SUPERSTABILITY OF F-STRUCTURES

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The following note is a supplement to [1]. We will maintain the notation of [1] throughout, and all numbered items to which we refer are from that paper. Theorem 6.11 states that if M is a finitely generated R-module then $\mathrm{Th}(M, \mathcal{F})$ is stable. As pointed out in Remark 6.12, the proof Theorem 6.11 yields a stronger conclusion: $\mathrm{Th}(M, \mathcal{F})$ is superstable. These notes consist of detailed arguments for this strengthening.

We wish to extend our notion of dimension for exponential F-sets (Definition 6.6) to arbitrary groupless F-sets. First of all, we point out that δ-dimension (unlike δ-degree) does not really depend on δ.

Lemma 0.1. *Suppose M is a finitely generated R-module, $\delta > 0$ is a multiple of δ_M, and $S \in \mathrm{Exp}_M(\delta)$. Suppose $\delta' > 0$ is another multiple of δ_M and $S \in \mathrm{Exp}_M(\delta')$. Then $\dim_\delta S = \dim_{\delta'} S$.*

Proof. Working instead with $\delta'\delta$, it suffices to deal with the case when δ' is a multiple of δ. Write $S = F^B\overline{a}$ where $\overline{a}$ is a tuple from M and B is a δ-closed set. Then B is also δ'-closed (see Remark 3.2). Now $\dim_\delta S$ is the Morley rank of $[B]_{\overline{a}}$ computed in $(\mathbb{N}, 0, \sigma, P_\delta)$, while $\dim_{\delta'} S$ is the Morley rank of $[B]_{\overline{a}}$ computed in $(\mathbb{N}, 0, \sigma, P_{\delta'})$. But these coincide. Indeed, both structures are of Morley rank 1 (Proposition $A.1$ and the bi-interpretation given by Proposition $A.2$), so that Morley rank is given by algebraic dimension. From quantifier elimination for these structures (again by Proposition $A.1$ and the bi-interpretation given by Proposition $A.2$), it follows that algebraic closure in $(\mathbb{N}, 0, \sigma, P_\delta)$ is the same as algebraic closure in $(\mathbb{N}, 0, \sigma, P_{\delta'})$. Thus, algebraic dimensions agree, and therefore, also Morley ranks. □

Lemma 0.1 allows us to define an absolute dimension for exponential F-sets:

Definition 0.2. If $S \in \mathrm{Exp}_M$ then the *dimension of S*, denoted by $\dim S$, is the δ-dimension of S for any $\delta > 0$ a multiple of δ_M such that $S \in \mathrm{Exp}_M(\delta)$.

Lemma 0.3. *Suppose $S_1, \ldots, S_n, T_1, \ldots, T_m \in \mathrm{Exp}_M$, and $\bigcup_{i=1}^n S_i = \bigcup_{j=1}^m T_j$. Then,* $\max\{\dim S_i : 1 \leq i \leq n\} = \max\{\dim T_j : 1 \leq j \leq m\}$.

Proof. First consider the case when $n = 1$, and let $S := S_1$. Let $\delta > 0$ be a multiple of δ_M such that $S, T_1, \ldots, T_m \in \mathrm{Exp}_M(\delta)$. Write $S = F^B\overline{a}$ and $T_j = F^{C_j}\overline{b}_j$ for each $j = 1, \ldots, m$, where B and the C_j's are δ-closed. Then, as each $T_j \subset S$, $T_j = F^{D_j}\overline{a}$ where D_j is the δ-closed subset of B given by

$$D_j := \{\overline{r} \in B : \text{ for some } \overline{s} \in C_j, (\overline{r}, \overline{s}) \in \log_{(\overline{a}, -\overline{b}_j)} 0\}.$$

Date: 29 June 2003.

So $\dim_\delta T_j = \mathrm{RM}[D_j]_{\overline{a}}$. Also, $B = \bigcup_{j=1}^m D_j$, and hence $[B]_{\overline{a}} = \bigcup_{j=1}^m [D_j]_{\overline{a}}$. Computing Morley rank in $(\mathbb{N}, 0, \sigma, P_\delta)$ we have $\dim_\delta S = \max\{\dim_\delta T_j : 1 \leq j \leq m\}$.

