# THE CONJECTURE OF TATE AND VOLOCH ON $p$-ADIC PROXIMITY TO TORSION

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Abstract. Tate and Voloch have conjectured that the $p$-adic distance from torsion points of semi-abelian varieties over $\mathbb{C}_p$ to subvarieties may be uniformly bounded. We prove this conjecture for torsion points on semi-abelian varieties over $\mathbb{Q}_p{}^{alg}$ using methods of algebraic model theory and a result of Sen on Galois representation of Hodge-Tate type.

As a generalization of their theorem on linear forms in $p$-adic roots of unity, Tate and Voloch conjectured:

**Conjecture:** (Tate, Voloch) *Let $G$ be a semi-abelian variety over $\mathbb{C}_p$. Let $X \subseteq G$ be a subvariety defined over $\mathbb{C}_p$. Then there is a constant $N \in \mathbb{N}$ such that for any torsion point $\zeta \in G(\mathbb{C}_p)_{\text{tor}}$ either $\zeta \in X$ or $\lambda(\zeta, X) \leq N$.*

In the above statement, $\mathbb{C}_p$ denotes the completion of the algebraic closure of $\mathbb{Q}_p$ and $\lambda(\cdot, X)$ is the $p$-adic proximity to $X$ function.

In [Sc] this conjecture was proved under the assumptions that $G$ is defined over $\mathbb{Q}_p{}^{alg}$ and that $\zeta$ is a torsion point of order prime-to-$p$. It was suggested that the same method of proof would work without the latter restriction. This suggestion is carried out in this note. The main theorem is the following.

**Main Theorem:** *Let $K$ be a finite extension of $\mathbb{Q}_p$. Let $G$ be a semi-abelian variety defined over $K$. Let $X \subseteq G$ be a closed subvariety defined over $\mathbb{C}_p$. There is a constant $N \in \mathbb{N}$ depending only on $X$ such that for any torsion point $\zeta \in G(\mathbb{C}_p)_{\text{tor}}$ either $\zeta \in X$ or $\lambda(\zeta, X) \leq N$.*

In the above statement, $\lambda(\cdot, X)$ is the $p$-adic proximity to $X$ function. In [Sc], this function was denoted by "$d(\cdot, X)$" and called a "distance function," but to match the notation common in the literature, we revert to "$\lambda$" instead of "$d$."

The proof of the Main Theorem passes through an analysis (due to Sen) of the action of the inertia group on the Tate module of the $p$-divisible group of $G$ and is completed by combining this argument with the main theorem of [Sc].

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## 1. Notation

Our notation is for the most part standard. $\mathbb{C}_p$ is the completion of the algebraic closure of $\mathbb{Q}_p$ equipped with the natural extension of the $p$-adic valuation, $v_p$. For $K$ a subfield of $\mathbb{C}_p$. $\mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K)$ denotes the group of $p$-adically continuous field automorphisms of $\mathbb{C}_p$ fixing $K$. Note that for $\sigma \in \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K)$ and $x \in \mathbb{C}_p$ one has $v_p(x) = v_p(\sigma(x))$. $\mathrm{Aut}(\mathbb{C}_p/K)$ denotes the group of all field automorphisms of $\mathbb{C}_p$ fixing $K$. By $K^{unr}$ we mean the maximal unramified extension of $K$ inside $\mathbb{C}_p$. For $H$ an abelian group and $n \in \mathbb{Z}$ an integer we denote the subgroup of $n$-torsion by $H[n] := \{h \in H : nh = 0\}$. The torsion subgroup of $H$ is $H_{\mathrm{tor}} := \bigcup_{n=1}^{\infty} H[n]$. For $p$ a prime number, the $p$-Tate module of $H$ is $T_p H := \varprojlim H[p^n]$ where as usual the maps in the inverse system are given by $p^m : H[p^{n+m}] \to H[p^n]$. $T_p H$ is naturally a $\mathbb{Z}_p$-module and we denote its rank by $\mathrm{rk}_{\mathbb{Z}_p} T_p H$. By $H_{p'-\mathrm{tor}}$ we mean the prime-to-$p$ torsion group of $H$, $\bigcup_{(n,p)=1} H[n]$. We denote the group of $p$-power torsion by $H[p^\infty]$.

