k(G)$ and σ be the representation of G in $k(G)$ defined by

$$\sigma_g(h) = hg^{-1}$$

for all $g, h \in G$. Then $k(G)$ has the structure of a $G \times G$-module and we have a decomposition

$$k(G) = \bigoplus_{i=1}^{r} V_i \boxtimes V_i^*,$$

where $V_1, \ldots, V_r$ are all up to isomorphism irreducible representations of G.

2.2. Duality between $GL(V)$ and S_n. Let V be a vector space over a field k of characteristic zero. Then it is an irreducible representation of the group $GL(V)$. We would like to understand $V^{\otimes n}$ as a $GL(V)$-module. Is it semisimple? If so, what are its simple component?

Let us define the representation $\rho : S_n \to \mathrm{GL}(V^{\otimes n})$ by setting

$$s(v_1 \otimes \cdots \otimes v_n) := v_{s(1)} \otimes \cdots \otimes v_{s(n)},$$

for all $v_1, \dots, v_n \in V$ and $s \in S_n$. One can easily check that the actions of $GL(V)$ and S_n in the space $V^{\otimes n}$ commute. We will show that $GL(V)$ and S_n form a dual pair.

THEOREM 2.4. *(Schur–Weyl duality) Let $m = \dim V$ and $\Gamma_{n,m}$ denote the set of all Young diagrams with n boxes such that the number of rows of λ is not bigger than m. Then*

$$V^{\otimes n} = \bigoplus_{\lambda \in \Gamma_{n,m}} V_\lambda \otimes S_\lambda(V),$$

where V_λ is the irreducible representation of S_n associated to λ and

$$S_\lambda := \mathrm{Hom}_{S_n}(V_\lambda, V)$$

is an irreducible representation of $\mathrm{GL}(V)$. *Moreover, $S_\lambda(V)$ and $S_\mu(V)$ are not isomorphic if $\lambda \neq \mu$.*

REMARK 2.5. If $\lambda \in \Gamma_{n,m}$ and $t(\lambda)$ is arbitrary Young tableau of shape λ, then the image of the Young symmetrizer $c_{t(\lambda)}$ in $V^{\otimes n}$ is a simple $GL(V)$-module isomorphic to $S_\lambda(V)$.

EXAMPLE 2.6. Let $n = 2$. Then we have a decomposition $V \otimes V = S^2(V) \oplus \Lambda^2(V)$. Theorem 2.4 implies that $S^2(V) = S_{(2)}(V)$ and $\Lambda^2(V) = S_{(1,1)}(V)$ are irreducible representations of $GL(V)$. More generally, $S_{(n)}(V)$ is isomorphic to $S^n(V)$ and $S_{(1,\dots,1)}(V)$ is isomorphic to $\Lambda^n(V)$.

Let us prove Theorem 2.4.

LEMMA 2.7. *Let $\sigma : k(GL(V)) \to \mathrm{End}_k(V^{\otimes n})$ be the homomorphism induced by the action of $GL(V)$ on $V^{\otimes n}$. Then*

$$\mathrm{End}_{S_n}\left(V^{\otimes n}\right) = \sigma(k(GL(V)).$$

PROOF. Let $E = \mathrm{End}_k(V)$. Then we have an isomorphism of algebras

$$\mathrm{End}_k\left(V^{\otimes n}\right) \simeq E^{\otimes n}.$$

We define the action of S_n on $E^{\otimes n}$ by setting

$$s(X_1 \otimes \cdots \otimes X_n) := X_{s(1)} \otimes \cdots \otimes X_{s(n)}$$

for all $s \in S$ and $X_1, \dots, X_n \in E$. Then $\mathrm{End}_{S_n}(V^{\otimes n})$ coincides with the subalgebra of S_n-invariants in $E^{\otimes n}$ that is with the n-th symmetric power $S^n(E)$ of E. Therefore it suffices to show that $S^n(E)$ is the linear span of $\sigma(g)$ for all $g \in GL(V)$.

We will need the following

EXERCISE 2.8. Let W be a vector space. Prove that for all $n \geq 2$ the following identity holds in the symmetric algebra $S(W)$

$$2^{n-1} n! x_1 \dots x_n = \sum_{i_2=0,1,\dots,i_n=0,1} (-1)^{i_2+\cdots+i_n} \left(x_1 + (-1)^{i_2} x_2 + \cdots + (-1)^{i_n} x_n\right)^n.$$

Let us choose a basis $e_1, \dots, e_{m^2}$ of E such that all non-zero linear combinations $a_1e_1+\dots+a_{m^2}e_{m^2}$ with coefficients $a_i \in \{-n, \dots, n\}$ belong to $GL(V)$. (The existence of such basis follows from density of $GL(V)$ in E in Zariski topology.) By the above exercise the set

$$\{(a_1e_{i_1} + \dots + a_ne_{i_n})^{\otimes n} \,|\, a_i = \pm 1,\, i_1, \dots i_n \leq N\}$$

spans $S^n(E)$. On the other hand, every non-zero $(a_1e_{i_1} + \dots + a_ne_{i_n})$ belongs to $GL(V)$. Therefore we have

$$(a_1e_{i_1} + \dots + a_ne_{i_n})^{\otimes n} = \sigma(a_1e_{i_1} + \dots + a_ne_{i_n}).$$

Hence $S^n(E)$ is the linear span of $\sigma(g)$ for $g \in GL(V)$. □

Lemma 2.9. *Let $\lambda = (\lambda_1, \dots, \lambda_p)$ be a partition of n. Then $S_\lambda(V) \neq 0$ if and only if $\lambda \in \Gamma_{n,m}$.*

Proof. Consider the tableau $t(\lambda)$ with entries $1, \dots, n$ placed in increasing order from top to bottom of the Young diagram λ starting from the first column. For instance, for $\lambda = \begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; & \; \\ \cline{1-2}\end{array}$ we consider the tableau $t(\lambda) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$. By Remark 2.5 $S_\lambda(V) \neq 0$ if and only if $c_{t(\lambda)}(V^{\otimes n}) \neq 0$.

If $\lambda^\perp = \mu = (\mu_1, \dots, \mu_r)$, then

$$b_{t(\lambda)}(V^{\otimes n}) = \otimes_{i=1}^r \Lambda^{\mu_i}(V).$$

If λ is not in $\Gamma_{n,m}$, then $\mu_1 > m$ and $b_{t(\lambda)}(V^{\otimes n}) = 0$. Hence $c_{t(\lambda)}(V^{\otimes n}) = 0$.

Let $\lambda \in \Gamma_{n,m}$. Choose a basis $v_1, \dots, v_m$ in V, then

$$B := \{v_{i_1} \otimes \dots \otimes v_{i_n} \,|\, 1 \leq i_1, \dots, i_n \leq m\}$$

is a basis of $V^{\otimes n}$. Consider the particular basis vector

$$u := v_1 \otimes \dots v_{\mu_1} \otimes \dots \otimes v_1 \otimes \dots v_{\mu_r} \in B.$$

One can easily see that in the decomposition of $c_{t(\lambda)}(u)$ in the basis B, u occurs with coefficient $\prod_{i=1}^{p} \lambda_i!$. In particular, $c_{t(\lambda)}(u) \neq 0$. Hence the statement. □

Lemma 2.7, Lemma 2.9 and Theorem 2.1 imply Theorem 2.4. Furthermore, Theorem 2.4 together with the Jacobson density theorem (Theorem 2.9 Chapter V) implies the double centralizer property:

Corollary 2.10. *Under assumptions of Theorem 2.4 we have*

$$\mathrm{End}_{GL(V)}(V^{\otimes n}) = \rho(k(S_n)).$$

Definition 2.11. Let λ be a partition of n. The Schur functor S_λ is the functor from the category of vector spaces to itself defined by

$$V \mapsto S_\lambda(V) = \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n}).$$

Note that S_λ is not an additive functor, in particular,

$$S_\lambda(V \oplus W) \neq S_\lambda(V) \oplus S_\lambda(W).$$

Schur–Weyl duality holds for an infinite-dimensional space in the following form.

PROPOSITION 2.12. *Let V be an infinite-dimensional vector space and Γ_n be the set of all partitions of n. Then we have the decomposition*

$$V^{\otimes n} = \bigoplus_{\lambda \in \Gamma_n} V_\lambda \otimes S_\lambda(V),$$

each $S_\lambda(V)$ is a simple $GL(V)$-module and $S_\lambda(V)$ is not isomorphic to $S_\mu(V)$ if $\lambda \neq \mu$.

PROOF. The existence of the decomposition is straightforward. For any finite-dimensional subspace W of V we have the embedding $S_\lambda(W) \hookrightarrow S_\lambda(V)$. Furthermore, $S_\lambda(W) \neq 0$ if $\dim W \geq \lambda_1^\perp$. Hence $S_\lambda(V) \neq 0$ for all $\lambda \in \Gamma_n$.

Furthermore, $S_\lambda(V)$ is the union of $S_\lambda(W)$ for all finite-dimensional subspaces $W \subset V$. Since $S_\lambda(W)$ is a simple $GL(W)$-module for sufficiently large $\dim W$, we obtain that $S_\lambda(V)$ is a simple $GL(V)$-module.

To prove the last assertion we notice that Corollary 2.10 holds by the Jacobson density theorem, hence $S_\lambda(V)$ is not isomorphic to $S_\mu(V)$ if $\lambda \neq \mu$. $\square$

Schur–Weyl duality provides a link between tensor product of $GL(V)$-modules and induction-restriction of representations of symmetric groups.

DEFINITION 2.13. Let λ be a partition of p and μ a partition of q. Note that $S_\lambda(V) \otimes S_\mu(V)$ is a submodule in $V^{\otimes(p+q)}$, hence it is a semisimple $GL(V)$-module and can be written as a direct sum of $S_\nu(V)$ with some multiplicities. These multiplicities are called *Littlewood-Richardson coefficients*. More precisely, we define $N^\nu_{\lambda,\mu}$ as the function of three partitions λ, μ and ν given by

$$N^\nu_{\lambda,\mu} := \dim \operatorname{Hom}_{GL(V)}(S_\nu(V), S_\lambda(V) \otimes S_\mu(V)).$$

Clearly, $N^\nu_{\lambda,\mu} \neq 0$ implies that ν is a partition of $p+q$.

PROPOSITION 2.14. *Let λ be a partition of p and μ a partition of q, $n = p+q$ and $\dim V \geq n$. Consider the injective homomorphism $S_p \times S_q \hookrightarrow S_n$ which sends S_p to the permutations of $1, \dots, p$ and S_q to the permutations of $p+1, \dots, n$. Then for any partition ν of n we have*

$$N^\nu_{\lambda,\mu} = \dim \operatorname{Hom}_{S_n}(V_\nu, \operatorname{Ind}^{S_n}_{S_p \times S_q}(V_\lambda \boxtimes V_\mu)) = \dim \operatorname{Hom}_{S_p \times S_q}(V_\nu, V_\lambda \boxtimes V_\mu).$$

PROOF. Let us choose three tableaux $t(\nu)$, $t'(\lambda)$ and $t''(\mu)$. We use the identification

$$S_\nu(V) \simeq c_{t(\nu)}(V^{\otimes n}),\ S_\nu(V) \simeq c_{t'(\lambda)}(V^{\otimes p}),\ S_\nu(V) \simeq c_{t''(\mu)}(V^{\otimes q}).$$

Since $V^{\otimes n}$ is a semisimple $GL(V)$-module we have

$$\operatorname{Hom}_{GL(V)}(c_{t(\nu)}(V^{\otimes n}), c_{t'(\lambda)} \otimes c_{t''(\mu)}(V^{\otimes n})) = c_{t(\nu)} k(S_n) c_{t'(\lambda)} c_{t''(\mu)}.$$

Now we use an isomorphism of S_n-modules

$$\mathrm{Ind}_{S_p\times S_q}^{S_n}(V_\lambda \boxtimes V_\mu)) = k(S_n)c_{t'(\lambda)}c_{t''(\mu)}.$$

Then by Lemma 1.21 we obtain

$$N_{\lambda,\mu}^{\nu} = \dim \mathrm{Hom}_{S_n}(V_\nu, \mathrm{Ind}_{S_p\times S_q}^{S_n}(V_\lambda \boxtimes V_\mu)) = \dim c_{t(\nu)}k(S_n)c_{t'(\lambda)}c_{t''(\mu)}.$$

The second equality follows by Frobenius reciprocity. □

3. Generalities on Hopf algebras

Let Z be a commutative unital ring.

Let A be a unital Z-algebra, we denote by $m : A\otimes A \to A$ the Z-linear multiplication. Since A is unital, there is an Z-linear map $e : Z \to A$. Moreover, we assume we are given two Z-linear maps $m^* : A \to A\otimes A$ (called the comultiplication) and $e^* : A \to Z$ (called the counit) such that the following axioms hold:

- (A): the multiplication m is associative, meaning the following diagram is commutative

$$\begin{array}{ccccc} & & m\otimes id_A & & \\ & A\otimes A\otimes A & \longrightarrow & A\otimes A & \\ id_A\otimes m & \downarrow & & \downarrow & m \\ & A\otimes A & \longrightarrow & A & \\ & & m & & \end{array}$$

- (A^*): the comultiplication is coassociative, namely the following diagram commutes:

$$\begin{array}{ccccc} & & m^* & & \\ & A & \longrightarrow & A\otimes A & \\ m^* & \downarrow & & \downarrow & id_A\otimes m^* \\ & A\otimes A & \longrightarrow & A\otimes A\otimes A & \\ & & m^*\otimes id_A & & \end{array}$$

  Note that this is the transpose of the diagram of (A).
- (U): The fact that $e(1) = 1$ can be expressed by the commutativity of the following diagrams:

$$\begin{array}{ccccc} & Z\otimes A & \simeq & A & \\ e\otimes 1 & \downarrow & & \downarrow & Id, \\ & A\otimes A & \longrightarrow & A & \\ & & m & & \end{array} \qquad \begin{array}{ccccc} & A\otimes Z & \simeq & A & \\ 1\otimes e & \downarrow & & \downarrow & Id \\ & A\otimes A & \longrightarrow & A & \\ & & m & & \end{array}$$

- (U^*): similarly, the following diagrams commute

$$\begin{array}{ccccc} & A & \simeq & A\otimes Z & \\ Id & \uparrow & & \uparrow & 1\otimes e^*, \\ & A & \longrightarrow & A\otimes A & \\ & & m^* & & \end{array} \qquad \begin{array}{ccccc} & A & \simeq & Z\otimes A & \\ Id & \uparrow & & \uparrow & e^*\otimes 1, \\ & A & \longrightarrow & A\otimes A & \\ & & m^* & & \end{array}$$

- (Antipode): there exists a Z-linear isomorphism $S : A \to A$ such that the following diagrams commute:

$$
m^* \begin{array}{ccc} A \otimes A & \xrightarrow{S \otimes Id_A} & A \otimes A \\ \uparrow & & \downarrow \\ A & \xrightarrow[e \circ e^*]{} & A \end{array} m, \quad m^* \begin{array}{ccc} A \otimes A & \xrightarrow{Id_A \otimes S} & A \otimes A \\ \uparrow & & \downarrow \\ A & \xrightarrow[e \circ e^*]{} & A \end{array} m,
$$

Definition 3.1. This set of data is called a *Hopf algebra* if the following property holds:

(H): the map $m^* : A \to A \otimes A$ is an homomorphism of Z-algebras.

Moreover, if the antipode axiom is missing, then we call it a *bialgebra*.

Exercise 3.2. Show that if an antipode exists, then it is unique. Moreover, if S is a left antipode and S' is a right antipode, then $S = S'$.

Remark 3.3. Assume that A is a commutative algebra, for any commutative algebra B, set $X_B := \mathrm{Hom}_{Z-alg}(A, B)$, then the composition with m^* induces a map $X_B \times X_B = \mathrm{Hom}_{Z-alg}(A \otimes A, B) \to X_B$ which defines a group law on X_B. This property characterizes commutative Hopf algebras.

Example 3.4. If M is a Z-module, then the symmetric algebra $S^\bullet(M)$ has a Hopf algebra structure, for the comultiplication m^* defined by: if Δ denotes the diagonal map $M \to M \oplus M$, then $m^* : S^\bullet(M \oplus M) = S^\bullet(M) \otimes S^\bullet(M)$ is the canonical morphism of Z-algebras induced by Δ.

Exercise 3.5. Find m^* when Z is a field and M is finite dimensional.

Definition 3.6. Let A be a bialgebra, an element $x \in A$ is called *primitive* if $m^*(x) = x \otimes 1 + 1 \otimes x$.

Exercise 3.7. Show that if k is a field of characteristic zero and if V is a finite dimensional k-vector space, then the primitive elements in $S^\bullet(V)$ are exactly the elements of V.

We say that a bialgebra A is *connected graded* if

1. $A = \bigoplus_{n \in \mathbb{N}} A_n$ is a graded algebra;
2. $m^* : A \to A \otimes A$ is a homomorphism of graded algebras, where the grading on $A \otimes A$ is given by the sum of gradings;
3. $A_0 = Z$;
4. the counit $e^* : A \to Z$ is a homomorphism of graded rings.