We have proved the $n = 1$ case of the Lemma. From this it follows that if $S \in \mathrm{Exp}_M$ and $S \subset \bigcup_{j=1}^m T_j$, then $\dim S \leq \max\{\dim T_j : 1 \leq j \leq m\}$. Indeed, $S = \bigcup_{j=1}^m (T_j \cap S)$, and each $T_j \cap S$ is a finite union of exponential F-sets of dimension at most $\dim T_j$ (see Remarks 6.5 and 6.7). It follows that if $\displaystyle\bigcup_{i=1}^n S_i = \bigcup_{j=1}^m T_j$, then $\max\{\dim S_i : 1 \leq i \leq n\} \leq \max\{\dim T_j : 1 \leq j \leq m\}$. By symmetry, we have $\max\{\dim S_i : 1 \leq i \leq n\} = \max\{\dim T_j : 1 \leq j \leq m\}$, as desired. □

We can thus extend the notion of dimension to cycle-free groupless F-sets:

Definition 0.4. Suppose M is a finitely generated R-module. For $U \in \mathrm{Orb}_M$, we define the *dimension of* U to be

$$\dim U = \max\{\dim S_i : 1 \leq i \leq n\},$$

for any $S_1, \ldots, S_n \in \mathrm{Exp}_M$ such that $U = \displaystyle\bigcup_{i=1}^n S_i$.

We wish to extend this definition to arbitrary groupless F-sets. Given $U \in \mathrm{Groupless}(M)$ there exists a finitely generated R-module $M' \geq M$ such that $U \in \mathrm{Orb}_{M'}$ (by Lemma 2.7). One can, of course, define the *dimension of* U *with respect to* M' to be the dimension of U computed as a cycle-free groupless F-set in M'. Suppose $M'' \geq M$ is another finitely generated R-module such that $U \in \mathrm{Orb}_{M''}$. Let N be a finitely generated R-module extending both M' and M''. Note that for any $S \in \mathrm{Exp}_{M'}$, the dimension of S as computed in M' coincides with that computed in N (since for any tuple $\overline{a} \in (M')^n$, $\overline{a}$-equivalence on $\mathbb{N}^n$ does not depend on whether one works with M' or N). It follows that the dimension of $U \in \mathrm{Orb}_{M'}$ is constant whether we compute it in M' or in N. Similarly for M'' and N. Hence the dimension of $U \in \mathrm{Groupless}(M)$ with respect to M' is equal to the dimension of U with respect to M''. We can therefore define:

Definition 0.5. Suppose $U \in \mathrm{Groupless}(M)$. The *dimension of* U, denoted by $\dim U$, is defined to be the dimension of U with respect to M', where $M' \geq M$ is any finitely generated R-module such that $U \in \mathrm{Orb}_{M'}$.

Lemma 0.6. (a) *If* $U \in \mathrm{Groupless}(M)$ *and* $c \in M$, *then* $\dim(c + U) = \dim U$.
(b) *If* $U, V \in \mathrm{Groupless}(M)$ *and* $U \subset V$, *then* $\dim U \leq \dim V$.
(c) *Suppose* $G \leq M$ *is a submodule,* $\pi : M \to M/G$ *is the quotient map, and* $U \in \mathrm{Groupless}(M)$. *Then* $\dim \pi U \leq \dim U$.

Proof. Let $M' \geq M$ be such that $U \in \mathrm{Orb}_{M'}$. Let $\delta > 0$ be a multiple of $\delta_{M'}$ such that $U \in \mathrm{Orb}_{M'}(\delta)$. Write $U = \bigcup_{i=1}^n S_i$, for some $S_1, \ldots, S_n \in \mathrm{Exp}_{M'}(\delta)$. Then $\dim U = \max\{\dim_\delta S_i : 1 \leq i \leq n\}$. On the other hand, $c + U = \bigcup_{i=1}^n (c + S_i)$, and $\dim_\delta(c + S_i) = \dim_\delta S_i$ for each i (see Remark 6.7). This proves part (a).