## 2. Proof of Main Theorem

In order to prove the main theorem we prove a refined version via induction. The refined version is the following.

**Main Theorem:** (inductive version) *Let $K$ be a finite extension of $\mathbb{Q}_p$ considered as a subfield of $\mathbb{C}_p$. Let $G$ be a semi-abelian variety over $K$. Let $\Gamma \leq G(\mathbb{C}_p)_{\mathrm{tor}}$ be a $\mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K^{unr})$-submodule of the torsion group of $G(\mathbb{C}_p)$. Let $X \subseteq G$ be a subvariety of $G$ defined over $\mathbb{C}_p$. There is an integer $N \in \mathbb{Z}_+$ such that for any $\zeta \in \Gamma$ either $\zeta \in X(\mathbb{C}_p)$ or $\lambda(\zeta, X) \leq N$.*

The main theorem is an instance of this version by taking $\Gamma = G(\mathbb{C}_p)_{\mathrm{tor}}$ and this version certainly follows from the main theorem. We will prove this refined version by induction on $\mathrm{rk}_{\mathbb{Z}_p} T_p(\Gamma/(\Gamma \cap G(K^{unr}))$ at the end of this section. Before proving this theorem we draw some consequences about the existence of continuous automorphism of $\mathbb{C}_p$ satifying reasonable equations on $\Gamma$ from a theorem of Sen.

For the reader's convenience we reproduce the statement of the theorem of Sen in the form we use (see Theorem 1 of [Sen] or Théorème 2 of [Ser]).

**Fact 2.1** (Sen)**.** *Let $K$ be a complete discretely valued subfield of $\mathbb{C}_p$. Let $H$ be a $p$-divisible group over $K$. Let $\rho : \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K^{unr}) \to \mathrm{Aut}_{\mathbb{Z}_p}(T_p H(\mathbb{C}_p))$ be the inertial Galois representation associated to $H$. Then the image of $\rho$ is an open subgroup of the $\mathbb{Q}_p$ points of an algebraic group. Moreover, the Zariski closure of the image of $\rho$ is generated (as an algebraic group) by the $\mathrm{Aut}(\mathbb{C}_p/K)$ conjugates of an algebraic torus defined over $\mathbb{C}_p$.*

**Lemma 2.2.** *Let $G$ be an algebraic subgroup of $\mathrm{GL}_n$ defined over $\mathbb{Q}_p$. Let $H \subseteq G(\mathbb{Q}_p)$ be an open subgroup of the $\mathbb{Q}_p$-points of $G$. Suppose that $G$*

*contains an algebraic torus of dimension $g$ over $\mathbb{C}_p$. Then there is some $h \in H$ such that*

(1) *The characteristic polynomial of $h$ is $P(T)(T-1)^m \in \mathbb{Q}[T]$ where for any root of unity $\zeta \in \mathbb{C}$ the polynomial $P$ does not vanish at $\zeta$ and $m \leq n-g$ and*
(2) *$h$ is semi-simple.*

Proof: $G$ contains a nontrivial maximal torus $T$ defined over $\mathbb{Q}_p$ (See section 34.4 of [Hu]). The dimension of $T$ is at least $g$ as $G$ contains a torus of dimension $g$ over $\mathbb{C}_p$. Since $H$ is open in $G(\mathbb{Q}_p)$, $H \cap T(\mathbb{Q}_p)$ is open in $T(\mathbb{Q}_p)$. Replace $H$ with $H \cap T(\mathbb{Q}_p)$ and work inside $T$.