Lemma 3.8. *Let A be a graded connected bialgebra, $I = \bigoplus_{n>0} A_n$. Then for any $x \in I$, $m^*(x) = x \otimes 1 + 1 \otimes x + m^*_+(x)$ for some $m^*_+(x) \in I \otimes I$. In particular every element of A of degree 1 is primitive.*

PROOF. From the properties (3) and (4) we have that $I = \operatorname{Ker} e^*$. Write

$$m^*(x) = y \otimes 1 + 1 \otimes z + m_+^*(x).$$

We have to check that $y = z = x$. But this immediately follows from the counit axiom. $\square$

PROPOSITION 3.9. *Let A be a connected graded bialgebra and $\mathcal{P}$ be the set of primitive elements of A. Assume that $I^2 \cap \mathcal{P} = 0$. Then A is commutative and has the antipode.*

PROOF. Let us prove first that A is commutative. Assume the opposite. Let $x \in A_k, y \in A_l$ be some homogeneous element of A such that $[x, y] \neq 0$ and $k + l$ minimal possible. Then $m^*([x, y]) = [m^*(x), m^*(y)]$. By minimality of $k + l$ we have that $[m_+^*(x), m_+^*(y)] = 0$, hence $[x, y]$ is primitive. On the other hand, $[x, y] \in I \otimes I$, hence $[x, y] = 0$. A contradiction.

Next, let us prove the existence of antipode. For every $x \in A_n$ we construct $S(x) \in A_n$ recursively. We set

$$S(x) := -x \text{ for } n = 1, \quad S(x) = -x - m \circ (\operatorname{Id} \otimes S) \circ m_+^*(x) \text{ for } n > 1.$$

EXERCISE 3.10. Check that S satisfies the antipode axiom.

$\square$

4. The Hopf algebra associated to the representations of symmetric groups

Let us consider the free $\mathbb{Z}$-module $\mathcal{A} = \oplus_{n \in \mathbb{N}} \mathcal{A}(S_n)$ where $\mathcal{A}(S_n)$ is freely generated by the characters of the irreducible representations (in $\mathbb{C}$-vector spaces) of the symmetric group S_n. (Note that since every S_n-module is semi-simple, $\mathcal{A}(S_n)$ is the Grothendieck group of the category $S_n - mod$ of finite dimensional representations of S_n). It is a $\mathbb{N}$-graded module, where the homogeneous component of degree n is equal to $\mathcal{A}(S_n)$ if $n \geq 1$ and the homogeneous part of degree 0 is $\mathbb{Z}$ by convention. Moreover, we equip it with a $\mathbb{Z}$-valued symmetric bilinear form, denoted $\langle ; \rangle$, for which the given basis of characters is an orthonormal basis, and with the positive cone $\mathcal{A}^+$ generated over the non-negative integers by the orthonormal basis.

In order to define the Hopf algebra structure on $\mathcal{A}$, we use the *induction* and *restriction* functors:

$$I_{p,q} : (S_p \times S_q) - mod \longrightarrow S_{p+q} - mod,$$
$$R_{p,q} : S_{p+q} - mod \longrightarrow (S_p \times S_q) - mod.$$

REMARK 4.1. Frobenius (see Theorem 4.3) observed that the induction functor is left adjoint to the restriction.

Since the restriction and induction functors are exact, they define maps in the Grothendieck groups. Moreover, the following lemma holds:

EXERCISE 4.2. Show that we have an group isomorphism

$$\mathcal{A}(S_p \times S_q) \simeq \mathcal{A}(S_p) \otimes_{\mathbb{Z}} \mathcal{A}(S_q).$$

We deduce, from the collections of functors $I_{p,q}, R_{p,q}$, $p, q \in \mathbb{N}$, two maps:

$$m : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A},$$

$$m^* : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}.$$

More explicitely, if M (resp N) is an S_p (resp. S_q) module and if $[M]$ (resp. $[N]$) denotes its class in the Grothendieck group,

$$m([M] \otimes [N]) = [I_{p,q}(M \otimes N)],$$

and if P is an S_n-module,

$$m^*([P]) = \sum_{p+q=n} [R_{p,q}(P)].$$

EXERCISE 4.3. Show that m is associative and hence m^* is coassociative (use adjunction), m is commutative and m^* is cocommutative (use adjunction).

The tricky point is to show the following lemma:

LEMMA 4.4. *The map m^* is an algebra homomorphism.*

PROOF. (Sketch) We will use Theorem 6.4 Chapter II to compute

$$\mathrm{Res}^{S_n}_{S_p \times S_q} \mathrm{Ind}^{S_n}_{S_k \times S_l} M \otimes N,$$

where $p + q = k + l = n$, M and N are representations of S_k and S_l respectively. The double cosets $S_p \times S_q \backslash S_n / S_k \times S_l$ are enumerated by quadruples $(a, b, c, d) \in \mathbb{N}^4$ satisfying $a + b = p, c + d = q, a + c = k, b + d = l$. So we have

$$\mathrm{Res}^{S_n}_{S_p \times S_q} \mathrm{Ind}^{S_n}_{S_k \times S_l} M \otimes N =$$

$$= \bigoplus_{a+b=p,c+d=q,a+c=k,b+d=l} \mathrm{Ind}^{S_p \times S_q}_{S_a \times S_b \times S_c \times S_d} \mathrm{Res}^{S_k \times S_l}_{S_a \times S_b \times S_c \times S_d} M \otimes N.$$

and

$$\mathrm{Res}^{S_k \times S_l}_{S_a \times S_b \times S_c \times S_d} M \otimes N = \mathrm{Res}^{S_k}_{S_a \times S_c} M \otimes \mathrm{Res}^{S_l}_{S_b \times S_d} N.$$

If

$$R_{a,c}(M) \otimes R_{b,d} N = \oplus_i A_i \otimes B_i \otimes C_i \otimes D_i,$$

then

$$R_{p,q} I_{k,l}(M \otimes N) = \sum_{a+b=p,c+d=q,a+c=k,b+d=l} \sum_i I_{a,b}(A_i \otimes C_i) \otimes I_{c,d}(B_i \otimes D_i). \tag{6.1}$$

The relation (6.1) is the condition

$$m^* m(a, b) = \sum_{i,j} m(a_i, b_j) \otimes m(a^i, b^j),$$

where $m^*(a) = \sum_i a_i \otimes a^i$, $m^*(b) = \sum_j b_j \otimes b^j$, in terms of homogeneous components. $\square$

The axiom (U) corresponds to the inclusion $\mathcal{A}_0 \subset \mathcal{A}$ and (U^*) is its adjoint, and finally the antipode of the class of a simple S_n-module $[M]$ is the virtual module $(-1)^n[\varepsilon \otimes M]$, where ε is the sign representation of S_n.

Hence we have a structure of Hopf algebra on $\mathcal{A}$, and the following properties are easily checked:

- *positivity*: the cone $\mathcal{A}^+$ is stable under multiplication (for m),
- *self-adjointness*: The maps m and m^* are mutually adjoint with respect to the scalar product on $\mathcal{A}$ and the corresponding scalar product on $\mathcal{A} \otimes \mathcal{A}$.

Definition 4.5. A graded connected bialgebra A over $\mathbb{Z}$ together with a homogeneous basis Ω, equipped with a scalar product $\langle\ ,\ \rangle$ for which Ω is orthonormal, which is positive (for the cone A^+ generated over $\mathbb{N}$ by Ω) and self-adjoint is called a *positive self-adjoint Hopf algebra*, PSH algebra for short.

Moreover, the elements of Ω are called *basic elements* of A.

Remark 4.6. Note that a PSH algebra is automatically commutative and cocommutative Hopf algebra by Proposition 3.9.

We have just seen that:

Proposition 4.7. *The algebra $\mathcal{A}$ with the basis Ω given by classes of all irreducible representations is a PSH algebra.*

Exercise 4.8. Show that for any $a_1, \dots, a_n$ in A, the matrix $Gram(a_1, \dots, a_n) = ((a_{ij}))$ such that $a_{i,j} := \langle a_i, a_j \rangle$ (called the Gram matrix) is invertible in $M_n(\mathbb{Z})$ (i.e. the determinant is ± 1) if and only if the a_i's form a basis of the sublattice of A generated by some subset of cardinal n of Ω. Note that if the Gram matrix is the identity, then, up to sign, the a_i's belong to Ω..

Exercise 4.9. Assume H is a Hopf algebra with a scalar product and assume H is commutative and self-adjoint. Let x be a primitive element in H and consider the map

$$d_x : H \longrightarrow H,\ y \mapsto \sum_i \langle y_i, x \rangle y^i$$

where $m^*(y) = \sum_i y_i \otimes y^i$.

- Show that d_x is a derivation (for all a, b in H, $d_x(ab) = ad_x(b) + d_x(a)b$).
- Show that if x and y are primitive elements in H, then $d_x(y) = \langle y, x \rangle$.

5. Classification of PSH algebras part 1: Zelevinsky's decomposition theorem

In this section we classify PSH algebras following Zelevinsky. Let A be a PSH with the specified basis Ω and positive cone A^+. Let us denote by Π the set of basic primitive elements of A that is primitive elements belonging to Ω.

A multi-index α is a finitely supported function from Π to $\mathbb{N}$. For such an α we denote by π^α the monomial $\prod_{p\in\Pi} p^{\alpha(p)}$. We denote by M the set of such monomials.

For $a \in A$ we denote by $Supp(a)$ (and call support of a) the set of basic elements which appear in the decomposition of a.

LEMMA 5.1. *The supports of π^α and π^β are disjoint whenever $\alpha \neq \beta$.*

PROOF. Since the elements of M belong to A^+, we just have to show that the scalar product $\langle \pi^\alpha, \pi^\beta\rangle$ is zero when $\alpha \neq \beta$. We prove this by induction on the total degree of the monomial π^α. Write $\pi^\alpha = \pi_1\pi^\gamma$ for some π_1 such that $\alpha(\pi_1) \neq 0$. Then (recall Exercise 4.9)

$$\langle \pi^\alpha, \pi^\beta\rangle = \langle \pi^\gamma, d_{\pi_1}(\pi^\beta)\rangle.$$

Since the total degree of π^γ is less than the degree of π^α. If the scalar product is not zero, we obtain by the induction assumption that $d_{\pi_1}(\pi^\beta)$ is a multiple of π^γ. This implies $\pi^\beta = \pi_1\pi^\gamma = \pi^\alpha$. □

For every monomial $\pi^\alpha \in M$, denote by A^α the $\mathbb{Z}$-span of $Supp(\pi^\alpha)$.

LEMMA 5.2. *For all π^α, π^β in M, one has:*

$$A^\alpha A^\beta \subset A^{\alpha+\beta}.$$

PROOF. We consider the partial ordering $\leq$ in A whose positive cone is A^+ (i.e. $x \leq y$ if and only if $y - x \in A^+$). Note that if $0 \leq x \leq y$ then $Supp(x) \subset Supp(y)$. Therefore if we pick up ω in $Supp(\pi^\alpha)$ and $\eta \in Supp(\pi^\beta)$ then $\omega\eta \leq \pi^{\alpha+\beta}$, hence the result. □

Let I be the ideal spanned by all elements of positive degree.

EXERCISE 5.3. (1) Show that if $x \in A$ is primitive, then $x \in I$.
(2) Show that if $x \in I$ then $m^*(x) - 1 \otimes x - x \otimes 1$ belongs to $I \otimes I$.

Moreover, $x \in I$ is primitive if and only if x is orthogonal to I^2. Indeed for y and z in I, $\langle m^*(x) - 1 \otimes x - x \otimes 1, y \otimes z\rangle = \langle x, yz\rangle$, hence the result by Exercise 5.3.

LEMMA 5.4. *One has:*

$$A = \bigoplus_{\pi^\alpha\in M} A^\alpha.$$

PROOF. Assume the equality doesn't hold, then there exists an $\omega \in \Omega$ which does not belong to this sum. We choose such an ω with minimal degree k. Since ω is not primitive, it is not orthogonal to I^2 and therefore belongs to the support of some $\eta\eta'$ with η, η' belonging to Ω. Hence $k = k' + k''$ where k' (resp. k'') is degree of η (resp. η'). By minimality of k, η and η' lie in the direct sum, thus, by Lemma 5.2, a contradiction. □

LEMMA 5.5. *Let π^α and π^β be elements in M which are relatively prime. Then the restriction of the multiplication induces an isomorphism $A^\alpha \otimes A^\beta \simeq A^{\alpha+\beta}$ given by a bijection between $Supp(\pi^\alpha) \times Supp(\pi^\beta)$ and $Supp(\pi^{\alpha+\beta})$.*

PROOF. We will prove that the Gram matrix (see Exercise 4.8)

$$Gram((\omega\eta)_{\omega\in Supp(\pi^\alpha),\eta\in Supp(\pi^\beta)})$$

is the identity. This will be enough since it implies that the products $\omega\eta$ are distinct elements of Ω (again, see Exercise 4.8), and they exhaust the support of $\pi^{\alpha+\beta}$ which belongs to their linear span.

Let ω_1, ω_2 (resp. η_1, η_2) be elements of $Supp(\pi^\alpha)$ (resp. $Supp(\pi^\beta)$), one has

$$\langle \omega_1\eta_1, \omega_2\eta_2\rangle = \langle m^*(\omega_1\eta_1), \omega_2 \otimes \eta_2\rangle = \langle m^*(\omega_1)m^*(\eta_1), \omega_2 \otimes \eta_2\rangle.$$

One has $m^*(\omega_1) \in \bigoplus_{\alpha'+\alpha''=\alpha} A^{\alpha'} \otimes A^{\alpha''}$ and $m^*(\eta_1) \in \bigoplus_{\beta'+\beta''=\beta} A^{\beta'} \otimes A^{\beta''}$ (this is just a transposed version of Lemma 5.2), hence

$$m^*(\omega_1)m^*(\eta_1) \in \sum_{\alpha'+\alpha''=\alpha,\beta'+\beta''=\beta} A^{\alpha'+\beta'} \otimes A^{\alpha''+\beta''}.$$

On the other hand, $\omega_2 \otimes \eta_2$ belongs to $A^\alpha \otimes A^\beta$. We must understand in which cases $A^{\alpha'+\beta'} \otimes A^{\alpha''+\beta''} = A^\alpha \otimes A^\beta$ and this occurs if and only if $\alpha' + \beta' = \alpha, \alpha'' + \beta'' = \beta$. Since π^α and π^β are relatively prime, this occurs if and only if $\beta' = 0 = \alpha''$.

The component of $m^*(\omega_1)$ in $A^\alpha \otimes A^0$ is $\omega_1 \otimes 1$ and the component of $m^*(\eta_1)$ in $A^0 \otimes A^\beta$ is $1 \otimes \eta_1$ (see Exercise 5.3), therefore

$$\langle \omega_1\eta_1, \omega_2\eta_2\rangle = \langle(\omega_1 \otimes 1)(1 \otimes \eta_1), \omega_2 \otimes \eta_2\rangle = \langle \omega_1 \otimes \eta_1, \omega_2 \otimes \eta_2\rangle = \langle \omega_1\eta_1, \omega_2\eta_2\rangle.$$

Hence the result. □

The following Theorem is a direct consequence of Lemmas 5.1, 5.2, 5.4, 5.5.

THEOREM 5.6. *(Zelevinsky's decomposition theorem). Let A be a PSH algebra with basis ω, and let Π be the set of basic primitive elements of A. For every $\pi \in \Pi$ we set $A_\pi := \bigoplus_{n\in\mathbb{N}} A^{\pi^n}$. Then*

(1) *A_π is a PSH algebra and its unique basic primitive element is π,*
(2) *$A = \bigotimes_{\pi\in\Pi} A_\pi$.*

REMARK 5.7. In the second statement, the tensor product might be infinite: it is defined as the span of tensor monomials with a finite number of entries non-equal to 1.

DEFINITION 5.8. The *rank* of the PSH algebra A is the cardinal of the set Π of basic primitive elements in A.

6. Classification of PSH algebras part 2: unicity for the rank 1 case

By the previous section, understanding a PSH algebra is equivalent to understanding its rank one components. Therefore, we want to classify the rank one cases.

Let A be PSH algebra of rank one with marked basis Ω, and denote π its unique basic primitive element. We assume that we have chosen the graduation of A so that π is of degree 1. We will construct a sequence $(e_i)_{i\in\mathbb{N}}$ of elements of Ω such that:

(1) $e_0 = 1$, $e_1 = \pi$ and e_n is of degree n (it is automatically homogeneous since it belongs to Ω),
(2) $A \simeq \mathbb{Z}[e_1, e_2, \dots, e_n, \dots]$ as graded $\mathbb{Z}$-algebras,
(3) $m^*(e_n) = \sum_{i+j=n} e_i \otimes e_j$.

Actually, we will find exactly two such sequences and we will denote the second one $(h_i)_{i\in\mathbb{N}}$. The antipode map exchanges those two sequences.

We denote by d the derivation of A which is adjoint to the multiplication by π (see Exercise 4.9).

