# Model theory, algebraic dynamics and local fields

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10 June 2010

# Zilber trichotomy

The fundamental qualitative result describing the definable sets in difference fields is the Zilber trichotomy (proven by Chatzidakis & Hrushovski in characteristic zero, Chatzidakis, Hrushovski & Peterzil in all characteristics, and by Pillay & Ziegler geometrically [though only in characteristic zero])

The correct formulation requires a few more definitions, but let me give an imprecise (and not entirely correct) version first.

For a one-dimensional definable set $X$ exactly one of the following is true:

- There is a finite-to-one definable map $f : X \to \mathbb{P}^1(F)$ for some definable field $F$ of dimension one.
- $X$ is in finite-to-finite definable correspondence with a group $G$ having the property that every (qf-)definable subset of $G^n$ is a finite Boolean combination of cosets of definable groups.
- $X$ is trivial in the sense that all definable relation on $X$ are reducible to binary relations.

# Definable fields

Up to definable isomorphism there are very few definable fields in a difference closed field $(K, \sigma)$.

- The full field $K$ itself (which is infinite dimensional).
- For $N \in \mathbb{Z}_+$, the fixed fields $\mathrm{Fix}(\sigma^N)(K) := \{a \in K : \sigma^N(a) = a\}$ have dimension $N$.
- If $K$ has characteristic $p > 0$ and $N \in \mathbb{Z} \smallsetminus \{0\}$ and $M \in \mathbb{Z}$, then the fields defined by $\{a \in K : \sigma^N(a) = a^{p^M}\}$ are finite dimensional.

Other than the finite fields, there are no other definable fields.

# What does the trichotomy mean for algebraic dynamics?

If $(X, f)$ is a "one-dimensional" AD, then provided that one knows which of the three cases the trichotomy theorem applies to $(X, f)$, we have strong restrictions on the class of $f$-(skew)-invariant varieties in $X^n$.

While there are one-dimensional ADs on higher dimensional varieties, it is certainly the case that if $X$ is a curve and $f : X \to X^\sigma$ is non-constant, then $(X, f)$ is one-dimensional.

# Triviality explained

- If $\rho : (X, f) \to (Y, g)$ is a morphisms of ADs (or difference varieties) and $Z \subseteq Y$ is $g$-invariant, then $\rho^{-1}Z$ is an $f$-invariant variety.
- If $U \subseteq X$ and $V \subseteq X$ are $f$-invariant then so is $U \cap V$ and the components of this intersection are $f$-pre-periodic.
- In particular, if we are given $(X_1, f_1), \ldots, (X_\ell, f_\ell)$ a sequence of ADs, then the projection maps $\pi_{i,j} : \prod_{k=1}^{\ell} X_k \to X_i \times X_j$ are maps of ADs and we may obtain $(f_1, \ldots, f_\ell)$-invariant varieties by taking some components of intersections of pullbacks by the $\pi_{i,j}$'s of $(f_i, f_j)$-invariant varieties.
- To say that each $(X_i, f_i)$ is trivial is to say that this is the only way to obtain such invariant varieties.
- It is a theorem of Medvedev that is $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational function of separable degree greater than one which is not a generalized Lattès map, then $(\mathbb{P}^1, f)$ is trivial.

# Generalized Lattès maps

By a generalized Lattès map over a difference field $(K, \sigma)$ I mean a rational map $f : X \dashrightarrow X^\sigma$ for which there is an algebraic group $G$ and a finite affine map $\phi : G \to G^\sigma$ and a dominant rational map $\pi : G \dashrightarrow X$ for which the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi^\sigma} \\ X & \xrightarrow{f} & X^\sigma \end{array}$$

- When $G = G^\sigma$ is an elliptic curve, $\phi(x) = [2]_G(x)$, and $\pi : G \to \mathbb{P}^1$ realizes the projective line (birationally) as $G/\pm 1$, then $f$ is a Lattès map in the usual sense.
- When $G = \mathbb{G}_m$, $\pi : \mathbb{G}_m \to \mathbb{A}^1$ is given by $x \mapsto x + \frac{1}{x}$, and $\phi(x) = x^N$, then $f$ is (conjugate to) the $N^{\text{th}}$ Chebyshev polynomial.
- With this definition, we allow $X = G$. One should really separate out this case.