Regard the function which assigns to an element of $\mathrm{GL}_n$ its characteristic polynomial as a regular function $\psi : \mathrm{GL}_n \to \mathbb{A}^n$. Let $C_T$ be the Zariski closure of $\psi(T(\mathbb{C}_p))$. Since $T(\mathbb{C}_p)$ is conjugate to an algebraic subgroup $\Delta(\mathbb{C}_p)$ of the diagonal subgroup of $\mathrm{GL}_n(\mathbb{C}_p)$ over $\mathbb{C}_p$ and the characteristic polynomial is invariant under conjugation, $C_T$ may also be described as the Zariski closure of $\psi(\Delta(\mathbb{C}_p))$. As $\psi|_\Delta$ is a finite map, the dimension of $C_T$ is that of $T$.

$\Delta$ is defined over $\mathbb{Q}$ as every connected subgroup of the diagonal is defined by character equations. $\Delta$ is a rational variety split over $\mathbb{Q}$ as it is defined over $\mathbb{Q}$ and has a point (namely the identity). Visibly, $\psi$ is defined over $\mathbb{Q}$. Hence, $C_T(\mathbb{Q})$ is dense in $C_T(\mathbb{Q}_p)$ in a neighborhood of $\psi(\mathrm{id})$.

The dimension of $T(\mathbb{Q}_p)$ as a $p$-adic Lie group is equal to its dimension as an algebraic group. Because $\psi|_T$ is a finite map, $\psi(T(\mathbb{Q}_p))$ contains a $p$-adic manifold of full dimension. In fact, $\psi(T(\mathbb{Q}_p))$ is of full dimension at every point in its image.

Let $m$ be the exponent of $(X-1)$ in the characteristic polynomial of a general element of $T(\mathbb{C}_p)$. Note that $m \leq n-g$. The Zariski closure of the subset of $C_T(\mathbb{Q})$ corresponding to polynomials with rational co-efficients and more than $m$ factors (counting multiplicity) of the form $(X-\zeta)$ for $\zeta$ a root of unity is a finite union, $\Sigma$, of subsets of co-dimension at least one in $C_T$. [The key point here is that there are only finitely many roots of unity which may satisfy a polynomial of degree $n$ over $\mathbb{Q}$.] Since $C_T(\mathbb{Q})$ is dense near $\psi(\mathrm{id})$ in $C_T(\mathbb{Q}_p)$, we can find $h \in H$ such that $\psi(h) \in C_T(\mathbb{Q}) \setminus \Sigma(\mathbb{Q})$. □

**Lemma 2.3.** *Let $K$ be a discretely valued subfield of $\mathbb{C}_p$. Let $G$ be a semi-abelian variety over $K$. Ler $\Gamma \leq G(\mathbb{C}_p)_{\mathrm{tor}}$ be a $\mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K^{unr})$-submodule. Assume that $\mathrm{rk}_{\mathbb{Z}_p} T_p\Gamma/T_p(\Gamma \cap G(K^{unr})) > 0$. Then there is a continuous automorphism $\sigma \in \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K^{unr})$ and a polynomial $P(T) \in \mathbb{Z}[T]$ with no cyclotomic roots such that*

(1) *$G(\mathrm{Fix}(\sigma)) + \{\zeta \in G(\mathbb{C}_p) : P(\sigma)(\zeta) = 0\} \supseteq \Gamma$ and*
(2) *$\sigma$ acts non-trivially on $T_p(\Gamma/(G(K^{unr}) \cap \Gamma))$.*