LEMMA 6.1. *There are exactly two elements in Ω of degree 2 and their sum is equal to π^2.*

PROOF. One has

$$\langle \pi^2, \pi^2 \rangle = \langle \pi, d(\pi^2) \rangle = 2\langle \pi, \pi \rangle = 2.$$

On the other hand, if we write π^2 in the basis Ω, $\pi^2 = \sum_{\omega\in\Omega} \langle \pi^2, \omega \rangle \omega$, we get

$$\langle \pi^2, \pi^2 \rangle = \sum_{\omega\in\Omega} \langle \pi^2, \omega \rangle^2,$$

but $\langle \pi^2, \omega \rangle$ is a non negative integer, hence the result. □

We will denote by e_2 one of those two basic elements and h_2 the other one. Furthermore, we set e_2^* (resp. h_2^*) to be the linear operator on A of degree -2 which is adjoint to the multiplication by e_2 (resp. h_2).

EXERCISE 6.2. Show that the operator e_2^* satisfies the identities

$$m^*(e_2) = e_2 \otimes 1 + \pi \otimes \pi + 1 \otimes e_2, \tag{6.2}$$

$$e_2^*(ab) = e_2^*(a)b + ae_2^*(b) + d(a)d(b). \tag{6.3}$$

LEMMA 6.3. *There is exactly one element e_n of degree n in Ω such that $h_2^*(e_n) = 0$. This element satisfies $d(e_n) = e_{n-1}$.*

PROOF. We prove this by induction on n. For $n = 2$, $h_2^*(e_2)$ is the scalar product $\langle h_2, e_2 \rangle$ which is zero because e_2 and h_2 are two distinct elements of Ω. We assume that the statement of the lemma holds for all $i < n$. If x is of degree n and satisfies $h_2^*(x) = 0$, then $d(x)$ (which is of degree $n-1$) is proportional to e_{n-1} by the induction hypothesis, since h_2^* and d commute. The scalar is equal to $\langle d(x), e_{n-1} \rangle = \langle x, \pi e_{n-1} \rangle$ since d is the adjoint of the multiplication by π. As in Lemma 6.1, we prove next that $\langle \pi e_{n-1}, \pi e_{n-1} \rangle = 2$: indeed $\langle \pi e_{n-1}, \pi e_{n-1} \rangle = \langle d(\pi e_{n-1}), e_{n-1} \rangle = \langle e_{n-1} + \pi e_{n-2}, e_{n-1} \rangle = 1 + \langle \pi e_{n-2}, e_{n-1} \rangle = 1 + \langle e_{n-2}, d(e_{n-1}) \rangle = 2$.

Therefore πe_{n-1} decomposes as the sum of two distinct basic elements $\omega_1 + \omega_2$. Besides, using Exercise 6.2 equation (6.3), we have $h_2^*(\pi e_{n-1}) = e_{n-2}$. Since h_2^* is a positive operator (i.e. preserves A^+), $h_2^*(\omega_1) + h_2^*(\omega_2) = e_{n-2}$ implies that one of the factors $h_2^*(\omega_i)$ ($i = 1$ or 2) is zero, so that ω_i can be choosen for $x = e_n$. □

PROPOSITION 6.4. *One has, for every $n \geq 1$,*

$$m^*(e_n) = \sum_{k=0}^{n} e_k \otimes e_{n-k}.$$

PROOF. If ω belongs to Ω, we denote by ω^* the adjoint of the multiplication by ω. We just need to show that $\omega^*(e_n) = 0$ except if $\omega = e_k$ for some $0 \leq k \leq n$, in which case $\omega^*(e_n) = e_{n-k}$.

Indeed, we can write $\pi^k = \sum_{\omega \in \Omega, deg(\omega)=k} C_\omega \omega$ where the coefficients C_ω are positive integers, hence $d^k = \sum_{\omega \in \Omega, deg(\omega)=k} C_\omega \omega^*$. Since $\sum_{\omega \in \Omega, deg(\omega)=k} C_\omega \omega^*(e_n) = d^k(e_n) = e_{n-k}$, all the terms in the sum are zero except one (by integrity of the coefficients). It remains to show that the non-zero term comes from the element e_k of Ω. But this is clear since $d^{n-k} e_k^*(e_n) = e_k^* d^{n-k}(e_n) = e_k^*(e_k) = 1$. □

EXERCISE 6.5.
(1) Show that for every $n \geq 0$, $e_n^*(ab) = \sum_{0 \leq k \leq n} e_k^*(a) e_{n-k}^*(b)$.
(2) We make the convention that $h_{-1} = 0$. Prove the following equality for any positive integer n and $i_1, \ldots, i_r$ non negative integers:

$$e_r^*(h_{i_1} \ldots h_{i_r}) = h_{i_1-1} \ldots h_{i_r-1}. \tag{6.4}$$

PROPOSITION 6.6. *Let t be an indeterminate, the two formal series $\sum_{i \geq 0} e_i t^i$ and $\sum_{i \geq 0} (-1)^i h_i t^i$ are mutually inverse.*

PROOF. Since A is a graded bialgebra over $\mathbb{Z}$, we know that it is equipped with a unique antipode S. Let us show that it exchanges e_n and $(-1)^n h_n$:

First, let us show that S is an isometry for the scalar product of A. Indeed, we have the following commutative diagram

$$\begin{array}{ccccc} & A \otimes A & \xrightarrow{Id_A \otimes S} & A \otimes A & \\ m^* & \uparrow & & \downarrow & m, \\ & A & \xrightarrow[e \circ e^*]{} & A & \end{array} \tag{6.5}$$

where $e : \mathbb{Z} \rightarrow$ is the unit of A (see section 3). By considering the adjoint of this diagram, we understand that S^* is also an antipode, and by uniqueness of the antipode, $S = S^{-1} = S^*$ hence S is an isometry and so for $\omega \in \Omega$, $S(\omega) = \pm\eta$, for some $\eta \in \Omega$ (ω and η have the same degree). Applying the diagram (6.5) to π, who is primitive, we check that $S(\pi) = -\pi$. In the same way, we obtain $S(\pi^2) = \pi^2$ and $(e_2) = h_2$. Since e_n is the unique basic element of degree n satisfying the relation $h_2^*(e_n) = 0$,we have $S(e_n) = \pm h_n$ and the sign coincides is $(-1)^n$ since $S(\pi^n) = (-1)^n \pi^n$.

The diagram (6.5) implies that $(m \circ Id_A \otimes S \circ m^*)(e_n) = 0$ for all $n \geq 1$. By Proposition 6.4, one has $m^*(e_n) = \sum_{0 \leq k \leq n} e_k \otimes e_{n-k}$ and so we have $\sum_{0 \leq k \leq n} (-1)^{n-k} e_k h_{n-k} = 0$. The result follows. □

We will now use the definitions and notations for partitions introduced in Section 1. For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, we denote by e_λ the product $e_\lambda = e_{\lambda_1} \ldots e_{\lambda_n}$ and set a similar definition for h_λ. Note that in general, the elements e_λ, h_λ do not belong to Ω.

DEFINITION 6.7. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ be two partitions of the same integer n. We say that λ *is greater or equal than* μ *for the dominance order* and denote it $\lambda \succeq \mu$ if, for every $k \leq \inf(r, s)$, $\lambda_1 + \ldots + \lambda_k \geq \mu_1 + \ldots + \mu_k$.

LEMMA 6.8. *Let λ and μ be partitions of a given integer n, define $M_{\lambda,\mu}$ as the number of matrices with entries belonging to $\{0, 1\}$ such that the sum of the entries in the i-th row (resp. column) is λ_i (resp. μ_i). Then one has:*

(1) $\langle e_\lambda, h_\mu \rangle = M_{\lambda,\mu}$,
(2) $M_{\lambda,\lambda^\perp} = 1$,
(3) $M_{\lambda,\mu} \neq 0$ *implies* $\lambda \preceq \mu^\perp$

PROOF. (Sketch) We write $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1 \ldots \mu_s)$. By Exercise 6.5, we have

$$e^*_{\lambda_1}(h_{\mu_1} \ldots h_{\mu_s}) = \sum_{\nu_i = 0,1 \sum \nu_i = \lambda_1} h_{\mu_1 - \nu_1} \ldots h_{\mu_s - \nu_s}.$$

Next, we apply $e^*_{\lambda_2}$ to this sum, $e^*_{\lambda_3}$ to the result, and so on. We obtain:

$$\langle e_\lambda, h_\mu \rangle = e^*_\lambda(h_\mu) = \sum_{\nu_{ij} \in \{0,1\}, \sum_j \nu_{ij} = \lambda_i} h_{\mu_1 - \sum_i \nu_{1i}} \ldots h_{\mu_s - \sum_i \nu_{si}}.$$

The terms in the sum of the right-hand side are equal to 0 except when $\mu_i = \sum_j \nu_{ij}$ for all i, in which case the value is 1. The statement (1) follows.

For statement (2), we see easily that the only matrix $N = (\nu_{i,j})$ with entries in $\{0, 1\}$ such that $\sum_j \nu_{ij} = \lambda_i$ and $\sum_i \nu_{ij} = \lambda^\perp_j$ is the one such that the entries decrease along both the rows and the columns, hence the result.

Finally, consider a matrix $N = (\nu_{ij})$ with entries in $\{0, 1\}$ such that $\sum_j \nu_{ij} = \lambda_i$ and $\sum_i \nu_{ij} = \mu_j$. The sum $\lambda_1 + \ldots + \lambda_i$ is the sum of the entries of the columns of index $\leq i$ of N. Furthermore, $\mu^\perp_1 + \ldots + \mu^\perp_i$ is equal to $\sum_{j \leq i} j l_j$ where l_j is the number of rows of N which have sum j. It is easy to check that statement (3) follows. $\square$

COROLLARY 6.9. *The matrix $(\langle e_\lambda, h_{\mu^\perp} \rangle)_{\lambda,\mu \vdash n}$ is upper triangular with 1's on the diagonal. In particular, its determinant is equal to 1.*

PROPOSITION 6.10. *When λ varies along the partitions of n, the collection of e_λ's is a basis of the homogeneous component of degree n, A_n, of A.*

PROOF. First we notice that every h_i is a polynomial with integral coefficients in the e_j's. This follows immediately from Proposition 6.6. Therefore the base change matrix P from $(h_\lambda)_{\lambda\vdash n}$ to $(e_\lambda)_{\lambda\vdash n}$ has integral entries. Then the Gram matrix G_e of $(e_\lambda)_{\lambda\vdash n}$ satisfies the equality

$$(\langle e_\lambda, h_\mu\rangle)_{\lambda,\mu\vdash n} = P^t G_e,$$

where P^t denotes the transposed P. The corollary 6.9 ensures that the left-hand side has determinant ± 1 (the corollary is stated for $\mu^\perp$ and $\mu \mapsto \mu^\perp$ is an involution which can produce a sign). Hence G_e has determinant ± 1: we refer to Exercise 4.8 to ensure that the $\mathbb{Z}$-module generated by $(e_\lambda)_{\lambda\vdash n}$ has a basis contained in Ω. Since the support of e_1^n is the set of all $\omega \in \Omega$ of degree n, we conclude that $(e_\lambda)_{\lambda\vdash n}$ is a basis of A_n. □

We deduce from the results of this section:

THEOREM 6.11. *(Zelevinsky) Up to isomorphism, there is only one rank one PSH algebra. It has only one non-trivial automorphism ι, which takes any homogeneous element x of degree n to $(-1)^n S(x)$ where S is the antipode.*

REMARK 6.12. The sets of algebraically independent generators (e_n) and (h_n) of the $\mathbb{Z}$-algebra A play symmetric roles, and they are exchanged by the automorphism ι of the theorem.

7. Bases of PSH algebras of rank one

Let A be a PSH algebra of rank one, with basis Ω and scalar product $\langle\ ,\rangle$, we will use the sets of generators (e_n) and (h_n). We keep all the notations of the preceding section.

We will first describe the primitive elements of A. We denote $A_\mathbb{Q} := A \otimes \mathbb{Q}$.

EXERCISE 7.1. Consider the algebra $A[[t]]$ of formal power series with coefficients in A. Let $f \in A[[t]]$ such that $m^*(f) = f \otimes f$ and the constant term of f is 1. Show that the logarithmic derivative $g := \frac{f'}{f}$ satisfies $m^*(g) = g \otimes 1 + 1 \otimes g$.

PROPOSITION 7.2.
(1) *For every $n \geq 1$, there is exactly one primitive element of degree n, p_n, such that $\langle p_n, h_n\rangle = 1$. Moreover, every primitive element of degree n is a integral multiple of p_n.*
(2) *In the formal power series ring $A_\mathbb{Q}[[t]]$, we have the following equality:*

$$exp\left(\sum_{n\geq 1} \frac{p_n}{n} t^n\right) = \sum_{n\geq 0} h_n t^n. \tag{6.6}$$

PROOF. We first show that the set of primitive elements of degree n is a subgroup of rank 1. Indeed, we recall that the primitive elements form the orthogonal complement of I^2 in I (see just below Exercise 5.3). Since all the elements $(h_\lambda)_{\lambda\vdash n}$ except h_n are in I^2, the conclusion follows. Moreover, A_n is its own dual with respect to the scalar product. Let denote by $(h^\lambda)_{\lambda\vdash n}$ the dual basis of $(h_\lambda)_{\lambda\vdash n}$. Clearly, h^n can be chosen as p_n. Hence statement (1).

Consider the formal series $H(t) := \sum_{n\geq 0} h_n t^n \in A[[t]]$, it satisfies the relation $m^(H) = H \otimes H$ by Proposition 6.4, re-written in terms of h's instead of e's. Hence, using Exercise 7.1, we get $P(t) := \frac{H'(t)}{H(t)}$ which satisfies $m^*(P) = P\otimes 1 + 1\otimes P$. Hence all the coefficients of P are primitive elements in A. Write $P(t) = \sum_{i\geq 1} \varpi_{i+1} t^i$. To prove statement (2), it remains to check that $\langle \varpi_n, h_n\rangle = 1$. We have $P(t)H(t) = H'(t)$, so when we compare the terms on both sides we get

$$\varpi_n + h_1\varpi_{n-1} + \ldots + h_{n-1}\varpi_1 - nh_n = 0. \tag{6.7}$$

By induction on n, this implies

$$\varpi_n = (-1)^n h_1^n + \sum_{\lambda\vdash n, \lambda\neq(1,\ldots,1)} c_\lambda h_\lambda,$$

where the c_λ's are integers. Now, we compute the scalar product with e_n: we apply Lemma 6.8 and find that there is no contribution from the terms indexed by λ if $\lambda \neq (1,\ldots,1)$. Therefore, $\langle \varpi_n, e_n\rangle = (-1)^n$. Finally, we use the automorphism of A to get the conclusion that $p_n = \varpi_n$ since $\iota(e_n) = h_n$ and $\iota(\varpi_n) = (-1)^n\varpi_n$ by Proposition 6.6. □

For every partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, we set $p_\lambda = p_{\lambda_1} \ldots p_{\lambda_r}$. Let us compute their Gram matrix:

Proposition 7.3. *The family (p_λ) is an orthogonal basis of $A_{\mathbb{Q}}$ and one has*

$$\langle p_\lambda, p_\lambda\rangle = \prod_j (\lambda_j^\perp - \lambda_{j+1}^\perp)! \prod_i \lambda_i$$

Proof. Since p_i is primitive, the operator p_i^* is a derivation of A. Moreover, since p_i is of degree i, p_i and p_j are orthogonal when $i \neq j$. We compute $\langle p_i, p_i\rangle$: we use the formula (6.7) (recall that we proved that $p_n = \varpi_n \forall n$) and since $p_i^*(h_r p_{i-r}) = p_i^*(h_r)p_{i-r} + h_r p_i^*(p_{i-r}) = 0$ if $1 \leq r \leq i-1$, we obtain $p_i^*(p_i) = \langle p_i, p_i\rangle = \langle p_i, ih_i\rangle = i$ by Proposition 7.2.

To show that p_λ is orthogonal to p_μ if $\lambda \neq \mu$, we repeat the argument of the proof of Lemma 5.1.

Finally, we compute $\langle p_i^r, p_i^r\rangle$: we use the fact that p_i^* is a derivation such that $p_i^*(p_i) = i$, hence $\langle p_i^r, p_i^r\rangle = r! i^r$. This implies the formula giving $\langle p_\lambda, p_\lambda\rangle$ for any λ. □

Now we want to compute the transfer matrices between the bases (h_λ) (or (e_λ)) and Ω.

Lemma 7.4. *Let λ be a partition, then the intersection of the supports $Supp(e_{\lambda^\perp})$ and $Supp(h_\lambda)$ is of cardinal one. We will denote this element ω_λ.*

Proof. By Lemma 6.8, one has $\langle e_{\lambda^\perp}, h_\lambda\rangle = 1$ and, by the positivity of those elements, this implies the statement. □

Our first goal is to express h_λ's in terms of ω_μ's. First, we compute $h_i^*(\omega_\lambda)$, and for this, we need to introduce some notations.

Let λ be a partition, or equivalently a Young diagram. We denote by $r(\lambda)$ (resp. $c(\lambda)$) the number of rows (resp columns) of λ.

We denote by $\mathbf{R}_i^\lambda$ the set of all μ's such that μ is obtained from λ by removing exactly i boxes, at most one in every row of λ. Similarly, $\mathbf{C}_i^\lambda$ is the set of all μ's such that μ is obtained from λ by removing exactly i boxes, at most one in every column of λ. In the specific case where $i = r(\lambda)$, there is only one element in the set $\mathbf{R}_i^\lambda$ and this element will be denoted by $\lambda^\leftarrow$, it is the diagram obtained by removing the first column of λ, similarly, if $i = c(\lambda)$ the unique element of $\mathbf{C}_i^\lambda$ will be denoted by $\lambda^\downarrow$ and is the diagram obtained by suppressing the first row of λ.

