Higher dimensional dynamical Mordell-Lang problems

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Conjecture

Let $f : X \to X$ be a regular self-map of a variety X over $\mathbb{C}$, $a \in X(\mathbb{C})$ a point in X and $Y \subseteq X$ a closed subvariety. Then the set

$$E(a, f, Y) := \{n \in \mathbb{N} \ : \ f^{\circ n}(a) \in Y(\mathbb{C})\}$$

is a finite union of points and arithmetic progressions.

With two notable exceptions, one having to do with the analytic geometry of the Hénon maps and the other for lines in the plane based on the theory of polynomial decompositions, the known proofs of special cases of this conjecture use Skolem's method.

Higher rank dynamical Mordell-Lang

The dynamical Mordell-Lang conjecture should generalize the Mordell-Lang proper, which is naturally a statement about higher rank monoids.

Theorem

Let X be an abelian variety over $\mathbb{C}$, $\Gamma < X(\mathbb{C})$ a finitely generated subgroup, and $Y \subseteq X$ a closed subvariety. Then $Y(\mathbb{C}) \cap \Gamma$ is a finite union of cosets of subgroups of Γ.

Reformulated dynamically:

Theorem

Let X be an abelian variety over $\mathbb{C}$, $\gamma_1, \ldots, \gamma_n \in X(\mathbb{C})$, and $Y \subseteq X$ a closed subvariety. Let $f_i : X \to X$ be defined by $x \mapsto x + \gamma_i$ and $g_i : X \to X$ be defined by $x \mapsto x - \gamma_i$. Then the exponent set

$$\{(\ell_1, \ldots, \ell_n, k_1, \ldots, k_n) \in \mathbb{N}^{2n} \ : \ f_1^{\circ \ell_1} \circ \cdots \circ f_n^{\ell_n} \circ g_1^{\circ k_1} \circ \cdots \circ g_n^{k_n}(0) \in Y(\mathbb{C})\}$$

is a finite union of translates of submonoids of $\mathbb{N}^{2n}$.

No naïve generalization

Example

Let $X = \mathbb{A}^n_{\mathbb{Z}}$ and let $f_i : X \to X$ be defined by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_n)$ for $i \leq n$. If $a = (0, \ldots, 0)$ and $Y \subseteq X$ is a closed subscheme, then

$$\{(\ell_1, \ldots, \ell_n) \in \mathbb{N}^n \ : \ f_1^{\circ \ell_1} \circ \cdots \circ f_n^{\circ \ell_n}(a) \in Y(\mathbb{C})\} = Y(\mathbb{Z}) \cap \mathbb{N}^n$$

In analogy with the Bombieri-Lang conjectures, one might seek geometric restrictions on the dynamical system to rule out phenomena coming from diophantine equations on the integers. As a first test problem, we might take the ambient variety to be a semiabelian variety and the monoid of operators to be maps of algebraic groups.

Problem

Let X be a semiabelian variety over $\mathbb{C}$, $\Gamma \subseteq \mathrm{Hom}(X, X)$ a finitely generated, commutative monoid of endomorphisms of X, $a \in X(\mathbb{C})$ a point and $Y \subseteq X$ a closed subvariety. Must the exponent set

$$\{\gamma \in \Gamma \ : \ \gamma \cdot a \in Y(\mathbb{C})\}$$

be a finite union of translates of submonoids of Γ?

Counter-examples

Ghioca-Tucker-Zieve studied this problem proving positive answers under various hypotheses, but also producing some counter-examples.

Proposition

Let $X = \mathbb{G}_m^3$ *be the Cartesian cube of the multiplicative group. Let* $\Phi : X \to X$ *be given by*

$$(x, y, z) \mapsto (x^2 y^{-1}, y^2 z^{-2}, z^2)$$

and $\Psi : X \to X$ *be give by*

$$(x, y, z) \mapsto (x^2 y^2, y^2 z^4, z^2) \ .$$

Let $\alpha := (1, \frac{1}{3}, 9)$ *and* $V \subseteq X$ *be defined by* $x = 1$. *Then*

$$\begin{aligned} \{(m,n) \in \mathbb{N}^2 \ : \ \Phi^{\circ m} \circ \Psi^{\circ n}(\alpha) \in V(\mathbb{Q})\} &= \\ \{(m,n) \in \mathbb{N}^2 \ : \ (2n - m)^2 = 6m\} &= \\ \{(12\ell^2 + 6\ell, 6\ell^2) \ : \ \ell \in \mathbb{Z}\} & \end{aligned}$$

Given an algebraic variety X, self-maps $\Phi_i : X \to X$ for $1 \leq i \leq n$, a point $a \in X$ and a subvariety $Y \subseteq X$, we define

$$E = E(a, \Phi_1, \ldots, \Phi_n, Y) := \{(\ell_1, \ldots, \ell_n) \in \mathbb{N}^n \ : \ \Phi_1^{\circ \ell_1} \circ \cdots \circ \Phi_n^{\circ \ell_n}(a) \in Y\}$$

While the exponent set as defined makes sense with any choice of self-maps, henceforth we shall assume that the various Φ_i commute.