- If $G$ is an algebraic group and $\Gamma < G \times G^\sigma$ is an algebraic subgroup for which both projections are finite-to-one, then $(G, \Gamma)$ is a finite dimensional definable group so that one may prove that it is modular (*ie* satisfies the conclusion of the Mordell-Lang conjecture) by checking that it is orthogonal to all fixed fields.
- For example, if $G$ is an abelian variety defined over the fixed field of $\sigma$ and $P(X) \in \mathbb{Z}[x]$ is a nonzero polynomial over the integers, then the definable group $\ker P(\sigma)(K) := \{a \in G(a) : P(\sigma)(a) = 0\}$ is finite dimensional and is modular just in case no complex root of $P$ is a root of unity (at least, in characteristic zero; positive characteristic is a little more complicated - Hrushovski/Chatzidakis)
- Modularity of such groups is a key step in Hrushovski's proof of Manin-Mumford and in my proofs of a Drinfeld module version of Manin-Mumford and a local version of the André-Oort conjecture.

# Trichotomy as pair of dichotomies

The trichotomy really breaks into two dichotomies of fundamentally different characters.

- Modular/non-modular: The content of this dichotomy is that non-modularity is always witnessed by the presence of a definable field, and hence, geometry arising from algebraic geometry.
- Trivial/Group within modular geometries

The second of these principles, namely that nontriviality for a modular geometry is witnessed by a group, holds very generally whereas the first one is a reflection of the geometry of difference varieties.

As the modular/non-modular dichotomy is tied up with questions about moduli of difference varieties/ADs, I shall focus on this part of the trichotomy.

# Families of definable sets

Given a definable set $X$, by a family of definable sets we shall mean a definable subset $Y \subseteq X \times B$ of the product of $X$ with some other definable set $B$. This is a normal family if $b \neq b'$ implies that $Y_b \neq Y_{b'}$. Provided that it makes sense to form quotients, we may convert any definable family into a normal family by quotienting by the equivalence relation $b \sim b' \iff Y_b = Y_{b'}$. To the extent that "dimension" makes sense, we define the dimension of a family of definable sets to the dimension of $B/\sim$.

**Definition**

We say that a finite dimensional difference variety $(X, \Gamma)$ is modular if there is bound on the dimensions of normal families of irreducible difference subvarieties of $(X, \Gamma)^2$.

There are many subtleties with the notion of modularity which we are ignoring. For example, the theory really works better generically, that is, at the level of types. In the literature, the word "one-based" is used for (essentially) the same concept.

# Example of non-modularity

Consider the fixed field $F$ of $\sigma$, which as an AD may be identified with $(\mathbb{A}^1, \mathrm{id})$.

If $B := F^N = (\mathbb{A}^N, \mathrm{id})(K)$ and $Y \subseteq F^2 \times B$ is defined by

$$(x, y, b_0, \ldots, b_{N-1}) \in Y(K) \iff y = x^N + \sum_{i=0}^{N-1} b_i x^i \\ \& \sigma(x) = x \& \bigwedge_{i=0}^{N-1} \sigma(b_i) = b_i$$

then $Y$ is a normal family of irreducible difference subvarieties of $F^2$ of dimension $N$.

Of course, any algebraic family of subvarieties of $\mathbb{A}^m$ would give a family of difference subvarieties of $F^m$.

The content of the modular/non-modular dichotomy theorem is that these are the only real examples of high dimensional families of difference varieties.