Remark 7.5. Note that if $\mu \in \mathbf{C}_i^\lambda$, then $\mu^\perp \in \mathbf{R}_i(\lambda^\perp)$.

Theorem 7.6. *(Pieri's rule) One has:*

$$h_i^*(\omega_\lambda) = \sum_{\mu \in \mathbf{C}_i^\lambda} \omega_\mu,$$

and

$$e_j^*(\omega_\lambda) = \sum_{\mu \in \mathbf{R}_j^\lambda} \omega_\mu.$$

We need several lemmas to show this statement.

Lemma 7.7. *One has, for all i, j in $\mathbb{N}$,*

$$e_i^* \circ h_j = h_j \circ e_i^* + h_{j-1} \circ e_{i-1}^*$$

Proof. From Exercise 6.5 statement (1), we obtain that

$$e_i^*(h_j x) = e_1^*(h_j) e_{i-1}^*(x) + h_j e_i^*(x) \; \forall x \in A,$$

hence the Lemma. □

Lemma 7.8. *Let p, q be two integers, let $a \in A$. Let us assume that $h_i^*(a) = 0$ for $i > p$ and $e_j^*(a) = 0$ for $j > q$, then*

$$h_p^* \circ e_q^*(a) = 0$$

and

$$h_{p-1}^* \circ e_q^*(a) = h_p^* \circ e_{q-1}^*(a).$$

Proof. Using a transposed version of Proposition 6.6, we get:

$$\sum_{i+j=n} (-1)^j h_i^* \circ e_j^* = 0.$$

The lemma follows. □

LEMMA 7.9. *One has:*

$$e^*_{r(\lambda)}(\omega_\lambda) = \omega_{\lambda^\leftarrow}.$$

PROOF. Applying Equation (6.4), we get

$$e^*_{r(\lambda)}(h_\lambda) = h_{\lambda^\leftarrow}.$$

Since $e^*_{r(\lambda)}$ is a positive operator and $\omega_\lambda < h_\lambda$ by definition, we have $Supp(e^*_{r(\lambda)}(\omega_\lambda)) \subset Supp(h_{\lambda^\leftarrow})$.

By definition of ω_λ, we know that $\langle e_{\lambda^\perp}, \omega_\lambda\rangle = 1$, so we have $\langle e_{(\lambda^\leftarrow)^\perp}, e^*_{r(\lambda)}(\omega_\lambda)\rangle = 1$. Therefore, $\omega_{\lambda^\leftarrow} \in Supp(e^*_{r(\lambda)}(\omega_\lambda))$.

It is sufficient to show now that $\langle e^*_{r(\lambda)}(\omega_\lambda), h_{\lambda^\leftarrow}\rangle = 1$: let us compute:

$$\langle e^*_{r(\lambda)}(\omega_\lambda), h_{\lambda^\leftarrow}\rangle = h^*_{\lambda^\leftarrow} e^*_{r(\lambda)}(\omega_\lambda),$$

assume $\lambda = (\lambda_1, \ldots, \lambda_r)$,

$$\begin{aligned}\langle e^*_r(\omega_\lambda), h_{\lambda^\leftarrow}\rangle &= h^*_{\lambda_r - 1} \circ \ldots \circ h^*_{\lambda_1 - 1} \circ e^*_r(\omega_\lambda)\\ &= h^*_{\lambda_r - 1} \circ \ldots \circ h^*_{\lambda_2 - 1} \circ e^*_{r-1} \circ h^*_{\lambda_1}(\omega_\lambda)\end{aligned}$$

by Lemma 7.8 (the hypothesis is satisfied because for all $i > \lambda_1$ one has $h^*_i(e_{\lambda^\perp}) = 0$ and for all $j > r$, $e^*_j(h_\lambda) = 0$). We use the same trick repeatedly, the enthusiastic reader is encouraged to check that the hypothesis of Lemma 7.8 is satisfied at each step by induction. We finally obtain

$$h^*_{\lambda^\leftarrow} \circ e^*_{r(\lambda)}(\omega_\lambda) = h^*_\lambda(\omega_\lambda) = 1.$$

□

For every i in $\mathbb{N}$ and for every partition λ, we set:

$$h^*_i(\omega_\lambda) = \sum_\mu a^i_{\lambda,\mu}\omega_\mu$$

which can also be written

$$h_i\omega_\mu = \sum_\lambda a^i_{\lambda,\mu}\omega_\lambda \tag{6.8}$$

the Theorem 7.6 amounts to computing the coefficients $a^i_{\lambda,\mu}$.

LEMMA 7.10. *For $i > 0$ and for every partitions λ and μ, one has*

$$a^i_{\lambda,\mu} = \begin{cases} a^i_{\lambda^\leftarrow,\mu^\leftarrow} & if \quad r(\lambda) = r(\mu)\\ a^{i-1}_{\lambda^\leftarrow,\mu^\leftarrow} & if \quad r(\lambda) = r(\mu) + 1\\ 0 & otherwise\end{cases}$$

REMARK 7.11. The first equality of Theorem 7.6 is obtained from this lemma by induction on $c(\lambda)$, the second one follows via the automorphism ι.

PROOF. First, let us prove that if $a^i_{\lambda,\mu} \neq 0$, then $r(\lambda) = r(\mu)$ or $r(\lambda) = r(\mu) + 1$.

Assume $r(\mu) > r(\lambda)$ and $a^i_{\lambda,\mu} \neq 0$: then we have $\omega_\mu \leq h^*_i(\omega_\lambda)$, therefore, applying Lemma 7.9, we get

$$\omega_{\mu^\leftarrow} = e^*_{r(\mu)}(\omega_\mu) \leq e^*_{r(\mu)} \circ h^*_i(\omega_\lambda) = h^*_i \circ e^*_{r(\mu)}(\omega_\lambda) = 0,$$

which gives a contradiction.

Assume $r(\mu) < r(\lambda) - 1$ and $a^i_{\lambda,\mu} \neq 0$: then applying the equation (6.8), we have $\omega_\lambda \leq h_i\omega_\mu$, therefore applying Lemma 7.9 and Lemma 7.7, we get

$$\omega_{\lambda^\leftarrow} = e^*_{r(\lambda)}(\omega_\lambda) \leq e^*_{r(\lambda)} \circ h_i(\omega_\mu) = h_i \circ e^*_{r(\lambda)}(\omega_\mu) + h_{i-1} \circ e^*_{r(\lambda)-1}(\omega_\mu) = 0,$$

which again gives a contradiction.

Next, we look at the case $r(\lambda) = r(\mu)$. We do a direct computation:

$$h^*_i(\omega_{\lambda^\leftarrow}) = e^*_{r(\lambda)} \circ h^*_i(\omega_\lambda) = \sum_\mu a^i_{\lambda,\mu} e^*_{r(\lambda)}(\omega_\mu) = \sum_{r(\lambda)=r(\mu)} a^i_{\lambda,\mu}\omega_{\mu^\leftarrow}.$$

Finally, we assume $r(\lambda) = r(\mu) + 1$. We apply Lemma 7.7,

$$e^*_{r(\mu)+1}(h_i\omega_\mu) = h_{i-1}e^*_{r(\mu)}(\omega_\mu) = h_{i-1}(\omega_{\mu^\leftarrow}) = \sum_\nu a^{i-1}_{\nu,\mu^\leftarrow}\omega_\nu.$$

On the other hand,

$$e^*_{r(\mu)+1}(h_i\omega_\mu) = \sum_\lambda a^i_{\lambda,\mu} e^*_{r(\mu)+1}(\omega_\lambda) = \sum_{r(\lambda)=r(\mu)+1} a^i_{\lambda,\mu}\omega_{\lambda^\leftarrow}.$$

Now we compare the coefficients and obtain that

$$a^i_{\lambda,\mu} = a^{i-1}_{\lambda^\leftarrow,\mu^\leftarrow}.$$

□

For a partition λ, we introduce the notion of *semistandard tableau of shape* λ: the Young diagram of shape λ is filled with entries wich are no longer distinct, with the condition that the entries are non decreasing along the rows and increasing along the columns of λ. For instance,

1	1	1
2	3	
3		
4		

is a semistandard tableau.

To such a semistandard tableau, we associate its *weight*, which is the sequence m_i consisting of the numbers of occurences of the integer i in the tableau: in our example, $m_1 = 2$, $m_2 = 1$, $m_3 = 2$, $m_4 = 1$ and all the other m_i's are zero.

PROPOSITION 7.12. *Let λ be a partition of n. Let $m_1, \ldots, m_r$ be a sequence of non negative integers such that $m_1 + \ldots + m_r = n$, then $\langle h_{m_1} \ldots h_{m_r}, \omega_\lambda \rangle$ is the number of semistandard tableaux of shape λ and weight $m_1, \ldots, m_r$.*

PROOF. We iterate Pieri's rule (see Theorem 7.6):

$$h^*_{m_r}(\omega_\lambda) = \sum_{\mu \in \mathbf{C}^\lambda_{m_r}} \omega_\mu,$$

$$(h_{m_{r-1}} h_{m_r})^*(\omega_\lambda) = \sum_{\mu_1 \in \mathbf{C}^\lambda_{m_r}} \sum_{\mu_2 \in \mathbf{C}^{\mu_1}_{m_{r-1}}} \omega_{\mu_2},$$

and eventually

$$\langle h_{m_1} \ldots h_{m_r}, \omega_\lambda \rangle = (h_{m_1} \ldots h_{m_{r-1}} h_{m_r})^*(\omega_\lambda) = \sum_{\mu_1 \in \mathbf{C}^\lambda_{m_1}} \sum_{\mu_2 \in \mathbf{C}^{\mu_1}_{m_2}} \cdots \sum_{\mu_r \in \mathbf{C}^{\mu_{r-1}}_{m_r}} 1,$$

because $\omega_{\mu_r} = 1$ due to the fact that $m_1 + \ldots + m_r = n$.

The sequences $\mu_1, \ldots, \mu_r$ indexing the sum in the right-hand side are in bijective correspondence with the semistandard tableaux of shape λ and weight $m_1, \ldots, m_r$, indeed given such a semistandard tableau, we set μ_i to be the union of the boxes filled with numbers $\leq i$: μ_i is a semistandard tableau. Hence the result. $\square$

REMARK 7.13. Since the product is commutative, $\langle h_{m_1} \ldots h_{m_r}, \omega_\lambda \rangle$ depends only on the non-increasing rearrangement μ of the sequence $\mu_1, \ldots, \mu_r$. Note that the partition μ we just obtained verifies $\lambda \succeq \mu$.

DEFINITION 7.14. Let λ, μ be two partitions of n, we define the *Kostka number* $K_{\lambda\mu}$ to be the number of semistandard tableaux of shape λ and weight μ.

THEOREM 7.15. *(Jacobi-Trudi) For any partition λ of n, one has*

$$\omega_\lambda = Det\left((h_{\lambda_i - i + j})_{1 \leq i,j \leq r(\lambda)} \right).$$

PROOF. The theorem is proved by induction on n. For $n = 1$ the statement is clear and we assume that the equality holds for all partitions μ of m with $m < n$.

We will use the automorphism H of the PSH algebra A defined by

$$H(h_i) = \sum_{j \leq i} h_j,$$

(the automorphism H is the formal sum $\sum_{k \in \mathbb{N}} h^*_k$).

First, we notice that the linear map $H - Id : I \to A$ is injective: indeed this amounts to saying that its adjoint restricts to the surjective linear map

$$\sum_{0 \leq j \leq n} A_j \longrightarrow A_n, \quad (a_0, \ldots, a_{n-1}) \mapsto a_0 h_n + \ldots + a_{n-1} h_1,$$

and this assertion is clear since A the polynomial algebra $\mathbb{Z}[(h_i)_{i \in \mathbb{N}}]$.

Let us explain the induction step: we denote by ϖ_λ the determinant of the theorem. We know (Pieri's rule, Theorem 7.6) that $H(\omega_\lambda) = \sum_{i\geq 0, \mu\in \mathbf{C}_i^\lambda} \omega_\mu$, and we will show that $H(\varpi_\lambda)$, when λ varies, satisfies the same equality (with obvious changes of notations). This will conclude the proof since $H - Id : I \to A$ is injective.

Since H is an algebra homomorphism, we have

$$H(\varpi_\lambda)(= H\left(Det\left(\left(h_{\lambda_i-i+j}\right)_{1\leq i,j\leq r(\lambda)}\right)\right)) = Det\left(\left(H(h_{\lambda_i-i+j})\right)_{1\leq i,j\leq r(\lambda)}\right).$$

We know that $H(h_{\lambda_i-i+j}) = \sum_{k\leq\lambda_i} h_{k-i+j}$ by definition of H. Hence

$$H(\varpi_\lambda) = Det\left(\left(\sum_{k_1\leq\lambda_1,\ldots,k_r\leq\lambda_r} h_{k_i-i+j}\right)_{1\leq i,j\leq r}\right).$$

We notice that, in this determinant, every entry is a partial sum of the entry which is just above it, we are led to substract the $i+1$th row from the ith row for all i. This doesn't affect the value of the determinant, therefore we obtain the equality

$$H(\varpi_\lambda) = Det\left(\left(\sum_{\lambda_2\leq k_1\leq\lambda_1,\ldots,\lambda_r\leq k_{r-1}\leq\lambda_{r-1},k_r\leq\lambda_r} h_{k_i-i+j}\right)_{1\leq i,j\leq r}\right).$$

Since the determinant is a multilinear function of its rows, we deduce

$$H(\varpi_\lambda) = \sum_{\lambda_2\leq k_1\leq\lambda_1,\ldots,\lambda_r\leq k_{r-1}\leq\lambda_{r-1},k_r\leq\lambda_r} Det\left(\left(h_{k_i-i+j}\right)_{1\leq i,j\leq r}\right).$$

Now each family of indices $k_1, \ldots, k_r$ gives rise to a partition μ belonging to $\mathbf{C}_m^\lambda$ for $m = n - k_1 - \ldots - k_r$, from which we deduce the result. $\square$

8. Harvest

In the last four sections, we defined and classified PSH algebras and we obtained precise results in the rank one case. Now it is time to see why this was useful. In this section, we will meet two avatars of the rank one PSH algebra, namely $\mathcal{A}$ of section 4, and the Grothendieck group of polynomial representations of the group GL_∞: this interpretation will give us precious information concerning the representation theory in both cases. The final section of this chapter will be devoted to another very important application of PSH algebras, in infinite rank case, associated to linear groups over finite fields. We will only state the main results without proof and refer the reader to Zelevinsky's seminal book.

8.1. Representations of symmetric groups revisited. We use the notations of section 4. We know by Proposition 4.7 that $\mathcal{A}$ is a PSH algebra.

PROPOSITION 8.1. *The PSH algebra $\mathcal{A}$ is of rank one with basic primitive element π, the class (in the Grothendieck group) of the trivial representation of the trivial group S_1.*

PROOF. Our goal is to show that every irreducible representation of S_n ($n \in \mathbb{N}\backslash\{0\}$) appears in π^n. It is clear that π^n is the regular representation of S_n, hence the result. $\square$

The choice of π gives us gives us two isomorphisms between $\mathcal{A}$ and A (one is obtained from the other by application of the automorphism ι). We choose the isomorphism which send h_2 to the trivial representation of S_2 (hence it sends e_2 to the sign representation of S_2).

Let us give an interpretation of the different bases (e_λ), (h_λ), (ω_λ), (p_λ) in this setting.

EXERCISE 8.2.
(1) Check that e_i corresponds to the sign representation of S_i and that h_i corresponds to the trivial representation of S_i.
(2) Show that ω_λ corresponds to the class of the irreducible representation V_λ defined in section 1.

REMARK 8.3. In the case of symmetric groups, the Grothendieck group is also the direct sum of $\mathbb{Z}$-valued central functions on S_n when n varies. See Chapter 1, associated with the fact that the characters of the symmetric groups take their values in $\mathbb{Z}$.

EXERCISE 8.4.
(1) Show that the primitive element $\frac{p_i}{i}$ is the characteristic function of the circular permutation of S_i.
(2) Interpreting the induction functors involved, show that, for every partition λ, p_λ is the characteristic function of the conjugacy class c_λ corresponding to λ times $\frac{|\lambda|!}{|c_\lambda|}$.

The following Proposition is now clear:

PROPOSITION 8.5. *The character table of S_n is just the transfer matrix expressing the p_λ's in terms of ω_λ's when λ varies along the partitions of n.*

Our goal now is to prove the Hook formula:

Let λ be a partition, let $a = (i, j)$ be any every box in the Young diagram λ, we denote $h(a)$ the number of boxes (i', j') of the Young diagram such that $i' = i$ and $j' \geq j$ or $i' \geq i$ and $j' = j$: $h(a)$ is called the *hook length* of a.