How complicated can E be?

From algebraic groups to linear algebra

In the case that $X = \mathbb{G}_m^g$ is an algebraic torus and the maps $\Phi_i : X \to X$ are commuting algebraic group endomorphisms, then the problem reduces to one of understanding intersection of orbits of points under finitely generated commutative groups of integer matrices with affine spaces.

Proposition

Let $\Phi_1, \ldots, \Phi_n$ be a finite sequence of commuting endomorphisms of $X = \mathbb{G}_m^g$, $a \in X(\mathbb{Q})$, and $Y \subseteq X$ a closed subvariety, then there is a finite union $Z \subseteq X$ of translates of algebraic subgroups for which

$$E(a, \Phi_1, \ldots, \Phi_n, Y) = E(a, \Phi_1, \ldots, \Phi_n, Z)$$

Proof.

- The set $\mathscr{O}(a) := \{\Phi_1^{\ell_1} \cdots \Phi_n^{\ell_n}(a) \ : \ (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n\}$ is a subset of $\langle a \rangle$, the group generated by a.
- Let $Z := \overline{\langle a \rangle \cap Y(\mathbb{Q})}$.
- $\mathscr{O}(a) \cap Y = \mathscr{O}(a) \cap Z$.

□

From algebraic tori to linear algebra

A map $\Phi : \mathbb{G}_m^g \to \mathbb{G}_m^g$ is given by a sequence of monomials

$$(x_1, \ldots, x_g) \mapsto (\prod_{j=1}^{g} x_j^{M_{1,j}}, \ldots, \prod_{j=1}^{g} x_j^{M_{g,j}})$$

where $(M_{i,j}) \in M_{g \times g}(\mathbb{Z})$ is an integer matrix. Let $\widetilde{\Phi} : \mathbb{Z}^g \to \mathbb{Z}^g$ be the corresponding linear map.

Likewise, if $Z \subseteq \mathbb{G}_m^g$ is a translate of an algebraic group, then it is defined by finitely many inhomogeneous monomial equations of the form

$$\prod x_j^{M_j} = c$$

Combining these observations with the proposition, we see that there is a subset $N \subseteq \mathbb{Z}^g$ which is a finite union of cosets of subgroups and a point $b \in \mathbb{Z}^g$ so that

$$E(a, \Phi_1, \ldots, \Phi_n, Y) = E(b, \widetilde{\Phi}_1, \ldots, \widetilde{\Phi}_n, N)$$

Jordan form

As $\widetilde{\Phi}_1, \ldots, \widetilde{\Phi}_n$ commute, after base changing to $\mathbb{C}$, we may choose a common basis with respect to which each $\widetilde{\Phi}_i$ is expressed in Jordan canonical form.

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \ddots & \ddots & \ddots & \cdots & \ddots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & \lambda^{n-1}n & \lambda^{n-2}\binom{n}{2} & \cdots & \lambda^{n-m+1}\binom{n}{m} \\ 0 & \lambda^n & \lambda^{n-1}n & \cdots & \lambda^{n-m+2}\binom{n}{m-1} \\ \ddots & \ddots & \ddots & \cdots & \ddots \\ 0 & 0 & 0 & \cdots & \lambda^n \end{pmatrix}$$

Thus, the entries of the matrices $\widetilde{\Phi}_1^{\ell_1} \cdots \widetilde{\Phi}_n^{\ell_n}$ will have the form

$$\lambda_1^{\ell_1} \cdots \lambda_n^{\ell_n} \binom{\ell_1}{k_1} \cdots \binom{\ell_n}{k_n}$$

for some eigenvalues λ_i of $\widetilde{\Phi}_i$ and $k_i \leq g$.