Proof: The hypothesis that $\Gamma[p^\infty] \subseteq G[p^\infty](\mathbb{C}_p)$ is $\mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K^{unr})$ stable implies that there is a $p$-divisible subgroup $H$ of $G[p^\infty]$ over $K^{unr}$

such that $H(\mathbb{C}_p)$ is of finite index in $\Gamma[p^\infty]$. Let $\rho : \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K^{unr}) \to \mathrm{Aut}_{\mathbb{Z}_p}(T_p H(\mathbb{C}_p))$ be the inertial Galois representation associated to $H$. The hypothesis that $\mathrm{rk}_{\mathbb{Z}_p} T_p \Gamma / T_p(G(K^{unr}) \cap \Gamma) > 0$ implies that the image of $\rho$ is infinite. By Sen's theorem this implies that the algebraic hull of the image of $\rho$ contains an algebraic torus of positive dimension over $\mathbb{C}_p$. Apply Lemma 2.2 to the algebraic hull of the image of $\rho$ to find some $\tau \in \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K^{unr})$ with $\rho(\tau)$ having characteristic polynomial $Q(T)(T-1)^m$ on $T_p H(\mathbb{C}_p)$ for $Q(T) \in \mathbb{Q}[T]$ having no cyclotomic factors and $m < \mathrm{rk}_{\mathbb{Z}_p}(T_p(\Gamma/(\Gamma \cap G(K^{unr}))))$. Replace $\tau$ by $\sigma := \tau^N$ with $N = \#\Gamma[p^\infty]/H(\mathbb{C}_p)$ so that $\tau$ acts trivially on $\Gamma/H(\mathbb{C}_p)$. If $Q(T)$ factors as $\prod(T-\alpha_i)$ over $\mathbb{C}$, then let $\tilde{P}(T) := \prod(T - \alpha_i^N) \in \mathbb{Q}[T]$. Let $P(T) = d\tilde{P}(T) \in \mathbb{Z}[T]$ where $d$ is the least common multiple of the denominators of the co-efficients of $\tilde{P}$. □

**Lemma 2.4.** *Let $\mathfrak{G}$ be a group and let $M$ be a $\mathfrak{G}$-module. For $s, g \in \mathfrak{G}$ write $s^g$ for $g^{-1}sg$. Let $s \in \mathfrak{G}$. Then $M^{s^{\mathfrak{G}}} := \{x \in M : (\forall g \in G) s^g x = x\}$, the set of elements of $M$ fixed by every conjugate of $s$, is a $\mathfrak{G}$-submodule of $M$.*

Proof: Since $\mathfrak{G}$ acts by endomorphisms of $M$, $M^{s^{\mathfrak{G}}}$ is certainly an abelian group. Let us check that it is closed under the action of $\mathfrak{G}$. Let $h \in \mathfrak{G}$ and $x \in M^{s^{\mathfrak{G}}}$. We need to show that $hx \in M^{s^{\mathfrak{G}}}$. Let $g \in \mathfrak{G}$. Because $x \in M^{s^{\mathfrak{G}}}$, $h^{-1}g^{-1}sghx = s^{gh}x = x$. Act by $h$ on each side of this equation to obtain $s^g hx = hx$. □

With the above lemmas in place we can now prove the Main Theorem in the inductive form.

Proof of Main Theorem: The truth value of the statement of the theorem does not change if for a given $G$ we replace $K$ by a finite extension or we replace $G$ with an isogenous semi-abelian variety. So, we may assume without loss of generality that $G$ is isomorphic over $K$ to a product of extensions of simple abelian varieties by split tori.

We work by induction on $\mathrm{rk}_{\mathbb{Z}_p} T_p(\Gamma/(\Gamma \cap G(K^{unr})))$. When this rank is zero, then after replacing $K$ with a finite extension we have $\Gamma \leq G(K^{unr}) + G(\mathbb{C}_p)_{p'-\mathrm{tor}}$. The theorem we aim to prove is, under the current hypotheses, the main theorem of [Sc]. [The theorem is stated there only for prime-to-$p$ torsion points but all unramified torsion points are contained in the group defined by the Frobenius equation, called $\Lambda(\mathbb{C}_p, \sigma)$ in [Sc] where $\sigma$ is a relative Frobenius.]

Large chunks of the remainder of this proof are identical to portions of the proof of Theorem 2.3 in [Sc]. That theorem was meant as a template from which the main theorem of this note could be deduced, but it is not quite general enough.