THEOREM 8.6. *(Hook formula) For every partition λ of n, the dimension of the S_n-module V_λ is equal to*

$$\dim V_\lambda = \frac{n!}{\prod_{a \in \lambda} h(a)}$$

PROOF. For any S_n-module V, let us denote by $rdimV$ the reduced dimension of V that is the quotient $\frac{\dim V}{n!}$: this defines a ring homomorphism from $\mathcal{A}$ to $\mathbb{Q}$, as one can easily see computing the dimension of an induced module.

We write $\lambda = (\lambda_1, \dots, \lambda_r)$. Set $L_i = \lambda_i + r - i$ and consider the new partition consisting of $(L_1, \dots, L_r) := L$. We apply Theorem 7.15 and notice that $rdim(h_p) = \frac{1}{p!}$: therefore one has

$$rdim(\omega_\lambda) = Det\left(\left(\frac{1}{(L_i - r + j)!}\right)_{1\le i,j\le r}\right).$$

Since $L_i! = (L_i - r + j)! P_{r-j}(L_i)$ where $P_k(X)$ is the polynomial $X(X-1)\dots(X-k+1)$, the right-hand side becomes

$$\frac{1}{L_1!\dots L_r!} Det\left((P_{r-j}(L_i))_{1\le i,j\le r}\right).$$

Now P_k is a polynomial of degree k with leading coefficient 1, hence this determinant is a Vandermonde determinant and is equal to $\prod_{1\le i<j\le r}(L_i - L_j)$ and we get

$$\dim V_\lambda = \frac{n!}{L_1!\dots L_r!} \prod_{1\le i<j\le r} (L_i - L_j).$$

Noting that $\frac{L_i!}{\prod_{i<j}(L_i - L_j)}$ is product of the hook lengths of boxes of the i-th row of λ, we obtained the wanted Hook formula. □

THEOREM 8.7. *For every partition λ of n, the restriction of V_λ to S_{n-1} is the direct sum $\oplus_\mu V_\mu$ where the Young diagram of μ is obtained from the Young diagram of λ by deleting exactly one box.*

PROOF. This restriction is $h_1^*(\omega_\lambda)$. Hence the result. □

EXERCISE 8.8. Compute the dimension of the S_6-module V_λ for $\lambda = (3, 2, 1)$. Calculate the restriction of V_λ to S_5.

8.2. Symmetric polynomials in infinitely many variables over $\mathbb{Z}$. Let R be a unital commutative ring, let us define the ring S_R of symmetric polynomials in a fixed infinite sequence $(X_i)_{i\in\mathbb{N}_{>0}}$ of variables with coefficients in R. Recall that the symmetric group S_n acts on the polynomial ring $R[X_1, \dots, X_n]$ by $\sigma(X_i) := X_{\sigma(i)}$, the ring of invariants consits of the symmetric polynomials in n variables. There is a surjective algebra homomorphism which preserves the degree

$$\psi_n : R[X_1, \dots, X_{n+1}]^{S_{n+1}} \to R[X_1, \dots, X_n]^{S_n}$$

$$P(X_1, \dots, X_{n+1}) \mapsto P(X_1, \dots, X_n, 0).$$

By definition, S_R is the projective limit of the maps $(\psi_n)_{n\in\mathbb{N}_{>0}}$ in the category of graded rings.

In order to be more explicit, we need to introduce the ring of formal power series $R[[X_1, \dots, X_n, \dots]]$ consisting of (possibly infinite) formal linear combinations, with coefficients in R, $\sum_\alpha a_\alpha X^\alpha$, where α runs along multi-indices $(\alpha_i)_{i\geq 1}$ of integers with finite support. There is no difficulty in defining the product since, for any multi-index α, there are only finitely many ways of expressing α into a sum $\alpha_1 + \alpha_2$. We set S_∞ to be the groups of permutations of all positive integers generated by the transpositions. Then S_R is the subring of $R[[X_1, \dots, X_n, \dots]]$ whose elements are invariant under S_∞ and such that the degrees of the monomials are bounded.

Let A be the PSH algebra of rank one.

THEOREM 8.9. *The map $\psi : A \to S_{\mathbb{Z}}$, given by, for all $a \in A$,*

$$\psi(a) = \sum_\alpha \langle a, \prod_i h_{\alpha_i} \rangle X^\alpha, \tag{6.9}$$

is an algebra isomorphism.

REMARK 8.10. We deduce immediately from the formula for ψ the following statements:

(1) $\psi(h_n) = \sum_{|\alpha|=n} X^\alpha$, where $|\alpha| := \sum_i \alpha_i$ if $\alpha = (\alpha_1, \dots, \alpha_i \dots)$,
(2) $\psi(e_n) = \sum_{\alpha=(\alpha_1,\dots)} X^\alpha$, where every α_i is either 0 or 1 and $|\alpha| = n$,
(3) $\psi(p_n) = \sum_{i\geq 1} X_i^n$.

Finally, if we denote by $h_\lambda^\diamondsuit$ the dual basis of h_λ with respect to the scalar product on A, one has

(4) $\psi(h_\lambda^\diamondsuit) = \sum_\alpha X^\alpha$ where α runs along the multi-indices whose non-increasing rearrangement is λ.

PROOF. We follow the proof given in Zelevinsky's book, attributed to Bernstein. Let us first define the homomorphism ψ: we iterate the comultiplication $A \to A \otimes A$ and obtain an algebra homomorphism $\mu_n : A \to A^{\otimes n}$ for any n (one has $\mu_2 = m^*$).

Furthermore, the counit ε induces a map $\varepsilon_n : A^{\otimes n+1} \to A^{\otimes n}$ such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[rr, "\mu_{n+1}"] \arrow[dr, "\mu_n"'] & & A^{\otimes n+1} \arrow[dl, "\varepsilon_n"] \\ & A^{\otimes n} & \end{tikzcd} \tag{6.10}$$

If B is a $\mathbb{N}$-graded commutative ring and t is an indeterminate, we can define a canonical homomorphistm $\beta_B : B \to B[t]$ by setting, for any $b \in B$ of degree k, $\beta_B(b) := bt^k$: thus we obtain a homomorphism $\beta_A^{\otimes n} : A^{\otimes n} \to A[X_1, \dots, X_n]$. Note that, in order to obtain a homogeneous homomorphism, we have to forget the grading of A for the definition of the degree in $A[X_1, \dots, X_n]$: in this algebra, the elements of A have degree 0. Note also that the image of μ_n is always contained in $(A^{\otimes n})^{S_n}$.

EXERCISE 8.11. Show that the following diagram is commutative (the symmetric group acts both on the set of variables and on the factors of $A^{\otimes k}$):

$$\begin{tikzcd}
(A^{\otimes n+1})^{S_{n+1}} \arrow[r, "\beta_A^{\otimes n+1}"] \arrow[d, "\varepsilon_n"] & (A^{\otimes n+1}[X_1, \dots, X_{n+1}])^{S_{n+1}} \arrow[d] \\
(A^{\otimes n})^{S_n} \arrow[r, "\beta_A^{\otimes n}"'] & (A^{\otimes n}[X_1, \dots, X_n])^{S_n}
\end{tikzcd} \tag{6.11}$$

Let $a \in A$, by definition of μ_n one has

$$\mu_n(a) = \sum_{\lambda_1,\dots,\lambda_n} \langle a, \omega_{\lambda_1} \dots \omega_{\lambda_n} \rangle \omega_{\lambda_1} \otimes \dots \otimes \omega_{\lambda_n}$$

where $\lambda_1, \dots, \lambda_n$ are partitions. Thus,

$$\beta_A^{\otimes n}(\mu_n(a)) = \sum_{\lambda_1,\dots,\lambda_n} \langle a, \omega_{\lambda_1} \dots \omega_{\lambda_n} \rangle \omega_{\lambda_1} \otimes \dots \otimes \omega_{\lambda_n} X_1^{|\lambda_1|} \dots X_n^{|\lambda_n|}. \tag{6.12}$$

LEMMA 8.12. *There are exactly two positive algebra homomorphisms from A to $\mathbb{Z}$, conjugate up to ι (see Theorem 6.11) which transform the basic primitive element π into 1. One of them, denoted by δ, is such that $\delta(h_i) = 1$ for all i and $\delta(\omega_\lambda) = 0$ whenever ω_λ is not one of the h_is.*

PROOF. Such a homomorphism maps π^2 onto 1, but $\pi^2 = e_2 + h_2$ and since it is positive, either e_2 or h_2 is sent to one 1 (and the other to 0). Since ι exchanges e_2 and h_2, we can assume that h_2 is sent to 1 (and e_2 to 0). We denote this homomorphism by δ. Let ω be a basic element of degree n in A, distinct from h_n. By Lemma 6.3, $e_2^*(\omega) \neq 0$, hence $\omega \preceq e_2\pi^{n-2}$ and since $\delta(e_2) = 0$ and δ is positive, $\delta(\omega) = 0$. Then, since $\delta(\pi^n) = 1$, we obtain $\delta(h_n) = 1$, hence the Lemma. □

Set $\psi_n = \delta^{\otimes n} \circ \beta_A^{\otimes n} \circ \mu_n : A \to \mathbb{Z}[X_1, \dots, X_n]$. Applying Lemma 8.12 and (6.12), we obtain

$$\psi_n(a) = \sum_{(i_1,\dots,i_n)\in\mathbb{N}^n} \langle a, h_{i_1} \dots h_{i_n} \rangle X_1^{i_1} \dots X_n^{i_n}. \tag{6.13}$$

Taking the projective limit, we get the morphism $\psi : A \to S_{\mathbb{Z}}$ we are looking for and the item (4) of Remark 8.10 ensures that ψ is an isomorphism. □

We now compute $\psi(\omega_\lambda)$ for any partition λ, and more precisely $\psi_n(\lambda)$ for any $n \geq |\lambda|$.

COROLLARY 8.13. *For any partition $\mu = (\mu_1, \dots, \mu_k)$ set*

$$X^\mu = \sum_{i_1 \neq i_2 \neq \dots i_k} X_{i_1}^{\mu_1} \dots X_{i_k}^{\mu_k}.$$

Then

$$\psi(\omega_\lambda) = \sum_{|\mu|=|\lambda|} K_{\lambda\mu} X^\mu.$$

We first introduce the following notation for the generalized Vandermonde determinant: let $\mu = (\mu_1, \dots, \mu_n)$ be a decreasing sequence of non-negative integers, we set $V_\mu(X_1, \dots, X_n) = \det((X_i^{\mu_j})_{1\le i,j\le n})$. Notice that $V_{(n-1,n-2,\dots,1,0)}(X_1, \dots, X_n)$ is the usual Vandermonde determinant.

PROPOSITION 8.14. *One has*

$$\psi_n(\omega_\lambda) = \frac{V_{(\lambda_1+n-1,\lambda_2+n-2,\dots\lambda_n)}(X_1, \dots, X_n)}{V_{(n-1,n-2,\dots,1,0)}(X_1, \dots, X_n)}. \tag{6.14}$$

EXERCISE 8.15. Prove Proposition 8.14. Hint: Let S_λ denote the right hand side of (6.14). Prove that

$$\psi_n(e_k)S_\lambda = \sum_{\mu\in T_k(\lambda)} S_\mu,$$

where $T_k(\lambda)$ is he set of all partitions obtained from λ be adding k boxes, at most one box in each row, satisfying the additional restriction that the number of rows of μ is not bigger than n. Check that it is consistent with dual Pieri formula. Then show that for any μ one can find $k > 0$ and λ such that $T_k(\lambda)$ contains only μ and partition less that μ in lexicographic order. Then prove the statement by induction on lexicographic order.

8.3. Complex general linear group for an infinite countable dimensional vector space. Let V be an infinite countable dimensional complex vector space, we consider the group $G = GL(V)$. Denote by $\mathcal{T}$ the full subcategory of the category of G-modules whose objects are submodules of direct sums of tensor powers of V. We saw in section 2 that $\mathcal{T}$ is a semisimple category. The simple modules are indexed by partitions and we denote by $S_\lambda(V)$ the simple module associated to the partition λ. We denote by $K(\mathcal{T})$ the Grothendieck group of $\mathcal{T}$.

Our aim is to equip $K(\mathcal{T})$ with a structure of PSH algebra of rank one.

We define the *multiplication*: $m([M], [N]) = [M \otimes N]$ for M and N in $\mathcal{T}$ (recall that if $M \in \mathcal{T}$, we denote by $[M]$ its class in the Grothendieck group).

We define the *scalar product*: $\langle [M], [N]\rangle = \dim \mathrm{Hom}_G(M, N)$, and the *grading*: by convention, the degree of $V^{\otimes n}$ is n.

Finally we proceed to define the *comultiplication* m^*, and it is a trifle more tricky. Since V is infinite dimensional, we can choose an isomorphism $\varphi : V \to V \oplus V$. By composition with φ, we obtain a group morphism $\Phi : G \times G$ to G,

$$\Phi \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \varphi^{-1} \circ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \circ \varphi.$$

We have two canonical projectors of $V \oplus V$ and we denote by V_1 (resp.V_2) the image of the first (resp. second) one.

EXERCISE 8.16. Show that $V_1^{\otimes p} \otimes V_2^{\otimes q}$ is a semisimple $G \times G$-module and that its irreducible components are of the form $S_\lambda(V_1) \otimes S_\mu(V_2)$, where λ is a partition of p and μ is a partition of q.

Therefore, if we denote $\tilde{\mathcal{T}}$ the full sucategory of the category of $G \times G$-modules whose objects are submodules of direct sums of $V_1^{\otimes p} \otimes V_2^{\otimes q}$, then its Grothendieck group is isomorphic to $K(\mathcal{T}) \otimes K(\mathcal{T})$.

Hence, the restriction functor *Res* (with respect to the inclusion of $G \times G$ in G) maps the category $\mathcal{T}$ to the category $\tilde{\mathcal{T}}$. Therefore it induces a linear map $m^* : K(\mathcal{T}) \to K(\mathcal{T}) \otimes K(\mathcal{T})$.

THEOREM 8.17. *The Grothendieck group $K(\mathcal{T})$, equipped the operations described above and the basis given by the classes of simple modules, is a PSH algebra of rank one, and the basic primitive element is the class of V, $[V]$.*

PROOF. The only axiom of the definition of PSH algebras which is not straightforward and needs to be checked is the self-adjointness, namely the fact that m and m^* are mutually adjoint with respect to the scalar product. For this, we have to find a functorial bijective map $\mathrm{Hom}_G(M \otimes N, P) \to \mathrm{Hom}_{G\times G}(M \otimes N, Res(P))$ (where M, N, P are objects of $\mathcal{T}$). Since any G-module is the direct sum of its homogeneous components, we may asume that M, N, P are homogeneous of degree p, q, n respectively, with $n = p + q$.

For any object $W \in \mathcal{T}$ homogeneous of degree r, set $\Pi_W := \mathrm{Hom}_G(V^{\otimes r}, W)$ which is an S_r-module; Schur-Weyl duality (see Proposition 2.12) can be reformulated in saying that there is a canonical isomorphism of G-modules $W \simeq \Pi_W \otimes_{\mathbb{C}(S_r)} V^{\otimes r}$. We set $M_1 := \Pi_M \otimes_{\mathbb{C}(S_p)} V_1^{\otimes p} \hookrightarrow M$, $N_2 := \Pi_N \otimes_{\mathbb{C}(S_q)} V_2^{\otimes q} \hookrightarrow N$.

Then we have an inclusion $M_1 \otimes N_2 \subset M \otimes N$, and the restriction defines a map $\mathrm{Hom}_G(M \otimes N, P) \to \mathrm{Hom}_{G\times G}(M_1 \otimes N_2, Res(P))$. This is the functorial map we where looking for.

In order to show this map is bijective, it is enough (by the semisimplicity of the categories $\mathcal{T}$ and $\tilde{\mathcal{T}}$) to check it for $M = V^{\otimes p}$, $N = V^{\otimes q}$ and $P = V^{\otimes n}$ with $p+q = n$ indeed, on one hand, one has:

$$\dim \mathrm{Hom}_G(V^{\otimes p} \otimes V^{\otimes q}, V^{\otimes n}) = \dim \mathrm{Hom}_{G\times G}(V_1^{\otimes p} \otimes V_2^{\otimes q}, V^{\otimes n}) = n!$$

the first equality coming from the Schur-Weyl duality and the second equality comes from the formula

$$V^{\otimes n} \simeq \oplus_{r=0}^{n} \left(V_1^{\otimes r} \otimes V_2^{\otimes (n-r)}\right)^{\oplus \frac{n!}{r!(n-r)!}}.$$

On the other hand, the map is injective because $V_1^{\otimes p} \otimes V_2^{\otimes q}$ spans the G-module $V^{\otimes n}$. □

EXERCISE 8.18. Let V have dimension n. Show that

$$\dim S_\lambda(V) = \psi_n(\omega_\lambda)(1, \ldots, 1) = \frac{\prod_{i<j}(\lambda_i - \lambda_j + j - i)}{\prod_{i<j}(j - i)}.$$

Hint: $\psi_n(\omega_\lambda)$ is the character of $GL(n)$.

CHAPTER 7

Introduction to representation theory of quivers

1. Representations of quivers

A *quiver* is an oriented graph. For example

$$\begin{tikzcd} \bullet^1 \arrow[r, bend left, "\alpha"] & \bullet^2 \arrow[l, bend left, "\beta"] \arrow[loop above, "\gamma"] & \bullet^3 \arrow[l, bend right, "\varepsilon"'] \arrow[l, bend left, "\delta"] \end{tikzcd}$$

is a quiver.