Exponential-polynomial sets

By an exponential monomial (over a commutative ring R), I will mean an expression of the form

$$cx_1^{M_1} \cdots x_n^{M_n} \cdot \alpha_1^{x_1} \cdots \alpha_n^{x_n}$$

where each M_i is a natural number, $c \in R$, and each α_i belongs to R. Each such exponential monomial gives a function $f : \mathbb{N}^n \to R$ defined by $(\ell_1, \ldots, \ell_n) \mapsto c\ell_1^{M_1} \cdots \ell_n^{M_n} \cdot \alpha_1^{\ell_1} \cdots \alpha_n^{\ell_n}$.

An exponential-polynomial function is a function $f : \mathbb{N}^n \to R$ given as the sum of finitely many exponential monomial functions.

An exponential-polynomial set is a subset of $\mathbb{N}^n$ defined by the vanishing of finitely many exponential-polynomial functions.

Remarks on exponential-polynomial sets

- Taking $\alpha_i = 1$ in the exponential monomials, one sees that every polynomial is an exponential-polynomial so that the class of exponential-polynomial sets includes all sets of the form $Y(\mathbb{Z}) \cap \mathbb{N}^n$ with $Y \subseteq \mathbb{A}^n_{\mathbb{Z}}$ a closed affine scheme over $\mathbb{Z}$.
- Likewise, the graph of any function built from exponentials is an exponential-polynomial set. For example, if F_n is the n^{th} Fibonacci number, then $\{(n, F_n) \ : \ n \in \mathbb{N}\}$ is an exponential-polynomial set.
- In work preceding Matiyasevich's solution of Hilbert's Tenth Problem, Davis-Putnam-Robinson showed that for every recursively enumerable set $X \subseteq \mathbb{N}$ there is some exponential-polynomial set $Y \subseteq \mathbb{N}^{1+n}$ for which X is the projection to the first coordinate of Y.

Strong failure of dynamical Mordell-Lang

As observed by Ghioca-Tucker-Zieve, the above considerations show

Proposition

If X is a semiabelian variety X over $\mathbb{C}$, $\Phi_1, \ldots, \Phi_n$ is a sequence of commuting algebraic group endomorphisms $\Phi_i : X \to X$, $a \in X(\mathbb{C})$ is any point, and $Y \subseteq X$ is a closed subvariety, then $E(a, \Phi_1, \ldots, \Phi_n, Y)$ is an exponential-polynomial set.

Our goal is to prove a converse showing that the higher rank dynamical Mordell-Lang conjecture fails as badly as one could imagine.

Theorem

For each exponential-polynomial set $S \subseteq \mathbb{N}^n$ there is a semiabelian variety X, a sequence of n commuting algebraic group endomorphisms $\Phi_i : X \to X$, a point $a \in X(\mathbb{C})$, and a closed subvariety Y for which $S = E(X, \Phi_1, \ldots, \Phi_n, a, Y)$.

Key ideas to proof

- First show that for any commutative ring R, any R-exponential-polynomial set (one in which the bases of the exponentials and the coefficients are taken from R) may be encoded as exponent sets for finite lists of R-linear maps on a free R-module. The basic monomials are easy to extract using the calculation of powers of Jordan matrices. General exponential-polynomials are then coded using direct sums.
- When R is a ring of integers in a number field, then a finite rank free R-module with a list of commuting R-linear maps may be regarded as a finite free $\mathbb{Z}$-module with a list of commuting $\mathbb{Z}$-linear maps. The exponent sets do not change in passing from R to $\mathbb{Z}$.
- Exponentiating the linear dynamical system over $\mathbb{Z}$ we obtain a dynamical system on a torus again with the same exponent set.

The most effective approach to the dynamical Mordell-Lang conjecture for a single morphism uses Skolem's method.

That is, we are given $\phi : X \to X$ defined over some finitely generated field k of characteristic 0, a starting point $a \in X(k)$ and a closed subvariety $Y \subseteq X$. Under various hypotheses, one can find an embedding $k \hookrightarrow \mathbb{Q}_p$ and a p-adic analytic map $\Phi : \mathbb{Z}_p \to X(\mathbb{Q}_p)$ so that $\Phi(0) = a$ and $\Phi(x+1) = \phi(\Phi(x))$.

To be honest, one usually finds some $N \geq 1$ and maps $\Phi_j : \mathbb{Z}_p \to Z(\mathbb{Q}_p)$ so that $\Phi_j(0) = \phi^j(a)$ and $\Phi_j(x+1) = \phi^{\circ N}(\Phi_j(x))$ for $0 \leq j < N$.

The set $\{\alpha \in \mathbb{Z}_p \ : \ \Phi(\alpha) \in Y(\mathbb{Q}_p)\}$ being the zero set of a p-adic analytic function is then either finite or all of $\mathbb{Z}_p$.