For part (b), let $M' \geq M$ be such that $U, V \in \mathrm{Orb}_{M'}$. Write $U = \bigcup_{i=1}^n S_i$ and $V = \bigcup_{j=1}^m T_j$, for some $S_1, \ldots, S_n, T_1, \ldots, T_m \in \mathrm{Exp}_{M'}$. So each $S_i \subset \bigcup_{j=1}^m T_j$. As in the proof of Lemma 0.3 (or by Remark 6.7), $\dim S_i \leq \max\{\dim T_j : 1 \leq j \leq m\} = \dim V$. Hence $\dim U \leq \dim V$, as desired.

For part (c), let $M' \geq M$ and $\delta > 0$ be a multiple of $\delta_{M'}$ such that $U \in \mathrm{Orb}_{M'}^M(\delta)$. Write $U = \bigcup_{i=1}^\ell S_i$, where each $S_i \in \mathrm{Exp}_{M'}^M(\delta)$. Write $S_i = F^{B_i}\overline{a}_i$, where $\overline{a}_i$ is a tuple from M' and B_i is δ-closed. Let $\pi' : M' \to M'/G$ be the quotient map. Then $\pi U = \bigcup_{i=1}^{\ell} \pi S_i$ and each $\pi S_i = F^{B_i}\pi'(\overline{a}_i)$. It suffices therefore to show that $\mathrm{RM}[B_i]_{\overline{a}_i} \geq \mathrm{RM}[B_i]_{\pi'(\overline{a}_i)}$ in $(\mathbb{N}, 0, \sigma, P_\delta)$, for each $i = 1, \ldots, \ell$. But this follows from the observation that $\overline{a}_i$-equivalence is finer than $\pi'(\overline{a}_i)$-equivalence. □

We now prove the superstability of F-structures. Given stability, this is equivalent to the assertion that there are no infinite forking chains. We prove this by defining an ordinal valued rank which drops when a type forks. We work in a sufficiently saturated elementary extension of M, ${}^*M$.

For $G \leq M$ a submodule, we define $d(G) := \dim_{\mathbb{Q}}(M \otimes \mathbb{Q})$. For $U \in \mathrm{Groupless}(M)$, we define $d_G(U)$ to be the dimension of $\pi U \in \mathrm{Groupless}(M/G)$, where π is the quotient map. We define

$$d(U, G) := \langle d(G), d_G(U)\rangle \in \mathbb{N} \times \mathbb{N} = \omega \cdot 2.$$

Here we regard $\mathbb{N} \times \mathbb{N}$ as the ordinal $\omega \cdot 2$ via the lexicographic ordering.

Let $q \in S_1(N)$ be a type over an elementary substructure $N \preceq {}^*M$. We extend the notation of the proof of Theorem 6.11 so that

$$D_{U,G}(q) := \{a \in {}^*M : q(x) \mid {}^*M \vdash x \in a + {}^*U + {}^*G\}$$

where $q \mid {}^*M$ denotes the unique nonforking extension of q to ${}^*M$. Note that as a consequence of stability, $D_{U,G}(q)$ is definable over N, and as $N \preceq {}^*M$, if $D_{U,G}(q)$ is non-empty then $D_{U,G}(q) \cap N$ is non-empty. We define

$$R(q) := \min\{d(U, G) : D_{U,G}(q) \neq \emptyset, U \in \mathrm{Groupless}(M), G \leq M \text{ a submodule}\}.$$

Proposition 0.7. *Let $N \preceq {}^*M$ be an elementary substructure and $p \in S_1({}^*M)$. If p forks over N, then $R(p \restriction N) > R(p)$.*

Proof. We show the contrapositive. Let $q := p \restriction N$, and assume that $R(p) \geq R(q)$. We will show that for any $G \leq M$ a submodule, and $U \in \mathrm{Groupless}(M)$, $D_{U,G}(p)$ is N-definable. As in Theorem 6.11, this will prove that p is N-definable, and hence p does not fork over N.