In this chapter we consider only finite quivers, namely quivers with finitely many vertices and arrows.

The *underlying graph* of a quiver Q is the graph obtained from Q by forgetting the orientation of the arrows.

If Q is a quiver, we denote by Q_0 the set of vertices of Q and by Q_1 the set of arrows of Q. In the example above, $Q_0 = \{1, 2, 3\}$ and $Q_1 = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$.

A quiver Q' is a *subquiver* of a quiver Q if $Q'_0 \subset Q_0$ and $Q'_1 \subset Q_1$.

For every arrow $\gamma \in Q_1 \colon i \xrightarrow{\gamma} j$ we define $s(\gamma) = i$ as the *source* or *tail* of γ and $t(\gamma) = j$ as the *target* or *head* of γ. In the example the vertex 1 is the source of α and the target of β.

An *oriented cycle* is a subgraph with vertices $C_0 := \{s_1, \dots, s_r\} \subset Q_0$ and arrows $C_1 = \{\gamma_1, \dots, \gamma_r\} \subset Q_1$ such that γ_i goes from s_i to s_{i+1} if $i < r$ and γ_r goes from s_r to s_1. In our example $\begin{tikzcd} \bullet^1 \arrow[r, bend left, "\alpha"] & \bullet^2 \arrow[l, bend left, "\beta"] \end{tikzcd}$ is a oriented cycle. A *loop* is an arrow with the same head and tail. In our example, there is only one loop $\begin{tikzcd} \bullet^2 \arrow[loop above, "\gamma"] \end{tikzcd}$.

Definition 1.1. Fix a field k. Let Q be a quiver. Consider a k-vector space

$$V = \bigoplus_{i \in Q_0} V_i$$

and a collection of k-linear maps

$$\rho = \{\rho_\gamma : V_i \to V_j \mid \gamma \in Q_1,\ s(\gamma) = i,\ t(\gamma) = j\}.$$

Then (V, ρ) is called a *representation* of Q. The *dimension* of the representation (V, ρ) is the vector $d \in \mathbb{Z}^{Q_0}$ such that $d_i = \dim V_i$.

We sometimes use a diagram to visualize a representation of a quiver. For example, if Q is of shape

$$\bullet^1 \xrightarrow{\alpha} \bullet^2 \xleftarrow{\beta} \bullet^3,$$

and if $V_1 = k^2, V_2 = k, V_3 = k$ and $\rho_\alpha = 0$, $\rho_\beta = \mathrm{id}$, we present is as the following diagram:

$$k^2 \xrightarrow{0} k \xleftarrow{\mathrm{id}} k.$$

Definition 1.2. Let (V, ρ) and (W, σ) be two representations of Q. A *morphism* of representations $\phi\colon (V, \rho) \to (W, \sigma)$ is a set of linear maps $\{\phi_i\colon V_i \to W_i \,|\, i \in Q_0\}$ such that the diagram

$$\begin{array}{ccc} V_j & \xleftarrow{\rho_\gamma} & V_i \\ \downarrow \phi_j & & \downarrow \phi_i \\ W_j & \xleftarrow{\sigma_\gamma} & W_i \end{array}$$

is commutative for every $\gamma \in Q_1$, where $i = s(\gamma), j = t(\gamma)$.

We say that two representations (V, ρ) and (W, σ) of Q are *isomorphic* if there exists a morphism $\phi\colon (V, \rho) \to (W, \sigma)$ such that ϕ_i is an isomorphism for every $i \in Q_0$.

The *direct sum* $(V \oplus W, \rho \oplus \sigma)$ of two representations (V, ρ) and (W, σ) of a quiver Q is defined in the obvious way.

A representaion (W, σ) is a *subrepresentation* of (V, ρ) if for every $i \in Q_0$ there is an inclusion $W_i \subset V_i$ such that for every $\gamma \in Q_1$ with $s(\gamma) = i$, the restriction of ρ_γ to W_i coincides with σ_γ.

A representation (V, ρ) is *irreducible* if it does not have non-trivial proper subrepresentations and is *indecomposable* if it can not be written as a direct sum of two non-trivial subrepresentations.

Example 1.3. Consider the quiver $1 \xrightarrow{\gamma} 2$ and the representation (V, ρ) which corresponds to the diagram $k \xrightarrow{\mathrm{id}} k$. Then (V, ρ) has only one non-trivial proper subrepresentation, namely the one given by the diagram $0 \xrightarrow{0} k$. Therefore (V, ρ) is indecomposable but not irreducible.

In many cases it is not difficult to classify irreducible representations of a given quiver. On the other hand, classifiying all indecomposable representations up to isomorphism is very hard. Many classical problems of linear algebra can be viewed as particular cases of this general problem. Let us see few examples.

Example 1.4. Let Q be the quiver of Example 1.3. A representation of Q can be seen as a pair of vector spaces V_1 and V_2 together with a linear map $\rho_\gamma : V_1 \to V_2$. Let us fix the dimension (d_1, d_2) and identify V_1 with k^{d_1}, V_2 with k^{d_2}. Classifying the representations of Q of dimension (d_1, d_2) is equivalent to the following problem of linear algebra. Consider the space of matrices of size $d_2 \times d_1$. Then the linear groups

$GL(d_1)$ and $GL(d_2)$ act on this space by multiplication on the left and on the right respectively. We would like to describe all the orbits for this action.

Consider a representation (V, ρ) of Q. Choose subspaces $W_1 \subset V_1$ and $W_2 \subset V_2$ such that $V_1 = \operatorname{Ker} \rho_\gamma \oplus W_1$ and $V_2 = \rho_\gamma(W_1) \oplus W_2$. Note that ρ_γ induces an isomorphism $\alpha : W_1 \to \rho_\gamma(W_1)$. Then (V, ρ) is the direct sum of the subrepresentations $\operatorname{Ker} \rho_\gamma \xrightarrow{0} 0$, $0 \xrightarrow{0} W_2$ and $W_1 \xrightarrow{\alpha} \rho_\gamma(W_1)$. It is clear that the first representation can be written as a direct sum of several copies of $k \xrightarrow{0} 0$, the second one is a direct sum of several copies of $0 \xrightarrow{0} k$. These decompositions are not unique, they depend on the choice of basis in $\operatorname{Ker} \rho_\gamma$ and W_2. Finally the representation $W_1 \xrightarrow{\alpha} \rho_\gamma(W_1)$ can be written as a direct sum of several copies of $k \xrightarrow{\mathrm{id}} k$.

Therefore there are three (up to isomorphism) indecomposable representations of Q. Their dimensions are $(1, 0)$, $(0, 1)$ and $(1, 1)$. Furthermore, in every dimension there are finitely many non-isomorphic representations. Quivers with the latter property are called *quivers of finite type*.

Example 1.5. Consider the quiver Q with one vertex and one loop. Then a finite-dimensional representation of Q is a pair (V, T), where V is a finite-dimensional vector space and T is a linear operator in V. Isomorphism classes of representations of this quiver are the same as conjugacy classes of $n \times n$ matrices when n is the dimension of V. If k is algebraically closed, this classification problem amounts to describing Jordan canonical forms of $n \times n$ matrices. In particular, indecomposable representations correspond to matrices with one Jordan block.

If k is not algebraically closed, the problem of classifying conjugacy classes of matrices is more tricky. This example shows that representation theory of quivers depends very much on the base field.

Example 1.6. Consider the Kronecker quiver $\bullet \leftleftarrows \bullet$. Classification of finite-dimensional representations of this quiver is also a classical problem of linear algebra. It amounts to the classification of pairs of linear operators $S, T : V_1 \to V_2$ up to multiplication by some $X \in GL(V_1)$ on the left and by some $Y \in GL(V_2)$ on the right. It is still possible to obtain this classification by brute force. We will solve this problem using general theory of quivers in the next chapter.

Example 1.7. Now let Q be the quiver with one vertex and two loops. Representation theory of Q is equivalent to classifying pairs of linear operators (T, S) in a vector space V up to conjugation. In contrast with all previous examples in this case the number of variables parametrizing indecomposable representations of dimension n grows as n^2. We call a pair (T, S) generic if T is diagonal in some basis $e_1, \ldots, e_n$ with distinct eigenvalues and the matrix of S in this basis does not have any zero entry.

EXERCISE 1.8. Check that if (T, S) is generic and $W \subset V$ is both T-stable and S-stable, then $W = 0$ or $W = V$. Thus, the corresponding representation of Q is irreducible.

Therefore every generic pair of operators (T, S) gives rise to an irreducible representation of Q. The eigenvalues of T give n distinct parameters. If T is diagonalized, we can conjugate S by linear operators diagonal in the eigenbasis of T. Thus, we have $n^2 - n$ parameters for the choice of S.

The situation which appears in this example is refered to as *wild*. There is a precise definition of wild quivers and we refer reader to ?? for further reading on this subject.

2. Path algebra

As in the case of groups, we can reduce the representation theory of a quiver to the representation theory of some associative ring. In the case of groups, this ring is the group algebra, while in the case of quivers it is the path algebra.

DEFINITION 2.1. Let Q be a quiver. A *path* p is a sequence $\gamma_1, \dots, \gamma_k$ of arrows such that $s(\gamma_i) = t(\gamma_{i+1})$. Set $s(p) = s(\gamma_k)$, $t(p) = t(\gamma_1)$. The number k of arrows is called the length of p.

DEFINITION 2.2. Let $p_1 = \gamma_1, \dots, \gamma_k$ and $p_2 = \delta_1, \dots, \delta_l$ be two paths of Q. We define the product of p_1 and p_2 to be the path $\delta_1, \dots, \delta_l, \gamma_1, \dots \gamma_k$ if $t(\gamma_1) = s(\delta_l)$ and zero otherwise.

Next we introduce elements e_i for each vertex $i \in Q_0$ and define the product of e_i and e_j by the formula

$$e_i e_j = \delta_{ij} e_i.$$

For a path p, we set

$$e_i p = \begin{cases} p, \text{ if } i = t(p) \\ 0 \text{ otherwise} \end{cases},$$

$$p e_i = \begin{cases} p, \text{ if } i = s(p) \\ 0 \text{ otherwise} \end{cases}.$$

The *path algebra* $k(Q)$ of Q is the vector space of k-linear combinations of all paths of Q and elements $\{e_i\}_{i \in Q_0}$, with the multiplication law obtained by extending of the product defined above by bilinearity.

Note that every $e_i,\ i \in Q_0$, is an idempotents in $k(Q)$ and that $\sum_{i \in Q_0} e_i = 1$.

EXAMPLE 2.3. Let Q be the quiver with one vertex and n loops then $k(Q)$ is the free associative algebra with n generators.

EXERCISE 2.4. Let Q be a quiver such that the underlying graph of Q does not contain any cycle or loop. Let $Q_0 = \{1, \ldots, n\}$. Show that the path algebra $k(Q)$ is isomorphic to the subalgebra of the matrix algebra $\mathrm{Mat}_n(k)$ generated by the subset of elementary matrices $\{E_{ii} \mid i \in Q_0\}$, $\{E_{ji} \mid \gamma \in Q_1,\ s(\gamma) = i, t(\gamma) = j\}$.

In particular, show that the path algebra of the quiver

$$\bullet \leftarrow \bullet \leftarrow \cdots \leftarrow \bullet$$

is isomorphic to the algebra B_n of upper triangular matrices, see Example 7.19 Chapter V.

LEMMA 2.5. *Let Q be a quiver.*

(1) *The path algebra $k(Q)$ is generated by the idempotents $\{e_i \mid i \in Q_0\}$ and the paths $\{\gamma \mid \gamma \in Q_1\}$ of length* 1.
(2) *The algebra $k(Q)$ is finite-dimensional if and only if Q does not contain an oriented cycle.*
(3) *If Q is the disjoint union of two quivers Q' and Q'', then $k(Q)$ is isomorphic to the direct product $k(Q') \times k(Q'')$.*
(4) *The path algebra has a natural $\mathbb{Z}$-grading*
$$k(Q) = \bigoplus_{n=0}^{\infty} k(Q)_{(n)},$$
where $k(Q)_{(0)}$ is the span of the idempotents e_i for all $i \in Q_0$ and $k(Q)_{(n)}$ is the span of all paths of length n.
(5) *For every vertex $i \in Q_0$ the element e_i is a primitive idempotent of $k(Q)$, and hence $k(Q)e_i$ is an indecomposable projective $k(Q)$-module.*

PROOF. The first four assertions are straightforward and we leave them to the reader as an exercise. Let us prove (5).

Let $i \in Q_0$. By Exercise 7.14 Chapter V, proving (5) amounts to checking that if $\varepsilon \in k(Q)e_i$ is an idempotent such that $e_i\varepsilon = \varepsilon e_i = \varepsilon$, then $\varepsilon = e_i$ or $\varepsilon = 0$. We use the grading of $k(Q)$ defined in (4). By definition, the left ideal $k(Q)e_i$ inherits this grading. Hence we can write

$$k(Q)e_i = \bigoplus_{n=0}^{\infty} k(Q)_{(n)} e_i,$$

where $k(Q)_{(0)}e_i = ke_i$ and, for $n > 0$, the graded component $k(Q)_{(n)}e_i$ is spanned by the paths of length n with sourse at i. We can write $\varepsilon = \varepsilon_0 + \cdots + \varepsilon_l$ with $\varepsilon_n \in k(Q)_{(n)}e_i$. Since ε is an idempotent, we have $\varepsilon_0^2 = \varepsilon_0$, which implies $\varepsilon_0 = e_i$ or $\varepsilon_0 = 0$. In the latter case let ε_p be the first non-zero term in the decomposition of ε. Then the first non-zero term in the decomposition of ε^2 has degree no less than $2p$. This implies $\varepsilon = 0$. If $\varepsilon_0 = e_i$, consider the idempotent $e_i - \varepsilon$ and apply the above argument again. $\square$

Given a representation (V,ρ) of a quiver Q, $V=\bigoplus_{i\in Q_0} V_i$ one can equip V with a structure of $k(Q)$-module in the following way

(1) The idempotent e_i acts on V_j by $\delta_{ij}\operatorname{Id}_{V_j}$.
(2) For $\gamma\in Q_1$ and $v\in V_i$ we set $\gamma v=\rho_\gamma(v)$ if $i=s(\gamma)$ and zero otherwise.
(3) We extend this action for the whole $k(Q)$ using Lemma 2.5 (1).

Conversly, every $k(Q)$-module V gives rise to a representation ρ of Q when one sets $V_i=e_iV$.

This implies the following Theorem. [1]

THEOREM 2.6. *The category of representations of Q over a field k is equivalent to the category of $k(Q)$-modules.*

EXERCISE 2.7. Let Q be a quiver and $J(Q)$ be the ideal of $k(Q)$ generated by all arrows $\gamma\in Q_1$. Then the quotient $k(Q)/J$ is a semisimple commutative ring isomorphic to k^{Q_0}.

EXERCISE 2.8. Let Q' be a subquiver of a quiver Q. Let $I(Q')$ be the ideal of $k(Q)$ generated by e_i for all $i\notin Q'_0$ and by all $\gamma\notin Q'_1$. Prove that $k(Q')$ is isomorphic to the quotient ring $k(Q)/I(Q')$.

LEMMA 2.9. *Let $A=\bigoplus_{i=0}^{\infty}A_{(i)}$ be a graded algebra and R be the Jacobson radical of A. Then*

(1) *R is a graded ideal, i.e. $R=\bigoplus_{i=0}^{\infty}R_{(i)}$, where $R_{(i)}=R\cap A_{(i)}$;*
(2) *If $u\in R_{(p)}$ for some $p>0$, then u is nilpotent.*

PROOF. Assume first that the ground field k is infinite. Let $t\in k^*$. Consider the automorphism φ_t of A such that $\varphi_t(u)=t^pu$ for all $u\in A_{(p)}$. Observe that $\varphi_t(R)=R$. Suppose that u belongs to R and write it as the sum of homogeneous components $u=u_0+\cdots+u_n$ with $u_j\in A_{(j)}$. We have to show that $u_i\in R$ for all $i=1,\dots,n$. Indeed,

$$\varphi_t(u)=u_0+tu_1+\cdots+t^nu_n\in R$$

for all $t\in k^*$. Since k is infinite, this implies $u_i\in R$ for all i. If k is finite, consider the algebra $A\otimes_k\bar{k}$ and use the fact that $R\otimes_k\bar{k}$ is included in the radical of $A\otimes_k\bar{k}$.

Let $u\in R_{(p)}$. Then $1-u$ is invertible. Hence there exists $a_i\in A_{(i)}$, for some $i=1,\dots,n$ such that

$$(a_0+a_1+\cdots+a_n)(1-u)=1.$$

This relation implies $a_0=1$ and $a_{pj}=u^j$ for all $j>0$. Thus $u^j=0$ for sufficiently large j. $\square$

[1]Compare with the analogous result for groups in Chapter 2.

Let us call a path p of a quiver Q a *one way path* if there is no path from $t(p)$ to $s(p)$.

EXERCISE 2.10. The span of all one way paths of Q is a two-sided nilpotent ideal in $k(Q)$.