In some cases where the p-adic flows do not exist, it is possible to find real analytic flows which are o-minimally definable (for example, when the map is conjugate to a monomial map for which the corresponding integer matrix has only real eigenvalues).

Definition

An expansion $(\mathbb{R}, +, \cdot, <, \ldots)$ of the real field is o-minimal if every definable subset of the line is a finite union of points and intervals.

For us, the one relevant structure is $\mathbb{R}_{\mathrm{an,exp}}$: the expansion of the real field by the globally defined real exponential function and all real analytic functions on closed boxes.

In the cases where an o-minimally definable flow exists, Skolem's method adapts immediately.

Pila-Wilkie counting theorem: set-up

- For $q = \frac{a}{b} \in \mathbb{Q}^\times$ written in lowest terms, we define $H(q) := \max\{|a|, |b|\}$ (and $H(0) := 0$).
- For $q = (q_1, \ldots, q_n) \in \mathbb{Q}^n$ we define $H(q) := \max\{h(q_i) \ : \ i \leq n\}$
- For $X \subseteq \mathbb{R}^n$ and $t > 0$ we define $X(\mathbb{Q}, t) := \{q \in X \cap \mathbb{Q}^n \ : \ H(q) \leq t\}$ and $N(X, t) := \#X(\mathbb{Q}, t)$.
- For $X \subseteq \mathbb{R}^n$ we define X^{alg} to be the union of all infinite, connected semialgebraic subsets of X and $X^{\mathrm{tr}} := X \smallsetminus X^{\mathrm{alg}}$.

Pila-Wilkie counting theorem

Theorem

Let $X \subseteq \mathbb{R}^n$ be definable in some o-minimal expansion of the real field. Then for every $\epsilon > 0$ there is a constant $C = C_\epsilon$ so that $N(X^{tr}, t) \leq Ct^\epsilon$.

Clucker-Comte-Loeser have proven an exact analogue the Pila-Wilkie counting theorem for sets definable using p-adic analytic functions.

Suppose that X is an algebraic variety over $\mathbb{C}$, $a \in X(\mathbb{C})$, $\phi_1, \ldots, \phi_n$ is a sequence of commuting regular maps $\phi_i : X \to X$, and $Y \subseteq X$ is a closed subvariety. Suppose moreover that there is an open set $U \subseteq X(\mathbb{C})$ containing the orbit of a and that there are o-minimally definable real analytic maps $\Phi_i : [0, \infty) \times U \to U$ satisfying $\Phi_i(0, b) = b$ and $\Phi_i(x + 1, b) = \phi_i(\Phi_i(x, b))$.

The set

$$\mathfrak{Y} := \{(x_1, \ldots, x_n) \in [0, \infty)^n \ : \ \Phi_n(x_n, \Phi_{n-1}(x_{n-1}, \ldots \Phi_1(x_1, a) \ldots)) \in Y(\mathbb{C})\}$$

is definable and by construction

$$E(a, \phi_1, \ldots, \phi_n, Y) = \mathbb{N}^n \cap \mathfrak{Y}$$

Consequently, the number of points in $E(a, \phi_1, \ldots, \phi_n, Y)$ of height $\leq t$ outside of the algebraic locus of $\mathfrak{Y}$ is $O(t^\epsilon)$ for all $\epsilon > 0$.

What is $\mathfrak{Y}^{\mathrm{alg}}$?

- For $X = \mathbb{A}^n$, $\phi_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_n)$, we have $\mathfrak{Y}^{\mathrm{alg}} = \mathfrak{Y}$.
- If $f : \mathbb{P}_1 \to \mathbb{P}_1$, $X = (\mathbb{P}_1)^n$, $\phi_i : X \to X$ is given by f on the i^{th} coordinate, and b is near a superattracting point for f and $a = (b, \ldots, b)$, then $\mathfrak{Y}^{\mathrm{alg}}$ is a finite union of sets defined by equations of the form $x_j = x_i + m$ for some natural number m.
- In general, the question is difficult.

Questions

- Almost all results on the dynamical Mordell-Lang conjecture, both positive and negative, use in essential ways a reduction to linear dynamics. Can we find an example where a complicated exponent set may be computed without passing to linear algebra?
- Are all exponent sets for algebraic actions of $\mathbb{N}^n$ exponential-polynomial sets?
- The Pila-Wilkie bounds on the size of exponent sets are almost certainly too large. Can we replace $O(t^\epsilon)$ with $O(\log(t)^K)$?