# Campana-Fujiki theorem on families of analytic varieties

**Theorem (Campana, Fujiki (independently and with slightly different statements))**

*Suppose that $X$ is a compact complex analytic space and that $Y \subseteq X \times B$ is a normal family of irreducible analytic subvarieties of $X$ all passing through some fixed point $a \in X$. Then $B$ is Moishezon: there is a generically injective map $f : B \to \mathbb{P}^n_{\mathbb{C}}$ for some $n \in \mathbb{Z}_+$.*

Pillay and Ziegler adapted the Campana-Fujiki proof to difference fields (and other theories of fields with operators)

**Theorem (Pillay-Ziegler)**

*Suppose that $(X, \Gamma)$ is a finite dimensional difference variety over a difference closed field $(K, \sigma)$ of characteristic zero and that $Y \subseteq X \times B$ is a normal family of irreducible difference subvarieties all passing through some fixed point $a \in (X, \Gamma)(K)$. Then there is a definable generically injective map $f : B \to F^n$ for some $n$ where $F = \mathrm{Fix}(\sigma)$.*

# Zilber trichotomy as consequence of Pillay-Ziegler

The modular/non-orthogonal to a field dichotomy (in characteristic zero) follows from the Pillay-Ziegler theorem.

- If $(X, \Gamma)$ is non-modular, then because one can find an arbitrarily high dimensional families of difference subvarieties of $(X, \Gamma)^2$, one may find such a family $Y \subseteq (X, \Gamma)^2 \times B$ passing through one point.
- The P-Z theorem gives a definable injective map $g : B \to F^n$ (where $F = \mathrm{Fix}(\sigma)$).
- It follows from considerations around definability of types (what is called "Shelah's reflection principle" in the first of the two difference fields and algebraic dynamics papers of Chatzidakis and Hrushovski) that $B$ may be realized as a quotient of a definable subset of $(X, \Gamma)$.
- Combining these relations, we obtain $(X, \Gamma) \not\perp F$.

# Proof of Pillay-Ziegler

- Recall that the $n^{\text{th}}$ jet space of $X$ at $a$ is $J^n(X)_a(K) := \mathrm{Hom}_K(\mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^{n+1}, K)$
- By noetherianity and a compactness argument, one sees that there is a number $n$ so that $Y_b = Y_{b'}$ if and only if $J^n(Y_b)_a = J^n(Y_{b'})_a$ as a subspace of $J^n(X)_a(K)$.
- There is a natural difference variety structure on $J^n(X)$ coming from $J^n(\Gamma)_a : J^n(X)_a \to J^n(X)_a^\sigma$ with respect to which $(J^n(X)_a, J^n(\Gamma)_a)(K)$ is a finite dimensional vector space over $F$.
- One shows that $j(b) := (J^n(X)_a, J^n(\Gamma)_a)(K) \cap J^n(Y_b)(K)$ is Zariski dense in $J^n(Y)_b$.
- Hence, the map $j : B(K) \to \mathrm{Gr}((J^n(X)_a, J^n(\Gamma)_a)(K)) \cong \mathrm{Gr}(M, N)(F)$ (for some $M$ and $N$) is the desired map.

**Theorem (Benedetto, Baker)**

*Let $k$ be an algebraically closed field, $L$ the function field of a curve over $k$, and $f : \mathbb{P}^1_L \to \mathbb{P}^1_L$ a nonconstant rational function of degree at least two, then either $f$ is conjugate to a rational function defined over $k$ of every point $P \in \mathbb{P}^1(L)$ of $f$-canonical height zero is $f$-pre-periodic.*

We heard on Monday about a generalization of this theorem due to Bhatnagar and Szpiro to higher dimensional polarized dynamical systems. I will discuss a generalization due to Chatzidakis and Hrushovski in which height considerations are replaced by an analysis of definability.

# Interpretations

If $L/K$ is a finite extension of fields, then using the Weil restriction of scalars construction we may functorially associate to any scheme $X$ over $L$ a scheme $R_{L/K}X$ over $K$ so that for any $K$-algebra $A$ we may identify $R_{L/K}X(A)$ with $X(A \otimes_K L)$.