LEMMA 2.11. *The Jacobson radical of the path algebra $k(Q)$ is the span of all one way paths of Q.*

PROOF. Let N be the span of all one way paths. By Exercise 2.10 R is contained in the radical of $k(Q)$.

Assume now that y belongs to the radical of $k(Q)$. Exercise 2.7 implies that $y \in J(Q)$ and moreover by Lemma 2.9(2) we may assume that y is a linear combination of paths of the same length. We want to prove that $y \in N$. Note that $e_i y e_j$ belongs to the radical for all $i, j \in Q_0$. Assume that the statement is false. Then there exists i and j such that $z := e_i y e_j$ is not in N, in other words there exists a path u with source j and target i. Furthermore zu is a linear combination of oriented cycles $u_1, \ldots, u_l$ of the same length. By Lemma 2.9(2) u must be nilpotent. But it is clearly not nilpotent. Contradiction. □

Lemma 2.11 implies the following

PROPOSITION 2.12. *Let Q be a quiver which does not contain oriented cycles. Then $k(Q)/\operatorname{rad} k(Q) \simeq k^n$, where n is the number of vertices. In particular, every simple $k(Q)$-module is one dimensional.*

PROOF. The assumption on Q implies that every path is a one way path. Hence the radical of $k(Q)$ is equal to $J(Q)$. □

3. Standard resolution and consequences

3.1. Construction of the standard resolution. A remarkable property of path algebras is the fact that every module has a projective resolution of length at most 2:

THEOREM 3.1. *Let Q be a quiver, A denote the path algebra $k(Q)$ and V be an A-module. Recall that $V = \bigoplus_{i \in Q_0} V_i$. Then the following sequence of A-modules*

$$0 \to \bigoplus_{\gamma \in Q_1} Ae_{t(\gamma)} \otimes V_{s(\gamma)} \xrightarrow{f} \bigoplus_{i \in Q_0} Ae_i \otimes V_i \xrightarrow{g} V \to 0,$$

where

$$f\left(ae_{t(\gamma)} \otimes v\right) = ae_{t(\gamma)}\gamma \otimes v - ae_{t(\gamma)} \otimes \gamma v$$

for all $\gamma \in Q_1, v \in V_{s(\gamma)}$, and

$$g\left(ae_i \otimes v\right) = av$$

for any $i \in Q_0, v \in V_i$, is exact. Hence it is a projective resolution of V.

REMARK 3.2. The structure of A-modules considered in the statement is defined by the action of A on the lefthand side of the tensor product.

PROOF. The fact that f and g are morphisms of A-modules is left to the reader. First, let us check that $g \circ f = 0$. Indeed,

$$g\left(f\left(ae_j \otimes v\right)\right) = g\left(ae_j\gamma \otimes v - ae_j \otimes \gamma v\right) = ae_j\gamma v - ae_j\gamma v = 0.$$

Since $V = \oplus_{i\in Q_0} V_i$ and $V_i = e_i V$, g is surjective.

Now let us check that f is injective. To simplify notations we set

$$X = \bigoplus_{\gamma\in Q_1} Ae_{t(\gamma)} \otimes V_{s(\gamma)}, \quad Y = \bigoplus_{i\in Q_0} Ae_i \otimes V_i.$$

Consider the $\mathbb{Z}$-grading

$$A \otimes V = \bigoplus_{p=0}^{\infty} A_{(p)} \otimes V.$$

Since all Ae_i for $i \in Q_0$ are homogeneous left ideals of A, there are induced gradings $X = \oplus_{p\geq 0} X_{(p)}$ and $Y = \oplus_{p\geq 0} Y_{(p)}$. Define $f_0 : X \to Y$ and $f_1 : X \to Y$ by

$$f_1\left(ae_{t(\gamma)} \otimes v\right) = ae_{t(\gamma)}\gamma \otimes v, \quad f_0\left(ae_{t(\gamma)} \otimes v\right) = ae_{t(\gamma)} \otimes \gamma v.$$

Note that for any $p \geq 0$ we have $f_1(X_{(p)}) \subset X_{(p+1)}$ and $f_0(X_{(p)}) \subset X_{(p)}$. Moreover, it is clear from the definition that f_1 is injective. Since $f = f_1 - f_0$, we obtain that f is injective by a simple argument on gradings.

It remains to prove that $\operatorname{Im} f = \operatorname{Ker} g$.

EXERCISE 3.3. Show that for any $p > 0$ and $y \in Y_{(p)}$ there exists $y' \in Y_{(p-1)}$ such that $y' \equiv y \mod \operatorname{Im} f$. (Hint: it suffices to check the statement for $x = u \otimes v$ where $v \in V_i$ and u is a path of length p with source i).

The exercise implies that for any $y \in Y$ there exists $y_0 \in Y_{(0)}$ such that $y \equiv y_0 \mod \operatorname{Im} f$. Let $y \in \operatorname{Ker} g$, then $y_0 \in \operatorname{Ker} g$. But g restricted to Y^0 is injective. Thus, $y_0 = 0$ and $y \in \operatorname{Im} f$. □

3.2. Extension groups. Let X and Y be two $k(Q)$-modules. We define a linear map

$$d : \bigoplus_{i\in Q_0} \operatorname{Hom}_k\left(X_i, Y_i\right) \to \bigoplus_{\gamma\in Q_1} \operatorname{Hom}_k\left(X_{s(\gamma)}, Y_{t(\gamma)}\right) \tag{7.1}$$

by the formula

$$d\phi\left(x\right) = \phi\left(\gamma x\right) - \gamma\phi\left(x\right) \tag{7.2}$$

for any $\gamma \in Q_1$, $x \in X_{s(\gamma)}$ and $\phi \in \operatorname{Hom}_k\left(X_{s(\gamma)}, Y_{s(\gamma)}\right)$. Theorem 3.1 implies that $\operatorname{Ext}^1\left(X, Y\right)$ is isomorphic to the cokernel of the map d.

According to Section 6.4 Chapter V, every non-zero $\psi \in \operatorname{Ext}^1(X,Y)$ induces a non-split exact sequence

$$0 \to Y \to Z \to X \to 0.$$

In our situation we can describe the $k(Q)$-module structure on Z precisely. Indeed, consider $\psi \in \bigoplus_{\gamma\in Q_1} \operatorname{Hom}_k\left(X_{s(\gamma)}, Y_{t(\gamma)}\right)$ and denote by ψ_γ the component of $\psi \in \operatorname{Hom}_k\left(X_{s(\gamma)}, Y_{t(\gamma)}\right)$. We set $Z_i = X_i \oplus Y_i$ for every $i \in Q_0$. Furthermore, for every $\gamma \in Q_1$ with source i and target j we set

$$\gamma(x,y) = (\gamma x, \gamma y + \psi_\gamma x).$$

Obviously we obtain an exact sequence of $k(Q)$-modules

$$0 \to Y \xrightarrow{i} Z \xrightarrow{\pi} X \to 0,$$

where $i(y) = (0,y)$ and $\pi(x,y) = x$. This exact sequence splits if and only if there exists $\eta \in \operatorname{Hom}_Q(X,Z)$ such that $\pi \circ \eta = \operatorname{Id}$. Note that $\eta = \oplus_{i\in Q_0}\eta_i$ with $\eta_i \in \operatorname{Hom}_k(X_i, Z_i)$ and for every $x \in X_i$ we have

$$\eta_i(x) = (x, \phi_i x),$$

for some $\phi_i \in \operatorname{Hom}_k(X_i, Y_i)$. The condition that η is a morphism of $k(Q)$-modules implies that for every arrow $\gamma \in Q_1$ with source i and target j we have

$$\gamma(x, \phi_i x) = (\gamma x, \gamma\phi_i x + \psi_\gamma x) = (\gamma x, \phi_j \gamma x).$$

Hence we have

$$\psi_\gamma x = \phi_j \gamma x - \gamma \phi_i x.$$

If we write $\phi = \oplus_{i\in Q_0}\phi_i$, then the latter condition is equivalent to $\psi = d\phi$.

Note also that Theorem 3.1 implies the following.

Proposition 3.4. *In the category of representations of Q one has*

$$\operatorname{Ext}^i(X,Y) = 0 \text{ for all } i \geq 2.$$

Corollary 3.5. *Let*

$$0 \to Y \to Z \to X \to 0$$

be a short exact sequence of representations of Q, then the maps

$$\operatorname{Ext}^1(V,Z) \to \operatorname{Ext}^1(V,X),\ \operatorname{Ext}^1(Z,V) \to \operatorname{Ext}^1(Y,V)$$

are surjective.

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Proof. Follows from Proposition 3.4 and the long exact sequence for extension groups, Theorem 5.7 Chapter V. ☐

Lemma 3.6. *If X and Y are indecomposable finite-dimensional $k(Q)$-modules and $\operatorname{Ext}^1(Y,X) = 0$, then every non-zero $\varphi \in \operatorname{Hom}_Q(X,Y)$ is either surjective or injective.*

PROOF. Consider the exact sequences

$$0 \to \operatorname{Ker}\varphi \to X \xrightarrow{\beta} \operatorname{Im}\varphi \to 0, \tag{7.3}$$

$$0 \to \operatorname{Im}\varphi \xrightarrow{\delta} Y \to S \cong Y/\operatorname{Im}\varphi \to 0. \tag{7.4}$$

Note that both sequences do not split. Let $\psi \in \operatorname{Ext}^1(S, \operatorname{Im}\varphi)$ be an element associated to the sequence (7.4). By Corollary 3.5 and (7.3) we have a surjective map

$$g : \operatorname{Ext}^1(S, X) \to \operatorname{Ext}^1(S, \operatorname{Im}\varphi).$$

Let $\psi' \in g^{-1}(\psi)$. Then ψ' induces a non-split exact sequence

$$0 \to X \xrightarrow{\alpha} Z \to S \to 0.$$

This exact sequence and the sequence (7.4) can be arranged in the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & X & \xrightarrow{\alpha} & Z & \to & S & \to & 0 \\
 & & \downarrow^{\beta} & & \downarrow^{\gamma} & & \downarrow^{\mathrm{Id}} & & \\
0 & \to & \operatorname{Im}\varphi & \xrightarrow{\delta} & Y & \to & S & \to & 0
\end{array}$$

here β and γ are surjective. We claim that the sequence

$$0 \to X \xrightarrow{\alpha+\beta} Z \oplus \operatorname{Im}\varphi \xrightarrow{\gamma-\delta} Y \to 0 \tag{7.5}$$

is exact. Indeed, $\alpha + \beta$ is obviously injective and $\gamma - \delta$ is surjective. Furthermore, $\dim Z = \dim X + \dim S$, $\dim \operatorname{Im}\varphi = \dim Y - \dim S$. Therefore,

$$\dim (Z \oplus \operatorname{Im}\varphi) = \dim X + \dim Y,$$

and therefore $\operatorname{Ker}(\gamma - \delta) = \operatorname{Im}(\alpha + \beta)$.

By the assumption $\operatorname{Ext}^1(Y, X) = 0$. Hence the exact sequence (7.5) splits, and we have an isomorphism

$$Z \oplus \operatorname{Im}\varphi \cong X \oplus Y.$$

By the Krull–Schmidt theorem either $X \cong \operatorname{Im}\varphi$ and hence φ is injective or $Y \cong \operatorname{Im}\varphi$ and hence φ is surjective. $\square$

3.3. Canonical bilinear form and Euler characteristic. Let Q be a quiver and X be a finite-dimensional $k(Q)$-module. We use the notation $x = \dim X \in \mathbb{Z}^{Q_0}$ where $x_i = \dim X_i$ for every $i \in Q_0$.

We define the bilinear form on $\mathbb{Z}^{Q_0}$ by the formula

$$\langle x, y\rangle := \sum_{i\in Q_0} x_i y_i - \sum_{\gamma\in Q_1} x_{s(\gamma)} y_{t(\gamma)} = \dim \operatorname{Hom}_Q(X, Y) - \dim \operatorname{Ext}^1(X, Y),$$

where the second equality follows from calculating Euler characteristic in (7.1). The symmetric form

$$(x, y) := \langle x, y\rangle + \langle y, x\rangle$$

is called the Tits form of the quiver Q. We also consider the corresponding quadratic form

$$q(x) := \langle x, x \rangle .$$

4. Bricks

Here we discuss further properties of finite-dimensional representations of a path algebra $k(Q)$. **In the rest of this chapter we assume that the ground field k is algebraically closed and all representations are finite-dimensional.**

DEFINITION 4.1. A $k(Q)$-module X is a *brick*, if $\mathrm{End}_Q(X) = k$.

EXERCISE 4.2. If X is a brick, then X is indecomposable. If X is indecomposable and $\mathrm{Ext}^1(X, X) = 0$, then X is a brick (by Lemma 3.6).

EXAMPLE 4.3. Consider the quiver $\bullet \to \bullet$. Then every indecomposable representation is a brick.

For the Kronecker quiver $\bullet \rightrightarrows \bullet$ the representation $k^2 \rightrightarrows k^2$ with $\rho_{\gamma_1} = \mathrm{Id}$, $\rho_{\gamma_2} = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ is not a brick because $\varphi = (\varphi_1, \varphi_2)$ with $\varphi_1 = \varphi_2 = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ is a non-scalar element in $\mathrm{End}_Q(X)$.

LEMMA 4.4. *Let X be an indecomposable $k(Q)$-module which is not a brick. Then X contains a brick W such that* $\mathrm{Ext}^1(W, W) \neq 0$.

PROOF. We will prove the lemma by induction on the length l of X. The base case $l = 1$ is trivial, since in this case X is irreducible and hence a brick by the Schur lemma.

Recall that if X is indecomposable and has finite length, then $\varphi \in \mathrm{End}_Q(X)$ is either isomorphism or nilpotent. Therefore, since k is algebraically closed and X is not a brick, the algebra $\mathrm{End}_Q(X)$ contains a non-zero nilpotent element. Let $\varphi \in \mathrm{End}_Q(X)$ be a non-zero operator of minimal rank. Then φ is nilpotent and $\mathrm{rk}\,\varphi^2 < \mathrm{rk}\,\varphi$, hence $\varphi^2 = 0$.

Let $Y := \mathrm{Im}\,\varphi$, $Z := \mathrm{Ker}\,\varphi$. Clearly, $Y \subset Z$. Consider a decomposition

$$Z = Z_1 \oplus \cdots \oplus Z_p$$

into a sum of indecomposable submodules. Denote by p_i the projection $Z \to Z_i$. Let i be such that $p_i(Y) \neq 0$. Set $\eta := p_i \circ \varphi$, $Y_i := p_i(Y) = \eta(Z)$. Note that by our assumption $\mathrm{rk}\,\eta = \mathrm{rk}\,\varphi$, therefore Y_i is isomorphic to Y. Let $Y_i = p_i(Y)$. Then $\mathrm{Ker}\,\eta = Z$ and $\mathrm{Im}\,\eta = Y_i$.

Note that the exact sequence

$$0 \to Z \to X \xrightarrow{\eta} Y_i \to 0$$

does not split since X is indecomposable. Let X_i be the quotient of X by the submodule $\oplus_{j \neq i} Z_j$ and $\pi : X \to X_i$ be the canonical projection. Then we have the exact sequence

$$0 \to Z_i \to X_i \xrightarrow{\bar{\eta}} Y_i \to 0,$$

where $\bar{\eta} := \eta \circ \pi^{-1}$ is well define since $\operatorname{Ker} \pi \subset \operatorname{Ker} \eta$. We claim that (7.6) does not split. Indeed, if it splits, then X_i decomposes into a direct sum $Z_i \oplus L$ for some submodule $L \subset X_i$ which is isomorphic to Y_i. But then $X = Z_i \oplus \pi^{-1}(L)$, which contradicts indecomposability of X.

Therefore we have shown that $\operatorname{Ext}^1(Y_i, Z_i) \neq 0$. Recall that Y_i is a submodule of Z_i. By Corollary 3.5 we have the surjection

$$\operatorname{Ext}^1(Z_i, Z_i) \to \operatorname{Ext}^1(Y_i, Z_i).$$

Hence $\operatorname{Ext}^1(Z_i, Z_i) \neq 0$.

The length of Z_i is less than the length of X. If Z_i is not a brick, then it contains a brick W by the induction assumption. □

COROLLARY 4.5. *Assume that Q is a quiver such that its Tits form is positive definite. Then every indecomposable representation X of Q is a brick with trivial* $\operatorname{Ext}^1(X, X)$. *Moreover, if $x = \dim X$, then $q(x) = 1$.*

PROOF. Assume that X is not a brick, then it contains a brick Y such that $\operatorname{Ext}^1(Y, Y) \neq 0$. Then

$$q(y) = \dim \operatorname{End}_Q(Y) - \dim \operatorname{Ext}^1(Y, Y) = 1 - \dim \operatorname{Ext}^1(Y, Y) \leq 0,$$

but this is impossible. Therefore X is a brick. Then

$$q(x) = \dim \operatorname{End}_Q(X) - \dim \operatorname{Ext}^1(X, X) = 1 - \dim \operatorname{Ext}^1(X, X) \geq 0.$$

By positivity of q we have $q(x) = 1$ and $\dim \operatorname{Ext}^1(X, X) = 0$. □

5. Orbits in representation variety

Fix a quiver Q. For arbitrary $x \in \mathbb{N}^{Q_0}$ consider the space

$$\operatorname{Rep}(x) := \prod_{\gamma \in Q_1} \operatorname{Hom}_k\left(k^{x_{s(\gamma)}}, k^{x_{t(\gamma)}}\right).$$

We can see every representation of Q of dimension x as a point $\rho \in \operatorname{Rep}(x)$ with components ρ_γ for every $\gamma \in Q_1$.

