Model theory, algebraic dynamics and local fields

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Difference fields and dynamical systems

Definition

A difference ring is a pair (R, σ) consisting of a (commutative) ring R and a ring endomorphism $\sigma : R \to R$. When R is a field, we say that (R, σ) is a *difference field*.

Definition

An algebraic dynamical (AD) system (X, f) over a field k consists of an algebraic variety X over k and a regular function $f : X \to X$. If we require merely that $f : X \dashrightarrow X$ be a rational function, then we say that (X, f) is a rational algebraic dynamical system (rAD).

(Rational) algebraic dynamical systems as difference fields

Proposition

If k is any field and $f : X \dashrightarrow X$ is a dominant rational self-map on an algebraic variety over k, then $(k(X), f^*)$ is a difference field. Conversely, if K is a finitely generated field and (K, σ) is difference field for which $k := \mathrm{Fix}(\sigma) := \{x \in K : \sigma(x) = x\}$ is relatively algebraically closed in K, then there is an algebraic variety X over k and a rational map $f : X \dashrightarrow X$ so that $(K, \sigma) = (k(X), f^*)$.

While finitely generated difference fields most directly encode rADs, they are far from being the only arithmetically or geometrically interesting difference fields and for our general theory we shall work with difference fields which are very far from being finitely generated.

More examples of difference fields

- (K, σ_q) where p is any prime number, q is a power of p, K is a field of characteristic p, and $\sigma_q : K \to K$ is the q-Frobenius $x \mapsto x^q$
- k any field, X an algebraic variety over k, $\Gamma \subseteq X \times X$ a correspondence (*ie* an irreducible subvariety for which each projection $\Gamma \to X$ is dominant and generically finite, $L = k(X)^{alg}$ and $\sigma : L \to L$ any automorphism extending Γ^*. We might take K be the the subfield of L generated by $k(X)$ and all of its iterates under σ and σ^{-1}. As an (inversive) difference field K is finitely generated and is of finite transcendence degree over k, but is usually not finitely generated as a field.

Definable sets in difference fields

We consider difference fields in the language of rings augmented by a unary function symbol σ for the distinguished endomorphism (and sometimes also σ^{-1} if we wish to enforce that σ be an automorphism.

General definable sets can be quite complicated, but the quantifier-free definable sets can be expressed as finite Boolean combinations of sets in definable bijection with difference varieties.

Difference varieties

Recall that if $\sigma : k \to k$ makes k into a difference ring and X is a scheme over k, then X^σ fits into the Cartesian square

$$\begin{array}{ccc} X^\sigma & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{\sigma^*} & \mathrm{Spec}(k) \end{array}$$

Definition

Let (k, σ) be a difference field. A difference variety over (k, σ) consists of a pair (X, Γ) where X is an algebraic variety over k and $\Gamma \subseteq X \times X^\sigma$ is a subvariety of $X \times X^\sigma$. For (A, σ) a difference ring extension of (k, σ), the set of (A, σ)-points (we often drop σ from the notation) of (X, Γ) is

$$(X, \Gamma)(A) := \{a \in X(A) : (a, \sigma(a)) \in \Gamma(A)\}$$

Density

In general, if (X, Γ) is a difference variety over a difference field (k, σ), then $(X, \Gamma)(k)$ need not be Zariski dense in X, even if k is algebraically closed. (Consider, for example, the case that $k = \mathbb{C}$, $\sigma : k \to k$ is the identity, X is any algebraic variety over $\mathbb{C}$, $f : X \to X$ is not the identity, and Γ is the graph of f.)

There is a geometric obstruction to density. The set $(X, \Gamma)(k)$ is contained in the image of $\Gamma(k) \to X(k)$ under the first projection map. Hence, we must require that this map be dominant. Likewise, if we want σ to be an automorphism, we must require that the second projection $\Gamma \to X^\sigma$ be dominant.