More naïvely, for any definable set $X$ in $n$-space over $L$, we may associate a definable set $\widetilde{X}$ in $n[L:K]$-space over $K$ so that we may identify $X(L) = \widetilde{X}(K)$. The point is that $L$ is interpreted in $K$: we may identify $L$ with a definable set in $K$, namely $K^{[L:K]}$, in such a way that the basic structure (the basic functions $+^L$, $\cdot^L$, $-^L$, *et cetera*) correspond to $K$-definable sets. It then follows that every definable set in a Cartesian power of $L$ corresponds to some definable set in a power of $K$.

When $L/K$ is an infinite degree extension a construction no longer works, but in some cases we may naturally represent $L$ as a countable union of definable sets.

# (ind)-Interpretations

If $L$ is a finitely generated extension of the field $K$, say, $L = K(a_1, \ldots, a_n)$, then we may naturally represent $L$ as a directly union of $K$-definable sets (possibly modulo definable equivalence relations).

For instance, for each $d$ we could take

$$X_d(K) := \{(c_\alpha, d_\alpha)_{\alpha \in n^{d+1}} \in K^{2n^{d+1}} : \sum d_\alpha a_1^{\alpha_1} \cdots a_n^{\alpha_n} \neq 0\}$$

with the equivalence relation $(c, d) \equiv (c', d') \Longleftrightarrow \frac{\sum c_\alpha a^\alpha}{\sum d_\alpha a^\alpha} = \frac{c'_\alpha a^\alpha}{d'_\alpha a^\alpha}$ and take the natural inclusions $X_d \hookrightarrow X_{d+1}$. We thereby obtain a natural surjective map $f : \bigcup X_d(K) \to L$

If $Y(L) \subseteq L^n$ is quantifier-free definable, then $(f^{\times n})^{-1} Y(L) \cap X_d(K)^n$ is definable in $K$.

# Limited sets

**Definition**

Relative to the presentation of $L = \bigcup f(X_d(K))$, if $Y$ is a (quasi-projective) algebraic variety over $L$, we say that a set $S \subseteq Y(L)$ is limited if (relative to some inclusion of $Y$ in projective space) the coordinates of the points in $S$ may be taken from $f(X_d(K))$ for some $d$.

- For example, if $\operatorname{tr.deg}(L/K) = 1$, then the set of $L$-rational points on $Y$ of Weil height bounded by any given positive number is limited.
- More importantly for the application, if $(Y, f, L)$ is a polarized dynamical system, then set of points of canonical height zero is limited.

# Primitive dynamics

While there is a version of the Chatzidakis-Hrushovski theorem for more general dynamical systems, the cleanest statement is for primitive dynamical systems.

**Definition**

Let $(X, f)$ be a dynamical system over a field $K$. We say that $(X, f)$ is primitive if $\dim(X) > 0$ and there does not exist a dynamical system $(Y, g)$ with $0 < \dim(Y) < \dim(X)$ and a dominant rational map of dynamical systems $h : (X, f) \dashrightarrow (Y, g)$.

### Theorem (Chatzidakis, Hrushovski)

*Let $L$ be a finitely generated regular extension of the field $K$ and $(Y, f)$ a primitive AD over $L$. Then either $(Y, f)$ constructibly descends to $K$ (there is an AD $(Z, g)$ over $K$ and a generically bijective ratinonal map $\gamma : (Y, f) \dashrightarrow (Z, g)$ — the inverse might require negative powers of the Frobenius) or for each limited subset $S \subseteq X(L)$ there is a number $n := n(S)$ and a proper (not necessarily irreducible) subvariety $W \subsetneq Y$ so that if $a \in S \smallsetminus W(L)$, then at least one of $f(a), \ldots, f^n(a)$ is not an element of $S$.*