Let us consider the group

$$G = \prod_{i \in Q_0} \operatorname{GL}(k^{x_i}),$$

and define an action of G on $\operatorname{Rep}(x)$ by the formula

$$g\rho_\gamma := g_{t(\gamma)} \rho_\gamma g_{s(\gamma)}^{-1} \quad \text{for every} \quad \gamma \in Q_1.$$

Two representations ρ and ρ' of Q are isomorphic if and only if they belong to the same orbit of G. In other words we have a bijection between isomorphism classes of representations of Q of dimension x and G-orbits in $\operatorname{Rep}(x)$. For a representation X we denote by O_X the corresponding G-orbit in $\operatorname{Rep}(x)$.

Note that
$$\dim \operatorname{Rep}(x) = \sum_{\gamma \in Q_1} x_{s(\gamma)} x_{t(\gamma)}, \quad \dim G = \sum_{i \in Q_0} x_i^2,$$
therefore
$$\dim \operatorname{Rep}(x) - \dim G = -q(x). \tag{7.6}$$

Let us formulate without proof certain properties of G-action on $\operatorname{Rep}(x)$. They follow from the general theory of algebraic groups, see for instance Humphreys We work in Zariski topology.

- Each orbit is open in its closure;
- if O and O' are two distinct orbits and O' belongs to the closure of O, then $\dim O' < \dim O$;
- If (X, ρ) is a representation of Q, then $\dim O_X = \dim G - \dim \operatorname{Stab}_X$, where Stab_X denotes the stabilizer of ρ.

Lemma 5.1. *For any representation (X, ρ) of dimension x we have*
$$\dim \operatorname{Stab}_X = \dim \operatorname{Aut}_Q(X) = \dim \operatorname{End}_Q(X).$$

Proof. The condition that $\phi \in \operatorname{End}_Q(X)$ is not invertible is given by the polynomial equations
$$\prod_{i \in Q_0} \det \phi_i = 0.$$
Since $\operatorname{Aut}_Q(X)$ is not empty and open in $\operatorname{End}_Q(X)$, we obtain that $\operatorname{Aut}_Q(X)$ and $\operatorname{End}_Q(X)$ have the same dimension. $\square$

Corollary 5.2. *If (X, ρ) is a representation of Q and $\dim X = x$, then*
$$\operatorname{codim} O_X = \dim \operatorname{Rep}(x) - \dim G + \dim \operatorname{Stab}_X = -q(x) + \dim \operatorname{End}_Q(X) = \dim \operatorname{Ext}^1(X, X).$$

Lemma 5.3. *Let (Z, τ) be a nontrivial extension of (Y, σ) by (X, ρ), i.e. there is a non-split exact sequence*
$$0 \to X \to Z \to Y \to 0.$$
Then $O_{X \oplus Y}$ belongs to the closure of O_Z and $O_{X \oplus Y} \neq O_Z$.

Proof. Following Section 3.2 for every $i \in Q_0$ consider a decomposition $Z_i = X_i \oplus Y_i$ such that for every $\gamma \in Q_1$ and $(x, y) \in X_{s(\gamma)} \oplus Y_{s(\gamma)}$
$$\tau_\gamma(x, y) = (\rho_\gamma(x) + \psi_\gamma(y), \sigma_\gamma(y))$$
for some $\psi_\gamma \in \operatorname{Hom}(Y_{s(\gamma)}, X_{t(\gamma)})$.

Next, for every $\lambda \in k \setminus 0$ define $g^\lambda \in G$ by setting for every $i \in Q_0$
$$g_i^\lambda|_{X_i} = \operatorname{Id}_{X_i}, \quad g_i^\lambda|_{Y_i} = \lambda \operatorname{Id}_{Y_j}.$$
Then we have
$$g^\lambda \tau_\gamma(x, y) = (\rho_\gamma(x) + \lambda \psi_\gamma(y), \sigma_\gamma(y)).$$

The latter formula makes sence even for $\lambda = 0$ and $g^0\tau$ lies in the closure of $\{g^\lambda \tau \mid \lambda \in k \setminus 0\}$. Furthermore $g^0\tau$ is the direct sum $X \oplus Y$. Hence $O_{X \oplus Y}$ belongs to the closure of O_Z.

It remains to check that $X \oplus Y$ is not isomorphic to Z. This follows immediately from the inequality

$$\dim \operatorname{Hom}_Q (Y, Z) < \dim \operatorname{Hom}_Q (Y, X \oplus Y) .$$

$\square$

The following corollary is straightforward.

COROLLARY 5.4. *If the orbit O_X is closed in* $\operatorname{Rep}(x)$*, then X is semisimple.*

COROLLARY 5.5. *Let (X, ρ) be a representation of Q and $X = \bigoplus_{j=1}^{m} X_j$ be a decomposition into the direct sum of indecomposable submodules. If O_X is an orbit of maximal dimension in* $\operatorname{Rep}(x)$*, then* $\operatorname{Ext}^1(X_i, X_j) = 0$ *for all $i \neq j$.*

PROOF. If $\operatorname{Ext}^1(X_i, X_j) \neq 0$, then by Lemma 5.3 we can construct a representation (Z, τ) such that O_X is in the closure of O_Z. Then $\dim O_X < \dim O_Z$. $\square$

6. Coxeter–Dynkin and affine graphs

6.1. Definition and properties. Let Γ be a *connected* non-oriented graph with vertices Γ_0 and edges Γ_1. We define the Tits form $(\cdot, \cdot)$ on $\mathbb{Z}^{\Gamma_0}$ by

$$(x, y) := \sum_{i \in \Gamma_0} (2 - 2l(i)) x_i y_i - \sum_{(i,j) \in \Gamma_1} x_i y_j ,$$

where $l(i)$ is the number of loops at i. If we equip all edges of Γ with orientation then the symmetric form coincides with the introduced earlier symmetric form of the corresponding quiver. We define the quadratic form q on $\mathbb{Z}^{\Gamma_0}$ by

$$q(x) := \frac{(x, x)}{2} .$$

By $\{\epsilon_i \mid i \in \Gamma_0\}$ we denote the standard basis in $\mathbb{Z}^{\Gamma_0}$. If Γ does not have loops, then $(\epsilon_i, \epsilon_i) = 2$ for all $i \in \Gamma_0$. If $i, j \in \Gamma_0$ and $i \neq j$, then (ϵ_i, ϵ_j) equals minus the number of edges between i and j. The matrix of the form $(\cdot, \cdot)$ in the standard basis is called the *Cartan matrix* of Γ.

EXAMPLE 6.1. The Cartan matrix of $\bullet - \bullet$ is $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. The Cartan matrix of the loop is (0).

DEFINITION 6.2. A connected graph Γ is called *Coxeter–Dynkin* if its Tits form $(\cdot, \cdot)$ is positive definite and *affine* if $(\cdot, \cdot)$ is positive semidefinite but not positive definite. If Γ is neither Coxeter–Dynkin nor affine, then we say that it is of *indefinite type*.

REMARK 6.3. For affine graph Γ the form $(\cdot,\cdot)$ is necessarily degenerate. Furthermore

$$\operatorname{Ker}(\cdot,\cdot) = \{x \in \mathbb{Z}^{Q_0} \mid (x,x) = 0\}. \tag{7.7}$$

LEMMA 6.4. *(a) If Γ is affine then the kernel of $(\cdot,\cdot)$ equals $\mathbb{Z}\delta$ for some $\delta \in \mathbb{N}^{\Gamma_0}$ with all $\delta_i > 0$.*

(b) If Γ is of indefinite type, then there exists $x \in \mathbb{N}^{\Gamma_0}$ such that $(x,x) < 0$.

PROOF. Let $x \in \mathbb{Z}^{Q_0}$. We define $\operatorname{supp} x$ to be the set of vertices $i \in Q_0$ such that $x_i \neq 0$. Let $|x|$ be defined by the condition $|x|_i = |x_i|$ for all $i \in Q_0$. Note that $\operatorname{supp} x = \operatorname{supp} |x|$ and by the definition of $(\cdot,\cdot)$ we have

$$(|x|,|x|) \leq (x,x). \tag{7.8}$$

To prove (b) we just notice that if Γ is of indefinite type then there exists $x \in \mathbb{Z}^{Q_0}$ such that $(x,x) < 0$. But then (7.8) implies $(|x|,|x|) < 0$.

Now let us prove (a). Let $\delta \in \operatorname{Ker}(\cdot,\cdot)$ and $\delta \neq 0$. Then (7.8) and (7.7) imply that $|\delta|$ also lies in $\operatorname{Ker}(\cdot,\cdot)$. Next we prove that $\operatorname{supp}\delta = Q_0$. Indeed, otherwise we can choose $i \in Q_0 \setminus \operatorname{supp}\delta$ such that i is connected with at least one vertex in $\operatorname{supp}\delta$. Then $(\epsilon_i,\delta) < 0$, therefore

$$(\epsilon_i + 2\delta, \epsilon_i + 2\delta) = 2 + 4(\epsilon_i,\delta) < 0$$

and Γ is not affine.

Finally let $\delta',\delta \in \operatorname{Ker}(\cdot,\cdot)$. Since $\operatorname{supp}\delta = \operatorname{supp}\delta' = Q_0$, one can find $a, b \in \mathbb{Z}$ such that $\operatorname{supp}(a\delta + b\delta') \neq Q_0$. Then by above $a\delta + b\delta' = 0$. Hence $\operatorname{Ker}(\cdot,\cdot)$ is one-dimensional and the proof of (a) is complete. □

Note that (a) implies the following

COROLLARY 6.5. *Let Γ be Coxeter–Dynkin or affine. Any proper connected subgraph of Γ is Coxeter–Dynkin.*

DEFINITION 6.6. A non-zero vector $x \in \mathbb{Z}^{Q_0}$ is called a *root* if $q\,(x) \leq 1$. Note for every $i \in Q_0$, ϵ_i is a root. It is called a *simple root*.

EXERCISE 6.7. Let Γ be a connected graph. Show that the number of roots is finite if and only if Γ is a Coxeter–Dynkin graph.

LEMMA 6.8. *Let Γ be Coxeter–Dynkin or affine. If x is a root, then either all $x_i \geq 0$ or all $x_i \leq 0$.*

PROOF. Assume that the statement is false. Let

$$I^+ := \{i \in Q_0 \mid x_i > 0\}, \quad I^- := \{i \in Q_0 \mid x_i < 0\}, \quad x^\pm = \sum_{i \in I^\pm} x_i \epsilon_i.$$

Then $x = x^+ + x^-$ and $(x^+, x^-) \geq 0$. Furthermore, since Γ is Coxeter–Dynkin or affine, we have $q(x^\pm) > 0$. Therefore

$$q(x) = q(x^+) + q(x^-) + (x^+, x^-) > 1.$$

$\square$

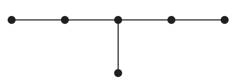
We call a root x *positive* (resp. *negative*) if $x_i \geq 0$ (resp. $x_i \leq 0$) for all $i \in Q_0$.

6.2. Classification. The following are all Coxeter–Dynkin graphs (below n is the number of vertices).

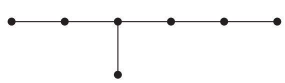
A_n

D_n

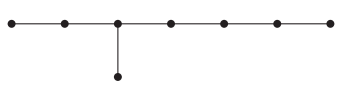
E_6



E_7



E_8

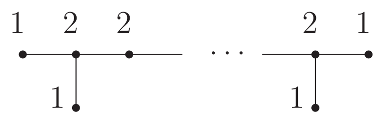


The affine graphs, except the loop , are obtained from the Coxeter–Dynkin graphs by adding a vertex (see Corollary 6.5). Here they are.

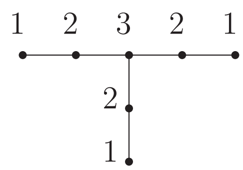
$\hat{A}_1$

For $n > 1$, $\hat{A}_n$ is a cycle with $n+1$ vertices. In this case $\delta = (1, \ldots, 1)$.
In what follows the numbers are the coordinates of δ.

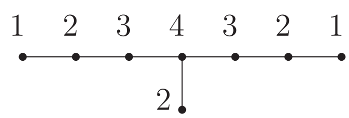
$\hat{D}_n$



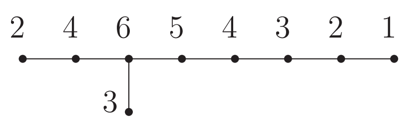
$\hat{E}_6$



$\hat{E}_7$



$\hat{E}_8$



The proof that the above classification is complete is presented below in the exercises.

Exercise 6.9. Check that A_n, D_n, E_6, E_7, E_8 are Coxeter–Dynkin using the Sylvester criterion and the fact that every subgraph of a Coxeter–Dynkin graph is Coxeter–Dynkin. One can calculate the determinant of a Cartan matrix inductively. It is $n+1$ for A_n, 4 for D_n, 3 for E_6, 2 for E_7 and 1 for E_8.

Exercise 6.10. Check that the Cartan matrices of $\hat{A}_n, \hat{D}_n, \hat{E}_6, \hat{E}_7, \hat{E}_8$ have corank 1 and every proper connected subgraph is Coxeter–Dynkin. Conclude that these graphs are affine.

Exercise 6.11. Let Γ be a Coxeter–Dynkin graph. Using Corollary 6.5 prove that Γ does not have loops, cycles and multiple edges. Prove that Γ has no vertices of degree 4 and at most one vertex of degree 3.

Exercise 6.12. Let a Coxeter–Dynkin graph Γ have a vertex of degree 3. Let p, q and r be the lengths of "legs" coming from this vertex. Prove that $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} > 1$. Use this to complete classification of Coxeter–Dynkin graphs.

Exercise 6.13. Complete classification of affine graphs using Corollary 6.5, Exercise 6.10 and Exercise 6.12.

7. Quivers of finite type and Gabriel's theorem

Recall that a quiver is of finite type if it has finitely many isomorphism classes of indecomposable representations.

Exercise 7.1. Prove that a quiver is of finite type if and only if all its connected components are of finite type.

Theorem 7.2. *(Gabriel) Let Q be a connected quiver and Γ be its underlying graph. Then*

1. *The quiver Q has finite type if and only if Γ is a Coxeter–Dynkin graph.*
2. *Assume that Γ is a Coxeter–Dynkin graph and (X, ρ) is an indecomposable representation of Q. Then $\dim X$ is a positive root.*
3. *If Γ is a Coxeter–Dynkin graph, then for every positive root $x \in \mathbb{Z}^{Q_0}$ there is exactly one indecomposable representation of Q of dimension x.*

Proof. Let us first prove that if Q is of finite type then Γ is a Coxeter–Dynkin graph. Indeed, if Q is of finite type, then for every $x \in \mathbb{N}^{Q_0}$, $\mathrm{Rep}\,(x)$ has finitely many G-orbits. Therefore $\mathrm{Rep}\,(x)$ must contain an open orbit. Assume that Q is not Coxeter–Dynkin. Then there exists a non-zero $x \in \mathbb{Z}^{Q_0}$ such that $q\,(x) \leq 0$. Let $O_X \subset \mathrm{Rep}\,(x)$ be an open orbit. Then $\mathrm{codim}\, O_X = 0$. But by Corollary 5.2

$$\mathrm{codim}\, O_X = \dim \mathrm{End}_Q\,(X) - q\,(x) > 0. \tag{7.9}$$

This is a contradiction.

Now assume that Γ is Coxeter–Dynkin. To show that Q is of finite type it suffices to prove assertions (2) and (3).

Note that (2) follows from Corollary 4.5.

Suppose that x is a positive root. Let (X, ρ) be a representation of Q such that $\dim O_X$ in $\operatorname{Rep}(x)$ is maximal. Let us prove that X is indecomposable. Indeed, let $X = X_1 \oplus \cdots \oplus X_s$ be a sum of indecomposable bricks. Then by Corollary 5.5 $\operatorname{Ext}^1(X_i, X_j) = 0$. Therefore $q(x) = s = 1$ and X is indecomposable.

Finally, if (X, ρ) is an indecomposable representation of Q, then (7.9) implies that O_X is an open orbit in $\operatorname{Rep}(x)$. By irreducibility, $\operatorname{Rep}(x)$ has at most one open orbit. Hence (3) is proved. $\square$

Remark 7.3. Gabriel's theorem implies that the propery of a quiver to be of finite type depends only on the underlying graph and does not depend on orientation.

Remark 7.4. Theorem 7.2 does not provide an algorithm for finding all indecomposable representations of quivers with Coxeter–Dynkin underlying graphs. We give such algorithm in the next chapter using the reflection functor.

Exercise 7.5. Let Q be a quiver whose underlying graph is A_n. Check that the positive roots are in bijection with connected subgraphs of A_n. For each positive root x give a precise construction of an indecomposable representation of dimension x.