We work by Noetherian induction on $X$. Since the proximity to a union of two varieties is the maximum of the proximities to each of the varieties separately, we may assume that $X$ is irreducible.

Let $H := \mathrm{Stab}^0_G(X)$ be the connected component of the stabilizer of $X$ in $G$ and let $\pi : G \to G/H$ be the quotient map. We may choose coverings so that for any $x \in G(\mathbb{C}_p)$ we have $\lambda(x, X) = \lambda(\pi(x), \pi(X))$. Since $G$ is split over $K$, we have $(G/H)(K^{unr}) = \pi(G(K^{unr}))$. Thus, $\mathrm{rk}_{\mathbb{Z}_p} T_p(\pi(\Gamma)/(\pi(\Gamma) \cap (G/H)(K^{unr}))) \leq \mathrm{rk}_{\mathbb{Z}_p} T_p(\Gamma/(G(K^{unr}) \cap \Gamma))$. So, if $\dim H > 0$, by induction we will have a bound on the proximity from points in $\pi(\Gamma)$ not in $\pi(X)$ to $\pi(X)$ which implies the same bound on the proximity from points in $\Gamma$ not in $X$ to $X$.

So we may assume now that $X$ has a finite stabilizer in $G$.

Let $\sigma \in \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K^{unr})$ and $P(T) \in \mathbb{Z}[T]$ be the field automorphism and integral polynomial produced by Lemma 2.3 for $\Gamma$. Let $\Lambda(\mathbb{C}_p, \sigma) := \ker P(\sigma) : G(\mathbb{C}_p) \to G(\mathbb{C}_p)$ and $\Phi(L, \sigma) := \ker(\sigma - 1) : G(\mathbb{C}_p) \to G(\mathbb{C}_p)$. With this notation, the proof of Theorem 2.3 of [Sc] goes through with just a couple of minor changes.

The first change worth remarking occurs in the penultimate multi-line paragraph. Our reduction to the case that $X$ has a finite stabilizer ensures that there is a finite set $A \subseteq \Gamma$ such that if $\zeta \in \Gamma \setminus X(\mathbb{C}_p)$ and $\lambda(\zeta, X) > \gamma'$ (in the notation used there), then for any conjugate $\tau$ of $\sigma$ it is possible to write $\zeta = a + b$ with $a, b \in \Gamma$, $\tau(b) = b$, and $a \in A$. Let $m$ be the least common multiple of the orders of $a \in A$. We conclude that if $\zeta \in \Gamma \setminus X(\mathbb{C}_p)$ and $\lambda(\zeta, X) > \gamma'$, then $[m]\zeta \in \bigcap_{g \in \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K^{unr})} \Phi(\mathbb{C}_p, g^{-1}\sigma g) \cap \Gamma = \Gamma^{s^{\mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K^{unr})}} =: \Psi$ which is a proper $\mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K^{unr})$-submodule of $\Gamma$ by Lemma 2.4. Since $\sigma$ acts non-trivially on $T_p(\Gamma/(\Gamma \cap G(K^{unr}))$ we have $\mathrm{rk}_{\mathbb{Z}_p} T_p(\Psi/(G(K^{unr}) \cap \Psi)) = \mathrm{rk}_{\mathbb{Z}_p} T_p(\Psi/(G(K^{unr}) \cap \Gamma)) \leq \mathrm{rk}_{\mathbb{Z}_p} T_p(\Gamma \cap \Phi(\mathbb{C}_p, \sigma)/(G(K^{unr}) \cap \Gamma) < \mathrm{rk}_{\mathbb{Z}_p} T_p(\Gamma/(G(K^{unr}) \cap \Gamma))$. So by induction, there is some $\delta \in \mathbb{Z}$ such that for any $\xi \in [m]^{-1}\Psi \setminus X(\mathbb{C}_p)$ we have $\lambda(\xi, X) \leq \delta$. The bound for $\Gamma$ is then $\max\{\delta, \gamma'\}$. ✠

## References


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