Fix $G \leq M$ submodule, and $U \in \mathrm{Groupless}(M)$ such that $D_{U,G}(p)$ is non-empty (else it is clearly N-definable). Also, let $V \in \mathrm{Groupless}(M)$ and $H \leq M$ be such that $D_{V,H}(q) \neq \emptyset$ and $d(V, G) = R(q)$. Fix $c \in D_{V,H}(q) \cap N$. Let $W_1, \ldots, W_\ell \in \mathrm{Groupless}(M)$ be such that for any $b \in {}^*M$,

$$(b + {}^*U + {}^*G) \cap (c + {}^*V + {}^*H) = \bigcup_{i \in J} d_j + {}^*W_j + {}^*(G \cap H) \tag{1}$$

for some $J \subset \{1, \ldots, \ell\}$ and $d_j \in {}^*M$ (using Corollary 5.5). Note that $b \in D_{U,G}(p)$ if and only if there exists $J \subset \{1, \ldots, \ell\}$ and $(d_j)_{j \in J}$ such that equation (1) holds and $d_j \in D_{W_j, G\cap H}(p)$ for some $j \in J$. As $c \in N$, it suffices to show that each $D_{W_j, G\cap H}(p)$ is N-definable. That is we may assume $G \leq H$.

As $d(G) \geq d(H)$, it must be that $d(G) = d(H)$ and G is of finite index in H. So for some $h_1, \ldots, h_n \in H$, ${}^*V + {}^*H = \bigcup_{i=1}^{n} {}^*(h_i + V) + {}^*G$. Hence $D_{h_i+V,G}(q) \neq \emptyset$ for some i. Also $d_G(h_i + V) = d_G(V) \leq d_H(V)$ by Lemma 0.6 parts (a) and (c). By the minimality of $d(V, H)$, we have that $d(h_i + V, G) = d(V, H) = R(q)$. That is, replacing V with $h_i + V$ and H with G, we may assume that $G = H$.

Letting $\overline{M} = M/G$, we work in ${}^*\overline{M} := {}^*M/{}^*G$ with the induced structure of an elementary extension of $\overline{M}$. Let $\overline{N} := N/({}^*G \cap N) \preceq {}^*\overline{M}$, and let π denote the quotient map. If $p = \mathrm{tp}(\alpha/^*M)$ and $q = \mathrm{tp}(\alpha/N)$, then let $\overline{p} := \mathrm{tp}(\pi\alpha/^*\overline{M})$, and its restriction $\overline{q} := \mathrm{tp}(\pi\alpha/\overline{N})$. In order to show that $D_{U,G}(p)$ is definable over N, it suffices to show that $D_{\pi U,0}(\overline{p})$ is $\overline{N}$-definable.

Let $M' \geq \overline{M}$ be a finitely generated R-module, and $\delta > 0$ a multiple of $\delta_{M'}$, such that $\pi U, \pi V \in \mathrm{Orb}^{\overline{M}}_{M'}(\delta)$. Write $\pi V = \bigcup_{i=1}^n T_i$ where $T_i \in \mathrm{Exp}^{\overline{M}}_{M'}(\delta)$. Using Proposition $A.6$, we may assume that each $\deg_\delta(T_i) = 1$. Since $D_{V,G}(q)$ is non-empty, for some $j \in \{1, \ldots, n\}$, $D_{T_j,0}(\overline{q})$ is also non-empty. We claim that it suffices to show:

(2) If $T \in \mathrm{Exp}^{\overline{M}}_{M'}(\delta)$ and $D_{T,0}(\overline{p}) \neq \emptyset$, then $(\dim_\delta, \deg_\delta)(T) \geq (\dim_\delta, \deg_\delta)(T_j)$.