Difference closed fields (ACFA)

A difference closed field or model of ACFA, is a difference field (K, σ) for which

- K is algebraically closed
- σ is an automorphism
- for any difference variety (X, Γ) over K for which both projections $\Gamma \to X$ and $\Gamma \to X^\sigma$ are dominant, $(X, \Gamma)(K)$ is Zariski dense in X

Theory of ACFA

Theorem (Chatizidakis, Hrushovski)

- *The theory of difference closed fields, ACFA, is first-order axiomatizable.*
- *ACFA is the model companion of the theory of difference fields.*
- *More specifically, every difference field embeds into some difference closed field.*
- *Difference closed fields satisfy a generalized version of the Hilbert Nullstellensatz (technically, they are existentially closed): If* $(K, \sigma) \subseteq (L, \sigma)$ *is an extension of difference fields for which* (K, σ) *is difference closed, and* $\phi(x_1, \ldots, x_n)$ *is a quantifier-free formula with parameters from* K *for which* $L \models (\exists x_1) \cdots (\exists x_n)\phi$*, then* $K \models (\exists x_1) \cdots (\exists x_n)\phi$*.*
- *Quantifier elimination fails, but (relative to ACFA), but every set is defined by a finite disjunction of formulae of the form* $(\exists y)[\phi(x_1, \ldots, x_n, y) \& \sum_{i=0}^{d} t_i(x) y^i = 0 \& t_d(x) \neq 0]$ *where each* t_i *is a term in* $x_1, \ldots, x_n$ *and* ϕ *is quantifier-free.*

Natural difference closed fields?

- As with the construction of the algebraic closure of a field K, one obtains a difference closed field extending K by successively (transfinitely) solving systems of difference equations and taking a direct limit.
- Unlike ordinary algebraic equations, in general a difference field does not have a unique "difference closure" even up to isomorphism.
- While $(\mathbb{C}, +, \times, 0, 1)$ is a "natural" algebraically closed field and the field of differerentially algebraic germs of meromorphic functions at a generic point of the complex plane is a "natural" differentially closed field, I do not know of a natural analytic construction of a difference closed field.

ACFA = Limit of Frobenius

If $n \in \mathbb{Z} \smallsetminus \{0\}$ is a nonzero integer, and $q = p^n$ then $(\mathbb{F}_p^{alg}, \sigma_q)$ is not difference closed as, for instance, if $\Gamma := V(y - x^q + 1) \subseteq \mathbb{A}^1 \times \mathbb{A}^1$, then the difference variety $(\mathbb{A}^1, \Gamma)$ has no (K, σ_q)-rational points. But is difference closed in the limit.

Theorem (Hrushovski)

If $\mathscr{U}$ is a nonprincipal ultrafilter on the set of prime powers, then the ultraproduct $\prod_{\mathscr{U}} (\mathbb{F}_p^{alg}, \sigma_q)$ is a difference closed field. Moreover, every difference closed field is elementarily equivalent to such an ultraproduct.

Density of periodic points

Theorem (Fakhruddin (who attributes the idea to Poonen who credits Hrushovski))

Let $f : X \to X$ be a dominant AD over a finite field $\mathbb{F}_q$. Then the set of f-periodic points (over $\mathbb{F}_p^{alg}$) is Zariski dense in X.

- One sees from the proof that we may assume merely that $f : X \dashrightarrow X$ is rational (and dominant) and even that we may replace the function f with a generically finite-to-finite (irreducible) correspondence.
- The proof (as we shall see) follows the same pattern as the proof of Ax's theorem on injective self-maps.
- In the published version, Hrushovski's refined Lang-Weil estimates are used directly rather than the assertion about the limit theory.