Indeed, from the proof of Theorem 6.11, if (2) holds, then for any $b \in D_{T_j,0}(\overline{p})$, $D_{\pi U,0}(\overline{p})$ will be b-definable. Since $D_{T_j,0}(\overline{q}) \cap \overline{N} \neq \emptyset$, we would have that $D_{\pi U,0}(\overline{p})$ is definable over $\overline{N}$, as desired.

To show (2), note that as $\deg_\delta T_j = 1$, we need only show that $\dim_\delta T \geq \dim_\delta T_j$ As $T \in \mathrm{Groupless}(\overline{M})$, there is $W \in \mathrm{Groupless}(M)$ such that $\pi W = T$. Moreover, as $D_{T,0}(\overline{p})$ is non-empty, so is $D_{W,G}(p)$. It follows that

$$\langle d(G), \dim_\delta T\rangle = d(W,G) \geq R(p) \geq R(q) = d(V,G) \geq \langle d(G), \dim_\delta T_j\rangle.$$

This proves (2) and hence the Proposition. □

Corollary 0.8. *For M a finitely generated R-module,* $\mathrm{Th}(M, \mathcal{F})$ *is superstable.*

Proof. By Proposition 0.7, there is no infinite forking chain of 1-types over elementary substructures. Superstability follows from the following lemma (which is probably well-known). □

Lemma 0.9. *Let T be a simple theory. Then T is supersimple just in case there is no infinite elementary chain $M_0 \preceq M_1 \preceq M_2 \preceq \cdots \bigcup M_i =: M \models T$ and type $p \in S_1(M)$ such that $p \upharpoonright M_{i+1}$ forks over M_i for all $i \in \omega$.*

Proof. Suppose that T is not supersimple and seek a contradiction. Then we can find some model $M \models T$, an increasing sequence $A_0 \subseteq A_1 \subseteq \cdots M$ of subsets of M and an element $a \in M$ such that $\mathrm{tp}(a/A_{i+1})$ forks over A_i for each i.

Working in a definitional expansion of the language of T we may assume that T eliminates quantifiers so that every extension of models of T is elementary.

Let $C := \{a\} \cup \bigcup_{i=0}^\infty A_i$. We build the sequence of models $\langle M_i \mid i \in \omega\rangle$ by recursion. Let M_0 be a model containing A_0 with M_0 free from C over A_0. For example, let M_0 realize a nonforking extension of $\mathrm{tp}(M/A_0)$ to C. At stage $i+1$ let M_{i+1} be a model containing $A_{i+1} \cup M_i$ which is free from C over $A_{i+1} \cup M_i$. Set $p := \mathrm{tp}(a/\bigcup_{i\in\omega} M_i)$.

We claim that $p \upharpoonright M_{i+1}$ forks over M_i for each i. We proceed by induction on i.

We start with the case of $i = 0$. If the claim were false in this case, then a would be free from M_1 over M_0. But as M_0 is free from C over A_0, we have (using symmetry, transitivity, and monotonicity) that a is free from A_1 over A_0, a contradiction.

We consider now the case of $i+1$. Suppose that a is free from M_{i+2} over M_{i+1}. We argue by induction on $j \leq i+2$ that a is free from M_{i+2} over $A_{i+1} \cup M_{i+1-j}$ where $M_{-1} := \varnothing$.

The case of $j = 0$ is given by hypothesis. Suppose now the result for j (with $j \leq i+1$). By construction, M_{i+1-j} is free from C over $M_{i+1-(j+1)} \cup A_{i+1-j}$. By monotonicity and symmetry, we see that a is free from M_{i+1-j} over $M_{i+1-(j+1)} \cup A_{i+1}$. By transitivity (using the inductive hypothesis) we have that a is free from M_{i+2} over $M_{i+1-(j+1)} \cup A_{i+1}$, as claimed.

Taking $j = i+2$, we see that a is free from M_{i+2} over A_{i+1}. By monotonicity, a is free from A_{i+2} over A_{i+1}, a contradiction. □

References

[1] R. Moosa and T. Scanlon. F-structures and integral points on semiabelian varieties over finite fields. Preprint, 2003.