- Let $f : X \to X$ be a dominant algebraic dynamical system over $\mathbb{F}_q$ and let $Y \subsetneq X_{\mathbb{F}_q^{alg}}$ be a proper subvariety defined over the algebraic closure, hence, over $\mathbb{F}_{q^n}$ for some n. We must find an f-periodic point in $X \smallsetminus Y$.
- If (K, σ) were a difference closed field containing $\mathbb{F}_q^{alg}$, then there would be some point $a \in (X, \Gamma_f)(K) \smallsetminus Y(K)$.
- In particular, if $\mathscr{U}$ is a nonprincipal ultrafilter on the set of prime powers containing $\{q^{nm} : m \in \mathbb{Z}_+\}$, then there is such a point rational over $(K, \sigma) = \prod_{\mathscr{U}} (\mathbb{F}_q^{alg}, \sigma_r)$.
- By Łoś's theorem on ultraproducts, the result is true with σ replaced by σ_r for b $\mathscr{U}$-measure one many r. In particular, there is some m such that the result is true for $\sigma_{q^{nm}}$, but such a point a is σ-periodic and hence f-periodic.

Stable?

The theory of difference closed fields is not stable, but it shares many of the features of stable theories and its analysis motivated the development of what Pillay had called "neo-stability theory."

For us, these key (interrelated) points are:

- (several) good theories of ranks or dimensions for types,
- a good theory of independence,
- a good theory of canonical bases/generalized moduli,
- fine structure theory for rank one types (Zilber trichotomy), and
- a strong structure theory for definable groups and internal automorphism groups.

Orthogonality

Definition

We say that the definable sets (relative to some theory) X and Y are (fully) orthogonal if every definable (even with parameters from any model) subset of $X \times Y$ is a finite Boolean combination of definable sets of the form $U \times V$ where $U \subseteq X$ and $V \subseteq Y$. We write $X \perp Y$.

If X and Y are algebraic varieties, then $X \perp Y$ only if one of X or Y is zero dimensional. Thus, the notion of orthogonality is almost vacuous in algebraic geometry, but for algebraic dynamical systems, or definable sets in difference fields more generally, orthogonal definable sets are the norm.

Orthogonal ADs: Trivial dynamics

Proposition

Let (K, σ) be a difference closed field and $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ a nonconstant rational function over K. Then $(\mathbb{P}^1, f) \perp (\mathbb{P}^1, \mathrm{id})$ if and only if $\deg(f) > 1$.

Proof.

- $(\Longleftarrow)$: If $\deg(f) = 1$, then f is an automorphism of $\mathbb{P}^1$ and solving the difference equation $\sigma(\gamma) \circ f = \gamma$ in $\mathrm{PSL}_2(k)$ we find that the graph of γ is a definable subset of $(\mathbb{P}^1, f)(K) \times (\mathbb{P}^1, \mathrm{id})(K)$ violating orthogonality.
- $(\Longrightarrow)$: Nonorthogonality would be witnessed by a curve $C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ which is mapped to C^σ by (f, id). The restriction $g := (f, \mathrm{id}) \upharpoonright C$ is a rational function $g : C \to C^\sigma$ for which $\pi_1 \circ g = f \circ \pi_1$ and $\pi_2 \circ g = \pi_2$. It follows that $\deg(f) = 1$.

□

Orthogonality for ADs: Between general systems?

The proof outlined above shows that in general if $f : X \to X^\sigma$ and $g : Y \to Y^\sigma$ are nonconstant rational maps on curves, then $(X, f) \not\perp (Y, g)$ implies that $\deg(f) = \deg(g)$, but this is far from a sufficient condition.

- As a test problem: for which parameters a and b do we have $(\mathbb{P}^1, x \mapsto x^2 + a) \not\perp (\mathbb{P}^1, x \mapsto x^2 + b)$?
- For the more general problem where f and g are polynomials (in characteristic zero), a complete answer to the question of when $(\mathbb{P}^1, f) \not\perp (\mathbb{P}^1, g)$ is given in my joint work with Medvedev who will explain some of the details.
- For dynamics on higher dimensional varieties, the situation is more delicate, but orthogonality is still ubiquitous.
- The real question should be: for an AD (X, f), or more generally a difference variety (X, Γ), what are the definable subsets of (X, Γ)? For example, while $(\mathbb{P}^1, f) \not\perp (\mathbb{P}^1, f)$, often the definable subsets of $(\mathbb{P}^1, f)^n$ are severely restricted.